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N-TIMES INTEGRATED C-SEMIGROUPS AND THE ABSTRACT CAUCHY PROBLEM

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Abstract. This paper is concerned with generation theorems for exponentially equicontinuous n-times integrated C-semigroups of linear operators on a sequentially complete locally convex space (SCLCS). The generator of a nondegenerate n-times integrated C-semigroup is characterized. The proofs will base on a SCLCS-version of the Widder-Arendt theorem about the Laplace transforms of Lipschitz continuous functions, and on some properties of a C-pseudoresolvent. We also discuss the existence and uniqueness of solutions of the abstract Cauchy problem: $u' = Au + f,\ u(0) = x$, for $x \in C(D(A^{n+1}))$ and suitable function f.

1. Introduction

The classical theory of (C_0)-semigroups of operators is a powerful method in studying the first order Cauchy problem: $u'(t) = Au(t), t > 0; u(0) = x \in D(A)$, where A is a closed linear operator satisfying certain conditions and having dense domain (see [8], [10], [22]). To deal with the Cauchy problem with an A satisfying weaker conditions or having nondense domain, recently two kinds of generalizations of (C_0)-semigroups have been developed by some authors.

The first kind is the so-called C-semigroups. A C-semigroup is a strongly continuous family $\{S(t); t \geq 0\}$ of bounded linear operators satisfying $S(0) = C$, a bounded injective operator, and $S(s+t)C = S(s)S(t)(s, t \geq 0)$. It was studied first by Da Prato [2] and later by Davies and Pang [3] for the case that C has dense range, and it was further generalized by Miyadera and Tanaka ([17], [18], [26], [27], [28]), and deLaubenfels ([5], [7]) for the case that the

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domain $D(A)$ of A may not be dense. See also [25] for representation formulas of C-semigroups.

The other kind of generalization is the so-called n-times integrated semigroups. A 0-times integrated semigroup is just a (C_0)-semigroup. When $n \geq 1$, an n-times integrated semigroup is defined as the special case: $C = I$ of our Definition 1.1, to be given below. This generalized semigroup was introduced by Arendt [1], and further developed by Kellerman and Hieber [11], Neubrander ([20], [21]), Tanaka and Miyadera ([27], [28]), and deLaubenfels [4]. Further extension to fractionally integrated semigroups were carried out by Hieber [9], and Miyadera, Okubo and Tanaka [19].

Both these theories are established in the setting of a Banach space, and neither of them completely covers the other. Moreover, two different approaches have been used; namely, the proof for C-semigroups, as an extension of Hille and Yosida's treatment for (C_0)-semigroups, is of operator theoretical nature, and Arendt's treatment for n-times integrated semigroups makes use of an integrated version of Widder's theorem on Laplace transform of Banach-space-valued functions.

This paper aims to study a natural generalization of the above two notions to a wider class of operator families, called exponentially equicontinuous n-times integrated C-semigroups. The ground space will be a sequentially complete locally convex space. The result in particular provides a unified treatment for both C-semigroups and integrated semigroups, and extends and improves many existing results.

To begin with, let X be a sequentially complete locally convex space, and let $L(X)$ denote the space of all continuous linear operators on X.

Definition 1.1. Let $n \geq 1$ and $C \in L(X)$. A strongly continuous family $\{S(t); t \geq 0\} \subset L(X)$ is called an n-times integrated C-semigroup on X if it satisfies:

(1.1) $S(t)C = CS(t)$ for $t \geq 0$ and $S(0) = 0$;

(1.2) $S(t)S(s)x = \frac{1}{(n-1)!}(\int_0^{s+t} - \int_0^s - \int_0^t)(s+t-r)^{n-1}S(r)Cx dr$ for $x \in X$ and $t, s \geq 0$.

$S(\cdot)$ is said to be nondegenerate if

(1.3) $S(t)x = 0$ for all $t > 0$ implies $x = 0$.

Finally, $S(\cdot)$ is called exponentially equicontinuous if

(1.4) there is $w \in R$ such that $\{e^{-wt}S(t);\ t \geq 0\}$ is equicontinuous.

A 0-times integrated C-semigroup is just a C-semigroup.

In Section 4, relations between an exponentially equicontinuous integrated C-semigroup and its Laplace transforms will be discussed. Sections 5 and 6 will concentrate on the generation of nondegenerate n-times integrated C-semigroups. Their generators will be defined and characterized. As will be seen in Section 8, for each $n \geq 1$ there exists an n-times integrated C-semigroup with $C \neq I$ such that its generator does not generate a $(n-1)$-times integrated C-semigroup. Section 7 will devote to the associated abstract Cauchy problem: $u' = Au + f, u(0) = x$.

The proofs of the generation theorems (Theorems 4.2 and 6.2) will base on a version of the Widder-Arendt theorem, for functions with values in a sequentially complete locally convex space, and also on some properties of C-pseudoresolvents. For this sake, these two subjects will be discussed in Sections 2 and 3 first. They also serve as tools in establishing a parallel theory of exponentially equicontinuous n-times integrated C-cosine functions ([13], [14]). However, this method does not apply to those n-times integrated C-semigroups which are not exponentially equicontinuous, because their Laplace transforms can not be defined. In general, integrated C-semigroups are not necessarily exponentially bounded (see e.g. [7], [9]). In [12], generators of non-exponentially bounded, fractionally integrated C-semigroups are characterized in terms of existence and uniqueness of solutions of the abstract Cauchy problem: $u' = Au + f, u(0) = x$ for suitable initial value $x \in X$ and function f. On the other hand, it is known that every n-times integrated C-semigroup of hermitian operators on a Banach space is exponentially bounded. This and some other interesting properties of n-times integrated C-semigroups of hermitian and positive operators can be found in [16].

2. The Widder-Arendt Theorem in LCS

In 1934 Widder [30] proved that a function $r : (0, \infty) \to R$ is the Laplace transform (i.e. $r(\lambda) = \int_0^\infty e^{-\lambda t} f(t)dt, \lambda > 0$) of a bounded function $f \in L^\infty(0, \infty)$ if and only if it satisfies

$$\sup\{|\lambda^{n+1} r^{(n)}(\lambda)/n!|; \lambda > 0, n \geq 0\} < \infty. \tag{2.1}$$

When R is replaced by a Banach space X, Arendt [1] proved that the above Widder's theorem holds if and only if X has the Radon-Nikodym property.

He showd that when X is a general Banach space what condition (2.1) characterizes is a larger class of functions as described in the following integrated version of Widder's theorem. For our purpose we extend the result of Arendt to sequentially complete locally convex spaces.

From now on X will denote such a space, and $S(X)$ will denote the set of all continuous seminorms on X.

Theorem 2.1. *Let $r : (0, \infty) \to X$ be a function. The following assertions are equivalent.*

(i) *r is infinitely differentiable and*

(2.2) $M_p := \sup\{p(\lambda^{n+1} r^{(n)}(\lambda))/n!; \lambda > 0, n = 0, 1, \cdots\} < \infty \ (p \in S(X))$.

(ii) *There exists a continuous function $g : [0, \infty) \to X$ satisfying $g(0) = 0$ and*

$$N_p := \sup\{p(g(t+\tau) - g(t))/\tau; t \geq 0, \tau > 0\} < \infty \ (p \in (X)) \tag{2.3}$$

such that

$$r(\lambda) = \int_0^\infty \lambda e^{-\lambda t} g(t) dt \ \ (\lambda > 0). \tag{2.4}$$

Moreover, we have $M_p = N_p$ for all $p \in S(X)$.

Proof. We shall only prove the theorem for the case that X is a complex vector space; the real case can be deduced from the complex case by complexification arguments. Assume that (i) holds. Then it is clear that r has an analytic continuation $r(z)$ on $\Omega := \{z \in C;\ \mathrm{Re}(z) > 0\}$, which can be defined iteratively by

$$r(z) := \sum_{k=0}^\infty \frac{(z-\lambda)^k}{k!} r^{(k)}(\lambda) \ (|\lambda - z| < \lambda,\ \lambda > 0).$$

Using (2.2) we can carry out the following estimation for $|\lambda - z| < \lambda, n = 0, 1, \cdots$, and $p \in S(X)$

$$\begin{aligned} p((\mathrm{Re}\ z)^{n+1} r^{(n)}(z)/n!) &= \frac{(\mathrm{Re}\ z)^{n+1}}{n!} p\left(\sum_{k=0}^\infty \frac{(z-\lambda)^k}{k!} r^{(n+k)}(\lambda)\right) \\ &\leq \left(\frac{\mathrm{Re}\ z}{\lambda}\right)^{n+1} \sum_{k=0}^\infty \frac{(n+k)!}{n!k!} \left|\frac{z-\lambda}{\lambda}\right|^k \\ &\quad \cdot p\Big(\lambda^{n+k+1} r^{(n+k)}(\lambda)/(n+k)!\Big) \\ &\leq M_p \left(\frac{\mathrm{Re}\ z}{\lambda}\right)^{n+1} \sum_{k=0}^\infty \begin{pmatrix} n+k \\ k \end{pmatrix} \left|\frac{z-\lambda}{\lambda}\right|^k \\ &= M_p \left(\frac{\mathrm{Re}\ z}{\lambda - |z-\lambda|}\right)^{n+1}. \end{aligned}$$

Since Re $z/(\lambda - |z-\lambda) \to 1$ as $\lambda \to \infty$ for each $z \in \Omega$, we have

(2.5) $$\sup\{p((\text{Re } z)^{n+1}r^{(n)}(z))/n!;\ z\in\Omega,\ n=0,1,\cdots\}=M_p.$$

In particular, the function $r(z), z\in\Omega$, is bounded on the half plane $\{z;\ \text{Re } z\ge\lambda\}$ for every $\lambda>0$. Therefore we can define a function $h:[0,\infty)\to X$ by

(2.6) $$h(t):=\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}e^{tz}z^{-2}r(z)dz\ \ (t\ge 0,\lambda>0),$$

the integral being well-defined and independent of $\lambda>0$.

For each $x^*\in X^*$, the classical Widder's theorem implies that there is a function $f(\cdot;x^*)\in L^\infty[0,\infty)$ with $\|f(\cdot;x^*)\|_\infty=\sup\{|\lambda^{n+1}\langle r^{(n)}(\lambda),x^*\rangle/n!|;\lambda>0,n=0,1,\cdots\}$ such that

(2.7) $$\begin{aligned}\langle r(z),x^*\rangle\ &=\int_0^\infty e^{-zt}f(t;x^*)dt\\ &=z^2\int_0^\infty e^{-zt}\int_0^t\int_0^u f(v;x^*)dvdudt,\ z\in\Omega.\end{aligned}$$

By the inversion of Laplace transform and (2.6) we have that

(2.8) $$\langle h(t),x^*\rangle=\int_0^t\int_0^u f(v;x^*)dvdudt\text{ for all }x^*\in X^*\text{ and }t\ge 0.$$

Next, we prove that $\lim\limits_{\tau\to 0}\frac{h(t+\tau)-h(t)}{\tau}$ exists uniformly for $t\ge 0$. Let V be any absolutely convex open neighborhood of 0, p be the Minkowski functional of V, and let $V^0:=\{x^*\in X^*;|\langle x,x^*\rangle|\le 1$ for all $x\in V\}$. We shall show that $\frac{h(t+\tau)-h(t)}{\tau}-\frac{h(t+s)-h(t)}{s}\in V$ for all $t\ge 0$ if τ and s are sufficiently srmall, for then the sequential completeness of X will ensure the uniform convergence of $\tau^{-1}(h(t+\tau)-h(t))$ as $\tau\to 0$.

Since p is the Minkowski functional of V, (2.2) implies that $\{\lambda^{n+1}r^{(n)}(\lambda)/n!;\ \lambda>0,n=0,1,\cdots\}\subset(M_p+\varepsilon)V$ for every $\varepsilon>0$. Thus, if $x^*\in V^0$, then $\|f(\cdot;x^*)\|_\infty\le M_p+\varepsilon$ for every $\varepsilon>0$ and hence $\|f(\cdot\,;x^*)\|_\infty\le M_p$. If $|\tau|,|s|<1/2M_p$, then we have

$$\begin{aligned}&\left|\left\langle\frac{h(t+\tau)-h(t)}{\tau}-\frac{h(t+s)-h(t)}{s},x^*\right\rangle\right|\\ =&\left|\frac{1}{\tau}\int_t^{t+\tau}\int_0^u f(v;x^*)dvdu-\frac{1}{s}\int_t^{t+s}\int_0^u f(v;x^*)dvdu\right|\\ =&\left|\int_0^1\int_{t+sw}^{t+\tau w}f(v;x^*)\mathrm{dvdw}\right|\le\|f(\cdot\,;x^*)\|_\infty\cdot\frac{|\tau-s|}{2}<\frac{1}{2}\end{aligned}$$

for all $t \geq 0$ and $x^* \in V^0$, and it follows from the bipolar theorem (cf. [29, p. 162]) that $\frac{h(t+\tau)-h(t)}{\tau} - \frac{h(t+s)-h(t)}{s} \in \frac{1}{2}\overline{V} \subset V$.

Thus we can define a function $g : [0, \infty) \to X$ by

$$g(t) := \lim_{s \to 0} \frac{h(t+s) - h(t)}{s}, t \geq 0.$$

For $x^* \in V^0$ we have, by (2.8), that $\langle g(t), x^* \rangle = \frac{d}{dt}\langle h(t), x^* \rangle = \int_0^t f(s; x^*)ds$ and

$$|\langle g(t+\tau) - g(t), x^* \rangle = \left| \int_0^{t+\tau} f(s; x^*)ds \right| \leq \tau \|f(\cdot\,; x^*)\|_\infty \leq \tau M_p.$$

It follows that $g(0) = 0$, and $\{\frac{g(t+\tau)-g(t)}{\tau}; t \geq 0, \tau > 0\} \subset M_p\bar{V}$, by the bipolar theorem. Therefore we have that $p(g(t+\tau) - g(t))/\tau \leq M_p$ for all $t \geq 0$ and $\tau > 0$ (cf. [29, Theorem 12.3]), i.e. $N_p \leq M_p$. This and the sequential completeness of X imply that the integral $\int_0^\infty \lambda e^{-\lambda t} g(t)dt$ exists for all $\lambda > 0$. Finally, from (2.7), (2.8) we see that

$$\begin{aligned}
\langle r(\lambda), x^* \rangle &= \int_0^\infty \lambda^2 e^{-\lambda t} \langle h(t), x^* \rangle dt \\
&= \int_0^\infty \lambda e^{-\lambda t} \frac{d}{dt} \langle h(t), x^* \rangle dt \\
&= \int_0^\infty \lambda e^{-\lambda t} \langle g(t), x^* \rangle dt = \left\langle \int_0^\infty \lambda e^{-\lambda t} g(t)dt, x^* \right\rangle
\end{aligned}$$

for all $x^* \in X^*$ and $\lambda > 0$. Hence (2.4) holds and we have proved the assertion (ii).

The converse implication is proved as easily as in the numerical case.

Theorem 2.2. *Let $R : (0, \infty) \to L(X)$ be a function. The following assertions are equivalent.*

(i) *$R(\cdot)x$ is infinitely differentiable for every $x \in X$ and the set $\{\lambda^{n+1}R^{(n)}(\lambda)/n!; \lambda > 0, n = 0, 1, \cdots\}$ is equicontinuous.*

(ii) *There exists a function $G : [0, \infty) \to L(X)$ satisfying*

(a) *$G(0) = 0$ and $G(\cdot)x$ is strongly continuous for every $x \in X$;*

(b) *$\{[G(t+\tau) - G(t)]/\tau; t \geq 0, \tau > 0\}$ is equicontinuous;*

(c) *$R(\lambda)x = \int_0^\infty \lambda e^{-\lambda t} G(t)xdt$ for all $\lambda > 0$ and $x \in X$.*

Proof. *(i) $\Rightarrow$ (ii).* By the assumption, for any $p \in S(X)$ there exists a $q \in S(X)$ such that $\sup\{p(\lambda^{n+1}R^{(n)}(\lambda)x)/n!; \lambda > 0, n = 0, 1, \cdots\} \leq q(x)$ for all $x \in X$. By Theorem 2.1, for each $x \in X$ there exists a continuous function $G(\cdots;x) : [0,\infty) \to X$ satisfying $G(0;x) = 0, \sup\{p([G(t+\tau;x) - G(t;x)]/\tau); t \geq 0, \tau > 0\} \leq q(x)$, and

$$R(\lambda)x = \int_0^\infty \lambda e^{-\lambda t} G(t;x)dt \ \ (\lambda > 0).$$

Since $G(\cdot\,;x)$ is continuous, the uniqueness theorem of Laplace transform implies that the map $G(t) : x \to G(t;x)$ is a linear operator on X. Moreover,

$$p(G(t+\tau)x - G(t)x)/\tau \leq q(x) \text{ for all } t \geq 0, \tau > 0 \text{ and } x \in X$$

This shows that $G(t) \in L(X)$ for all $t \geq 0$ and that $G(\cdot)$ satisfies (a), (b), and (c). That is, (ii) holds. The converse implication is proved as easily as in the numerical case.

3. C-pseudoresolvents

In this section we first investigate some properties of a C-pseudoresolvent, which is a generalization of pseudoresolvent (cf. [32]).

Definition 3.1. Let X be a locally convex space and let $C \in L(X)$. A function $R(\cdot)$ defined on a subset $D(R)$ of the complex plane with values in $L(X)$ is called a C-pseudoresolvent if it commutes with C and satisfies the equation:

$$(\lambda - \mu)R(\mu)R(\lambda) = R(\mu)C - R(\lambda)C \ \ (\lambda, \mu \in D(R)). \tag{3.1}$$

Moreover, $R(\cdot)$ is said to be nodegenerate if $R(\lambda)x = 0$ for all except one $\lambda \in D(R)$ implies $x = 0$.

From (3.1) one sees that $R(\cdot)$ is a commutative family, and that if $R(\cdot)$ is non-degenerate then C is injective. The converse is not true in general as shown by the I-pseudoresolvent $R(\lambda) := \begin{bmatrix} (\lambda-1)^{-1} & 0 \\ 0 & 0 \end{bmatrix}$, $\lambda > 1$.

The following lemmas and theorem generalize the correspoding results [32, pp. 215-216] on pseudoresolvents.

Lemma 3.2. *Suppose C is an injection. Then the null space $N(R(\lambda))$ and the preimage $C^{-1}[R(R(\lambda))]$ of the range $R(R(\lambda))$ of $R(\lambda)$ under C are independent of λ in $D(R)$.*

Proof. If $x \in N(R(\lambda))$, then by (3.1) $CR(\mu)x = (\lambda-\mu)R(\mu)R(\lambda)x + CR(\lambda)x = 0$. Since C is injective, we have $R(\mu)x = 0$ for all $\mu \in D(R)$. Hence $N(R(\lambda)) = N(R(\mu))$ for all λ and μ.

If $x \in C^{-1}[R(R(\lambda))]$, then $Cx = R(\lambda)y$ for some $y \in X$. Let $z := y - (\lambda - \mu)x$. Then $C[R(\mu)z - R(\lambda)y] = C[R(\mu)y - R(\lambda)y] - (\lambda - \mu)CR(\mu)x = (\lambda - \mu)R(\mu)R(\lambda)y - (\lambda - \mu)R(\mu)Cx = 0$. Since C is injective, we have $R(\lambda)y = R(\mu)z$ and hence $x \in C^{-1}[R(R(\mu))]$.

Lemma 3.3. *Suppose C is an injection. Then $N(C - \lambda R(\lambda))$ and $C^{-1}[R(C - \lambda R(\lambda))]$ are independent of λ in $D(R)$.*

Proof. For any $\lambda, \mu \in D(R)$ we have, by (3.1), that

$$\begin{aligned}
&(C - (\lambda - \mu)R(\lambda))(C - \mu R(\mu)) \\
&\quad = C^2 - \mu CR(\mu) - (\lambda - \mu)R(\lambda)C + \mu(\lambda - \mu)R(\lambda)R(\mu) \\
&\quad = C^2 - \mu CR(\mu) - (\lambda - \mu)R(\lambda)C + \mu[CR(\mu) - CR(\lambda)] \\
&\quad = C(C - \lambda R(\lambda)).
\end{aligned} \tag{3.2}$$

Since C is injective, we have $N(C - \mu R(\mu)) \subset N(C - \lambda R(\lambda))$ and hence also the inverse inclusion.

If $x \in C^{-1}[R(C - \lambda R(\lambda))]$, then $Cx = (C - \lambda R(\lambda))y$ for some $y \in X$. Let $z := \mu y + (\lambda - \mu)x$. Using (3.2) we have

$$\begin{aligned}
&C[(C - \mu R(\mu)z - \lambda Cx] \\
&\quad = (\lambda - \mu)(C - \mu R(\mu))Cx + \mu C(C - \mu R(\mu))y - \lambda C^2 x \\
&\quad = (\lambda - \mu)(C - \mu R(\mu))(C - \lambda R(\lambda))y + \mu C(C - \mu R(\mu))y - \lambda C^2 x \\
&\quad = \lambda(C - \mu R(\mu))Cy - \lambda(\lambda - \mu)R(\lambda)(C - \mu R(\mu))y - \lambda C^2 x \\
&\quad = \lambda(C - \mu R(\mu))Cy + \lambda C(C - \lambda R(\lambda))y - \lambda C(C - \mu R(\mu))y - \lambda C^2 x \\
&\quad = 0,
\end{aligned}$$

which implies that $Cx = \lambda^{-1}(C - \mu R(\mu))z \in R(C - \mu R(\mu))$. Hence $C^{-1}[R(C - \lambda R(\lambda))] = C^{-1}[R(C - \mu R(\mu))]$ for any $\lambda, \mu \in D(R)$.

Theorem 3.4. *Let $R(\cdot)$ be a C-pseudoresolvent with C injective.*

(i) *There exists some linear operator B such that $\lambda - B$ is injective and*

$$\begin{aligned}
&R(R(\lambda)) \subset D(B) \text{ and} \\
&R(\lambda)(\lambda - B) \subset (\lambda - B)R(\lambda) = C \text{ for all } \lambda \in D(R)
\end{aligned} \tag{3.3}$$

if and only if $N(R(\lambda)) = \{0\}$.

(ii) *The largest operator which satisfies* (3.3) *is the closed operator* A *defined by*

$$\text{(3.4)} \qquad \begin{cases} D(A) := C^{-1}[R(R(\lambda))] = \{x \in X; Cx \in R(R(\lambda))\}; \\ Ax := (\lambda - R(\lambda)^{-1}C)x \ for \ x \in D(A), \end{cases}$$

which is independent of $\lambda \in D(R)$.

(iii) *If* B *satisfies* (3.3), *then* $C^{-1}BC = A$, *where* $D(C^{-1}BC) := \{x \in X; Cx \in D(B)$ *and* $BCx \in R(C)\}$. *In particular,* $C^{-1}AC = A$.

Proof. The necessity part of (i) is clear. For the sufficiency, suppose $N(R(\lambda)) = \{0\}$. Lemma 3.2 has shown that the definition of $D(A)$ is independent of λ. Since $R(\lambda)$ and $R(\mu)$ are injective and

$$\begin{aligned} &R(\lambda)R(\mu)[\lambda - R(\lambda)^{-1}C - \mu + R(\mu)^{-1}C] \\ &\qquad = (\lambda - \mu)R(\lambda)R(\mu) - R(\mu)C + R(\lambda)C = 0, \end{aligned}$$

we have $\lambda - R(\lambda)^{-1}C = \mu - R(\mu)^{-1}C$ for any $\lambda, \mu \in D(R)$. Hence the operator A in (3.4) is a well-defined closed operator. If $x \in D(A)$, then $(\lambda - A)x = R(\lambda)^{-1}Cx$ and so $R(\lambda)(\lambda - A)x = Cx$. Since $R(\lambda)$ commutes with C, for any $x \in X$ we have $R(\lambda)x \in D(A)$ and

$$(\lambda - A)R(\lambda)x = R(\lambda)^{-1}CR(\lambda)x = R(\lambda)^{-1}R(\lambda)Cx = Cx.$$

Hence $\lambda - A$ is injective and $R(R(\lambda)) \subset D(A)$ and $R(\lambda)(\lambda - A) \subset (\lambda - A)R(\lambda) = C$. Since (3.3) implies that $D(B) \subset C^{-1}[R(R(\lambda))] = D(A)$, A is the largest operator which satisfies (3.3). This proves (i) and (ii).

To prove (iii), let $x \in D(A)$. By the definition of the domain of A we have $Cx = R(\lambda)y$ for some $y \in X$ and $\lambda \in D(R)$. By (3.3) we have $Cx \in D(B)$ and

$$\begin{aligned} (\lambda - B)Cx &= (\lambda - B)R(\lambda)y = Cy = (\lambda - A)R(\lambda)y \\ &= (\lambda - A)Cx \\ &= \lambda - A)[R(\lambda)(\lambda - A)x] \\ &= [(\lambda - A)R(\lambda)](\lambda - A)x \\ &= C(\lambda - A)x, \end{aligned}$$

hence $Cx \in D(B)$ and $BCx = CAx$ for $x \in D(A)$. This shows $A \subset C^{-1}BC$.

If $x \in D(C^{-1}BC)$, then $Cx \in D(B)$ and $BCx \in R(C)$ so that by the commutativity of C and $R(\lambda)$ we have

$$R(\lambda)(\lambda - C^{-1}BC)x = C^{-1}R(\lambda)(\lambda C - BC)x = C^{-1}C^2x = Cx \text{ for } \lambda \in D(R).$$

Hence $x \in D(A)$ and $Ax = (\lambda - R(\lambda)^{-1}C)x = C^{-1}BCx$. This proves (iii).

4. Laplace Transforms of N-times Integrated C-semigroups

Let X be a sequentially complete locally convex space, let $C \in L(X)$, and let $S : [0,\infty) \to L(X)$ be strongly continuous such that $\{e^{-wt}S(t); t \le 0\}$ is equicontinuous for some w. For $n = 0, 1, \cdots$, let $R_n : (w,\infty) \to L(X)$ be defined by $R_n(\lambda)x := \int_0^\infty \lambda^n e^{-\lambda t}S(t)xdt (x \in X, \lambda > w)$. In Definition 1.1 we have defined n-times integrated C-semigroups. The following lemma characterizes the integrated C-semigroup property (i.e. (1.2)) in terms of $R_n(\cdot)$ being a C-pseudoresolvent. The proof is almost the same with those in Proposition 2.2 and Theorem 3.1 of [1], where n-times integrated I-semigroups on Banach spaces have been considered.

Lemma 4.1. *$R_0(\cdot)$ (resp. $R_n(\cdot), n \ge 1$), is a C-pseudoresolvent if and only if $S(\cdot)$ satisfies $S(t)S(s) = S(t+s)C$ (resp. (1.2)), $t, s \ge 0$. Thus, when $S(0) = C$ or 0 (depending on $n = 0$ or $n \ge 1$), $S(\cdot)$ is an n-times integrated C-semigroup if and only if $R_n(\cdot)$ is a C-pseudoresolvent.*

Conversely, for a given function $R(\cdot) : (w,\infty) \to L(X)$, the next theorem describes when $\lambda^{-n-1}R$ is the Laplace transform of some $(n+1)$-times integrated C-semigroup.

Theorem 4.2. *Let $C \in L(X)$. The following assertions are equivalent.*

(i) *$R(\cdot)$ is a C-pseudoresolvent and*

(4.1) $$\{(\lambda - w)^{k+1}\left(\frac{d}{d\lambda}\right)^k (R(\lambda)/\lambda^n)/k!;\ \lambda > w, k = 0, 1, \cdots\}$$
is equicontinuous.

(ii) *There exists an $(n+1)$-times integrated C-semigroup $S(\cdot)$ such that $R(\lambda) = \int_0^\infty \lambda^{n+1}e^{-\lambda t}S(t)dt (\lambda > w)$ and such that*

(4.2) $\{e^{-wt}(S(t+\tau) - S(t))/\tau; t \ge 0, 0 < \tau \le 1\}$ *is equicontinuous.*

Proof. (i) $\Rightarrow$ (ii). Let $\tilde{R}(\lambda) := (\lambda + w)^{-n}R(\lambda + w), \lambda > 0$, and replace the $R(\cdot)$ in Theorem 2.2 by $\tilde{R}(\cdot)$. Then it follows that there is a continuous function $G : [0,\infty) \to L(X)$ such that $G(0) = 0$, $\{[G(t+\tau) - G(t)]/\tau; t \ge 0, \tau > 0\}$ is equicontinuous, and $\tilde{R}(\lambda) = \int_0^\infty \lambda e^{-\lambda t}G(t)dt, \lambda > 0$. Hence

$$R(\lambda) = \lambda^n(\lambda - w)\int_0^\infty e^{-(\lambda-w)t}G(t)dt = \lambda^{n+1}\int_0^\infty e^{-\lambda t}S(t)dt \quad (\lambda > w),$$

where $S(t) := e^{wt}G(t) - w\int_0^t e^{ws}G(s)ds, t \geq 0$. Since $R(\cdot)$ is assumed to be a C-pseudoresolvent and $S(0) = 0$, Lemma 4.1 tells that $S(\cdot)$ is an $(n+1)$-times integrated C-semigroup.

For any continuous seminorm $p \in S(X)$, let $q \in S(X)$ be such that $p([G(t+\tau) - G(t)]x) \leq \tau q(x)$ for all $\tau > 0, t \geq 0$, and $x \in X$. If $\tau \in (0,1]$, then

$$\begin{aligned}
&p(e^{-wt}[S(t+\tau) - S(t)]x) \\
&\quad \leq p[(e^{w\tau} - 1)G(t+\tau)x - w\int_0^\tau e^{ws}G(t+s)xds] \\
&\qquad +p([G(t+\tau) - G(t)]x) \\
&\quad \leq p(w\int_0^\tau e^{ws}[G(t+\tau) - G(t+s)]xds) + \tau q(x) \\
&\quad \leq [w\int_0^\tau e^{ws}(\tau - s)ds + \tau]q(x) \\
&\quad \leq [\tau(e^{w\tau} - 1) + \tau]q(x) \leq \tau e^w q(x)
\end{aligned}$$

for all $x \in X$, which shows (4.2).

(ii) $\Rightarrow$ (i). Since $\lambda^{-n-1}R(\lambda)$ is the Laplace transform of the $(n+1)$ times integrated semigroup $S(\cdot)$, Lemma 4.1 asserts that $R(\cdot)$ is a C-pseudoresolvent. To show (4.1) we shall apply Theorem 2.2 to the function

$$\begin{aligned}
\tilde{R}(\lambda) &= (\lambda + w)^{-n}R(\lambda + w) \\
&= (\lambda + w)\int_0^\infty e^{-(\lambda+w)t}S(t)dt \\
&= \lambda\int_0^\infty e^{-\lambda t}[e^{-wt}S(t) + w\int_0^t e^{-ws}S(s)ds]dt.
\end{aligned}$$

Let $G(t) := e^{-wt}S(t) + w\int_0^t e^{-ws}S(s)ds, t \geq 0$. Then $G(0) = S(0) = 0$. Also, by an estimation similar to the above one can deduce from (4.2) that the set $\{[G(t+\tau) - G(t)]/\tau; t \geq 0, 0 < \tau \leq 1\}$ is equicontinuous. Since we can write

$$[G(t+\tau) - G(t)]/\tau = \frac{1}{m}\sum_{k=1}^{m}\left[G\left(t + k\frac{\tau}{m}\right) - G\left(t + (k-1)\frac{\tau}{m}\right)\right] / \left(\frac{\tau}{m}\right)$$

for any $\tau > 0$ and $m = 1, 2, \cdots,$ it is clear that $\{[G(t+\tau) - G(t)]/\tau; \tau > 0, t \geq 0\}$ is equicontinuous. Hence one can deduce from Theorem 2.2 that (4.1) holds. This completes the proof.

5. Subgenerators and Generators

In this and the next sections we are concerned with the generation of nondegenerate n-times integrated C-semigroups. In this case C must be injective as easily seen from (1.2). (But this is not a sufficiency for nondegeneracy except when $n = 0$.)

First we define subgenerators of an n-times integrated C-semigroup $S(\cdot)$ to be those closed operators B which satisfy the following two conditions:

(5.1) $S(t)x \in D(B)$ and $BS(t)x = S(t)Bx$ for all $x \in D(B)$ and $t \geq 0$;

$$\begin{aligned} &\int_0^t S(s)xds \in D(B) \text{ and} \\ &B\int_0^t S(s)xds = S(t)x - t^n Cx/n! \text{ for } \ x \in X, \ t \geq 0. \end{aligned} \tag{5.2}$$

Let $\mathbf{G}$ be the set of all subgenerators of $S(\cdot)$. The following lemma shows that when $S(\cdot)$ is nodegenerate $\mathbf{G}$ is nonempty and is invariant under the map: $B \to C^{-1}BC$.

Lemma 5.1. *Let $S(\cdot)$ be a nondegenerate n-times integrated C-semigroup.*

(i) *The operator A_0, defined by*

$$\begin{cases} D(A_0) := \left\{ \sum_{k=1}^m \int_0^{t_k} S(s)x_k ds; x_k \in X, t_k \geq 0, k = 1, \cdots, m, m \geq 1 \right\} \\ A_0 \left[\sum_{k=1}^m \int_0^{t_k} S(s)x_k ds \right] := \sum_{k=1}^m [S(t_k)x_k - t_k^n Cx_k/n!] \ (x_k \in X, t_k \geq 0), \end{cases}$$

is closable, and its closure $\overline{A_0}$ is the smallest subgenerator of $S(\cdot)$.

(ii) $C^{-1}\mathbf{G} \subset \mathbf{G}$.

Proof. (i) Let $T(t) := \int_0^t S(r)dr$. To show that A_0 is well-defined it sufflces to show that $\sum_{k=1}^m T(t_k)x_k = 0$ implies $\sum_{k=1}^m [S(t_k)x_k - t_k^n Cx_k/n!] = 0$.

By straightforward computation involving integration of (1.2) with respect to t and s, one can see that $T(\cdot)$ is a nondegenerate $(n+1)$-times integrated

C-semigroup. Hence we have for all $t \geq 0$

$$\begin{aligned}
0 &= T(t)\sum_{k=1}^{m} T(t_k)x_k \\
&= \frac{1}{n!}\sum_{k=1}^{m}\left\{\int_t^{t_k+t}(t_k+t-r)^n T(r)dr - \int_0^{t_k}(t_k+t-r)^n T(r)dr\right\} Cx_k \\
&= \frac{1}{(n+1)!}\sum_{k=1}^{m}\left\{\int_t^{t_k+t}(t_k+t-r)^{n+1}S(r)dr + t_k^{n+1}T(t)- \right. \\
&\quad \left. \int_0^{t_k}(t_k+t-r)^{n+1}S(r)dr + t^{n+1}T(t_k)\right\} Cx_k \\
&= \frac{1}{(n=1)!}\sum_{k=1}^{m}\left\{\int_t^{t_k+t}(t_k+t-r)^{n+1}S(r)dr + t_k^{n+1}T(t)- \right. \\
&\quad \left. \int_0^{t_k}(t_k+t-r)^{n+1}S(r)dr\right\} Cx_k + 0,
\end{aligned}$$

and so its second derivative $S(t)\sum_{k=1}^{m}[S(t_k)x_k - t_k^n Cx_k/n!]$ also vanishes for all $t \geq 0$. Since $S(\cdot)$ is nondegenerate, we have $\sum_{k=1}^{m}[S(t_k)x_k - t_k^n Cx_k/n!] = 0$.

To see that A_0 is closable, let $\{y_\alpha\} \subset D(A_0)$ be a net such that $y_\alpha \to 0$ and $A_0 y_\alpha \to y (\in X)$. Then by the definition of A and the commutativity of $T(t)$ with $S(t)$, we have $T(t)y = \lim_\alpha T(t)A_0 y_\alpha = \lim_\alpha A_0 T(t)y_\alpha = \lim_\alpha [S(t)y_\alpha - t^n Cy_\alpha/n!] = 0$ and hence $y = 0$.

By the definition of A_0, every B in $\mathbf{G}$ is an extension of A_0 and hence an extension of $\overline{A_0}$. Next, we show that $\overline{A_0}$ is itself a subgenerator and accordingly is the smallest subgenerator. Let $x \in D(\overline{A_0})$ and let $\{x_\alpha\} \subset D(A_0)$ be a net such that $x_\alpha \to x$ and $A_0 x_\alpha \to \overline{A_0}x$. Then the definition of $\overline{A_0}$ implies that $S(t)x_\alpha \in D(A_0) \subset D(\overline{A_0})$ and $\overline{A_0}S(t)x_\alpha = A_0 S(t)x_\alpha = S(t)A_0 x_\alpha$ for all α.

Since $S(t)x_\alpha \to S(t)x$ and $\overline{A_0}S(t)x_\alpha = S(t)Ax_\alpha \to S(t)\overline{A_0}x$, the closedness of $\overline{A_0}$ implies that $S(t)x \in D(\overline{A_0})$ and $\overline{A_0}S(t)x = S(t)\overline{A_0}x$, i.e. (5.1) holds for $B = \overline{A_0}$. Since (5.2) holds for $B = A_0$ (by the definition of A_0), it certainly holds for $B = \overline{A_0}$.

(ii) Let $B \in \mathbf{G}$ and $B_1 = C^{-1}BC$. If $x \in D(B_1)$, then $Cx \in D(B)$ and $BCx \in R(C)$. Using (5.1) we have $CS(t)x = S(t)Cx \in D(B)$ and $BCS(t)x = BS(t)Cx = S(t)BCx = S(t)C^{-1}CBCx = CS(t)B_1x \in R(C)$. This means that $S(t)x \in D(B_1)$ and $B_1S(t)x = C^{-1}BCS(t)x = S(t)B_1x$, that is, (5.1) holds with B replaced by B_1. Next, by (5.2) we have that $C\int_0^t S(s)xds \in D(B)$ and $BC\int_0^t S(s)xds = B\int_0^t S(s)Cxds = S(t)Cx - t^nC^2x/n! = C[S(t)x - t^n x/n!] \in R(C)$ for all $x \in X$. Hence $\int_0^t S(s)xds \in D(B_1)$ and $B_1\int_0^t S(s)xds = S(t)x - t^nCx/n!$ for all $x \in X$, i.e. (5.2) also holds for B_1. Finally, we show that B_1 is closed. Indeed, let $\{x_\alpha\} \subset D(B_1)$ be a net such that $x_\alpha \to x$ and $B_1 x_\alpha \to y(\in X)$. Then $Cx_\alpha \to Cx$ and $BCx_\alpha = CBx_\alpha \to Cy$. The closedness of B implies that $Cx \in D(B)$ and $BCx = Cy$ so that $x \in D(B_1)$ and $B_1 x = y$. Hence B_1 is a subgenerator of $S(\cdot)$.

The next lemma gives a characterization of subgenerators of an exponentially equicontinuous n-times integrated C-semigroup in terms of its Laplace transforms.

Lemma 5.2. *Let $S : [0,\infty) \to L(X)$ be a strongly continuous function such that $\{e^{-wt}S(t); t \geq 0\}$ is equicontinuous for some $w \geq 0$, and let $C \in L(X)$.*

(i) *A closed linear operator B satisfies conditions* (5.1) *and* (5.2) *if and only if the function $R_n(\lambda) := \int_0^\infty \lambda^n e^{-\lambda t} S(t)dt (\lambda > w)$ satisfies:*

$$R(R_n(\lambda)) \subset D(B) \text{ and}$$
$$R_n(\lambda)(\lambda - B) \subset (\lambda - B)R_n(\lambda) = C \text{ for } \lambda > w. \tag{5.3}$$

(ii) *If $S(\cdot)$ commutes with C, and if there is a closed operator B satisfying* (5.1) *and* (5.2) (*or equivalently* (5.3)), *then $S(\cdot)$ is an n-times integrated C-semigroup with B a subgenerator.*

Proof. (i) Suppose (5.1) and (5.2) are satisfied. Then for all $x \in X$ we have, on the one hand,

$$\begin{aligned}\lambda^{n+1}\int_0^\infty e^{-\lambda t} B\int_0^t S(s)xdsdt &= \lambda^{n+1}\int_0^\infty e^{-\lambda t}[s(t) - t^n C/n!]xdt \\ &= \lambda R_n(\lambda)x - Cx,\end{aligned}$$

and on the other hand,

$$\lambda^{n+1}\int_0^\infty e^{-\lambda t}\int_0^t S(s)xdsdt = \lambda^n\int_0^\infty e^{-\lambda t}S(t)xdt = R_n(\lambda)x.$$

Since B is closed, we must have that $R_n(\lambda)x \in D(B)$ and $BR_n(\lambda)x = \lambda R_n(\lambda)x - Cx$ for all $x \in X$. When $x \in D(B)$, (5.1) and the closedness of B imply that $BR_n(\lambda)x = R_n(\lambda)Bx$. Hence (5.3) holds.

To prove the sufficiency we shall use the Post inversion formula, which says that if $L(f,\lambda) = \int_0^\infty e^{-\lambda t}f(t)dt$, with f an exponentially bounded, continuous function on $[0,\infty)$, then $\lim\limits_{m\to\infty} \frac{(-1)^m}{m!}\lambda^{m+1}D^m\lambda(f,\lambda)\Big|_{\lambda=m/t} = f(x)$ uniformly on bounded subsets of $[0,\infty)$. (see [31, Chapt. 7] or [24, p. 250] for the proof).

Dividing (5.3) by λ^n and λ^{n+1} we have

$$(5.4)\quad \lambda \int_0^\infty e^{-\lambda t}[S(t) - t^n C/n!]xdt = \begin{cases} B\displaystyle\int_0^\infty e^{-\lambda t} S(t)xdt, & x \in X; \\ \displaystyle\int_0^\infty e^{-\lambda t} S(t)Bxdt, & x \in D(B), \end{cases}$$

and

$$(5.5)\quad \int_0^\infty e^{-\lambda t}[S(t) - t^n C/n!]xdt = \begin{cases} B\displaystyle\int_0^\infty e^{-\lambda t} \int_0^t S(s)xdsdt, & x \in X; \\ \displaystyle\int_0^\infty e^{-\lambda t} \int_0^t S(s)Bxdsdt, & x \in D(B). \end{cases}$$

Since B is closed, differentiating the two terms on the right hand side of (5.4) twice shows that $BD^m \int_0^\infty e^{-\lambda t} S(t)xdt = D^m \int_0^\infty e^{-\lambda t} S(t)Bxdt$ for $x \in D(B)$. By the Post inversion formula we see that $\left\{ \frac{(-1)^m}{m!} \lambda^{m+1} D^m \int_0^\infty e^{-\lambda t} S(t)xdt|_{\lambda=m/t} \right\}$ converges weakly to $S(t)x$ and $\left\{ \frac{(-1)^m}{m!} \lambda^{m+1} D^m \int_0^\infty e^{-\lambda t} S(t)xdt|_{\lambda=m/t} \right\}$ converges weakly to $S(t)Bx$. Again by the weak closedness of B one can assert that $S(t)x \in D(B)$ and $BS(t)x = S(t)Bx$ for $x \in D(B)$, i.e. (5.1) holds. Similarly, m-times differentiation of two sides of the identity (5.5) followed by invoking the Post inversion formula and the closedness of B will show (5.2).

(ii) will follow from Lemma 4.1 once we show that $R_n(\cdot)$ is a C-pseudoresolvent. Indeed, the commutativity of $S(\cdot)$ and C implies that of $R_n(\cdot)$ and C. Then by (5.3) we have

$$\begin{aligned} R_n(\mu)C &= R_n(\mu)(\lambda - B)R_n(\lambda) \\ &= (\lambda - \mu)R_n(\mu)R_n(\lambda) + R_n(\mu)(\mu - B)R_n(\lambda) \\ &= (\lambda - \mu)R_n(\mu)R_n(\lambda) + R_n(\lambda)C \end{aligned}$$

for all $\lambda, \mu > w$.

If $S(\cdot)$ is a nodegenerate n-times integrated C-semigroup and if it is exponentially equicontinuous, then it follows from Lemma 4.1 and the uniqueness theorem of Laplace transform that $R_n(\cdot)$ is a nondegenerate C-pseudoresolvent. Thus one can deduce the folloing theorem as a consequence of Lemmas 5.1, 5.2, and Theorem 3.4.

Theorem 5.3. *Let $S(\cdot)$ be an exponentially equicontinuous, nondegenerate n-times integrated C-semigroup and let A be as defined in Lemma* 5.1. *Then*

(i) $\overline{A_0}$ *and $A := C^{-1}\overline{A_0}C$ are respectively the smallest and the largest subgenerators, and $\overline{a_0} \subset B \subset C^{-1}BC = A$ for any subgenerator B. In particular, $C^{-1}AC = A$.*

(ii) *The above defined operator A is identical to the operator defined by*

$$\begin{cases} D(A) := \{x \in X; Cx \in R(R_n(\lambda))\}; \\ Ax := [\lambda - R_n(\lambda)^{-1}C]x \ for \ x \in D(A). \end{cases}$$

This operator A will be called the generator of $S(\cdot)$.

Since $R_n(\lambda)x = \int_0^\infty \lambda^n e^{-\lambda t} S(t)xdt = \int_0^\infty \lambda^{n+1} e^{-\lambda t} \int_0^t S(s)xdsdt$, from Lemma 4.1 and (ii) of Theorem 5.3 follows the next corollary.

Corollary 5.4. *If $S(\cdot)$ is an n-times integrated C-semigroup with generator A, then the family $\{T(t) = \int_0^t S(s)ds; t \geq 0\}$ is an $(n+1)$-times integrated C-semigroup with the same generator A.*

Remark. However, not every $(n+1)$-times integrated C-semigroup is the integral of some n-times integrated C-semigroup. For instance, the weak*-Riemman integral of the dual $S^*(\cdot)$ of an n-times integrated C-semigroup $S(\cdot)$ on a Banach space X is an $(n+1)$-times integrated C^*-semigroup on the dual space X^*. It is not the integral of some n-times integrated C^*-semigroup; if it were, $S^*(\cdot)$ would have to be strongly continuous on X^*, but this is not always the case.

The next proposition is readily read from Theorem 5.3.

Proposition 5.5. *We have the following* (5.6) – (5.8)*:*

$$S(t)x \in D(A) \text{ and } AS(t)x = S(t)Ax \ for \ x \in D(A) \text{ and } t \geq 0; \tag{5.6}$$

$$\begin{aligned} &\int_0^t S(s)xds \in D(A) \text{ and} \\ &A\int_0^t S(s)xds = S(t)x - t^n Cx/n! \ for \ x \in Xt \geq 0; \end{aligned} \tag{5.7}$$

$$\begin{aligned} &R(R_n(\lambda)) \subset D(A) \text{ and} \\ &R_n(\lambda)(\lambda - A) \subset (\lambda - A)R_n(\lambda) = C \ for \ \lambda > w. \end{aligned} \tag{5.8}$$

Corollary 5.6. *For all $x \in X$ we have $S(t)x \in \overline{D(A)}$, $t \geq 0$. If $S(\cdot)x$ is right-sided differentiable at some $t \geq 0$, then $S(t)x \in D(A)$ and*

$$\frac{d}{dt}S(t)x = \begin{cases} AS(t)x + t^{n-1}Cx/(n-1)! & if \ n > 0, \\ AS(t)x & if \ n = 0. \end{cases}$$

In particular, this is true for $x \in D(A)$.

Proof. Using (5.7) one has that

$$S(t)x = \lim_{h\to 0+} h^{-1}\int_t^{t+h} S(s)xds \in \overline{d(A)}$$

and

$$\frac{S(t+h)x - S(t)x}{h} = Ah^{-1}\int_t^{t+h} S(s)xds + \frac{(t+h)^n - t^n}{h} Cx/n!,$$

from which the second assertion follows since A is closed.

When $S(\cdot)$ is a nondegenerate (0-times) C-semigroup, let G and A_i be the operators defined as follows

$$\begin{cases} D(G) := \{x \in X; S([0,\infty))x \subset R(C) \text{ and } \lim\limits_{t\to 0+} t^{-1}[C^{-1}S(t)x - x] \text{ exists}\}, \\ Gx := \lim\limits_{t\to +} t^{-1}[C^{-1}S(t)x - x] \text{ for } x \in D(G); \end{cases}$$

and

$$\begin{cases} D(A_i) := \{x \in X; \lim\limits_{t\to 0+} t^{-1}[S(t)x - Cx] \in R(C)\}, \\ A_i x := C^{-1} \lim\limits_{t\to 0+} t^{-1}[S(t)x - Cx] \text{ for } x \in D(A_i). \end{cases}$$

A_i is called the infinitesimal generator of $S(\cdot)$ (see [28]). Note that this G is a little bigger than that defined in [3] and [28].

Corollary 5.7. *For a nondegenerate C-semigroup we have $A = A_i, R(C) \subset \overline{D(A)}$, and $\overline{G} \in \mathbf{G}$.*

Proof. To prove $A_i \subset A$ let $x \in D(A_i)$. Then by (5.7) and the closedness of A we have $CA_i x = \lim\limits_{t\to 0+} t^{-1}[S(t)x - Cx] = A \lim\limits_{t\to 0+} \int_0^t S(s)xds = ACx$. Hence (i) of Theorem 5.3 implies $x \in D(C^{-1}AC) = D(A)$ and $Ax = C^{-1}ACx = A_i x$. Conversely, let $x \in D(A)$. Then by (5.6), (5.7), the closedness of A, and $S(0) = C$, we have

$$\lim_{t\to 0+} t^{-1}[S(t)x - Cx] = \lim_{t\to 0+} t^{-1}\int_0^t S(s)Axds = CAx$$

so that $x \in D(A_i)$ and $A_i x = Ax$. The inclusion $R(C) \subset \overline{D(A)}$ follows from the first assertion of Corollary 5.6.

Since $G \subset A_i$ and $A_i(= A)$ is closed, G is closable and $\overline{G} \subset A$. To show that $B = G$ satisfies (5.1), let $x \in D(\overline{G})(\subset D(A))$. Then by Corollaries 5.5 and 5.6 we have that $S(t)x \in D(A)$ and $\lim\limits_{h\to 0+} h^{-1}[C^{-1}S(h)S(t)x - S(t)x] =$

$\lim_{h\to 0+} h^{-1}[S(t+h)x - S(t)x] = \frac{d}{dt}S(t)x = AS(t)x = S(t)Ax = S(t)\overline{G}x$, that is, $S(t)x \in D(G) \subset D(\overline{G})$ and $\overline{G}S(t)x = GS(t)x = S(t)\overline{G}x$. Lastly, we show that (5.2) is satisfied by $B = \overline{G}$ so that G is a subgenerator. In fact,

$$\begin{aligned} G\int_0^t S(s)xds &= \lim_{h\to 0+} h^{-1}[C^{-1}S(h) - I]\int_0^t S(s)xds \\ &= \lim_{h\to 0+} h^{-1}\left(\int_t^{t+h} - \int_0^h\right) S(s)xds \\ &= S(t)x - Cx. \end{aligned}$$

Remarks. (i) That $A = A_i$ was previously proved by Tanaka and Miyadera [28] using different argument.

(ii) It follows from $R(C) \subset \overline{D(A)}$ that A is densely defined when C has dense range. For different proof of this fact see Davies and Pang's [3]. But, it is possible that $D(A)$ is dense while $R(C)$ is not. For instance, the operator $C : x \to \int_0^t x(s)ds$ is an injective operator on C [0,1] with nondense range. For the C-semigroup $S(\cdot)$ defined by $S(\cdot) \equiv C$ (cf. [28, P. 102]) we have $A_0 = 0|_{R(C)}$ and $G = A_i = A = 0$ on C [0,1].

(iii) When $n \geq 1$, the inclusion $R(C) \subset \overline{D}(A)$ is in general no longer true. In fact, there is an once integrated I-semigroup with a nondensely defined generator (see [1, Example 6.4]).

6. Generation of Nondegenerate Integrated C-semigroups

The generator A of an exponentially equicontinuous, nondegenerate n-times integrated C-semigroup has been defined as above, we now proceed to characterize it. Notice that the integral of an exponentially equicontinuous n-times integrated C-semigroup is an $(n+1)$-times integrated C-semigroup satisfying (4.2). Hence the conclusion of the following proposition holds for generators of exponentially equicontinuous n-times integrated C-semigroups.

Proposition 6.1. *Let A be the generator of a nondegenerate $(n+1)$-times integrated C-semigroup $S(\cdot)$ which satisfies* (4.2). *Then A has the following properties:*

(6.1) $\lambda - A$ *is injective for* $\lambda > w$;

(6.2) $R(C) \subset D((\lambda - A)^{-m})$ *for* $\lambda > w$ *and* $m \geq 1$;

(6.3) $(\lambda - A)^{-1}C$ *is infinitely differentiable for* $\lambda > w$ *and* $\{(\lambda - w)^{k+1}D^k[\lambda^{-n}(\lambda - A)^{-1}C]/k!;\ \lambda > w,\ k \geq 0\}$ *is equicontinuous*;

(6.4) $\quad C^{-1}AC = A.$

Proof. Since C is injective, (5.8) implies that $\lambda - A$ is also injective and $(\lambda - A)^{-1}C = R(\lambda) = \int_0^\infty \lambda^{n+1} e^{-\lambda t} S(t)dt$. Furthermore, $(\lambda - A)^{-1}C$ is infinitely differentiable and $R^{(m)}(\lambda) \in L(X)$ for all $\lambda > w$ and $m \geq 0$. Now (6.3) follows from Theorem 4.2.

By the closedness of A we can differentiate (5.8) m times to obtain that $R(R^{(m)}(\lambda)) \subset D(A)$ and $(\lambda - A)R^{(m)}(\lambda) + mR^{(m-1)}(\lambda) = 0$. Then it follows by induction that $R(R^{(m)}(\lambda)) \subset D(A^{m+1})$ and

$$(\lambda - A)^{m+1} R^{(m)}(\lambda) = (-1)^m m!(\lambda - A)R(\lambda) = (-1)^m m! C$$

for all $\lambda > w$ and $m \geq 0$. Hence we have $R(C) \subset D((\lambda - A)^{-m-1})$ and $(-1)^m m!(\lambda - A)^{-m-1}C = R^{(m)}(\lambda)$ for $\lambda > w$ and $m \geq 0$. This proves (6.2). Finally, we note that the validity of $C^{-1}AC = A$ has been asserted in Theorem 5.3.

Remark. If A is replaced by any subgenerator B, (5.6)-(5.8) and (6.1)-(6.3) still hold but (6.4) does not. Instead, we only have that $B \subset C^{-1}BC$.

A complete characterization of generators of nondegenerate integrated C-semigroups is given by the following

Theorem 6.2. *Let $C \in L(X)$ be an injective operator and let A be a closed linear operator. The following are true:*

(i) *A satisfies condition* (6.1) – (6.4) *if and only if A generates a nondegenerate $(n+1)$-times integrated C-semigroup $S(\cdot)$ such that*

(6.5) $\quad \{e^{-wt}[S(t+\tau) - S(t)]/\tau; \tau(0,1], t \geq 0\}$ *is equicontinuous.*

(ii) *$S(\cdot)x$ is continuously differentiable on $[0,\infty)$ for $x \in X_1 := \overline{D(A)}$, and the family $\{S_1(t) := \frac{d}{dt}S(t)|_{X_1}; t \geq 0\}$ of operators on X_1 is an exponentially equicontinuous n-times integrated C-semigroup with generator A_1, where $C_1 := C|_{X_1}$ and A_1 is the part of A in X_1, i.e. $D(A_1) = \{x \in D(A); Ax \in X_1\}$ and $A_1 x = Ax$ for $x \in D(A_1)$.*

Proof. (i) The sufficiency is Proposition 3. To prove the necessity, let $R(\lambda) := (\lambda - A)^{-1}C, \lambda > w$. Then $R(R(\lambda)) \subset D(A)$ and $(\lambda - A)R(\lambda) = C$. Furthermore, if $x \in D(A)$, then by (6.4) we have $Cx \in D(A)$ and $ACx = CAx$ so that $R(\lambda)(\lambda - A)x = (\lambda - A)^{-1}C(\lambda - A)x = Cx$. Thus we have

(6.6) $\quad R(R(\lambda)) \subset D(A)$ and $R(\lambda)(\lambda - A) \subset (\lambda - A)R(\lambda) = C$ for $\lambda > w$.

Also, $R(\lambda)C = (\lambda - A)^{-1}C^2 = (\lambda - A)^{-1}C(\lambda - A)R(\lambda) = CR(\lambda)$. Hence, as shown in the proof of (ii) of Lemma 5.2, $R(\cdot)$ is a C-pseudoresolvent. This with the assumption (6.3) implies that there exists an $(n+1)$-times integrated C-semigroup $S(\cdot)$ such that $R(\lambda) = \int_0^\infty \lambda^{n+1}e^{-\lambda t}S(t)dt$ and such that $\{e^{-wt}[S(t+\tau) - S(t)]/\tau; \tau \in (0,1], t \geq 0\}$ is equicontinuous (Theorem 4.2). Since C is injective, (6.6) shows that $R(\lambda)$ is injective and so $S(\cdot)$ is nondegenerate. Now A being a subgenerator of $S(\cdot)$ such that $C^{-1}AC = A$, it is the generator of $S(\cdot)$, by Theorem 5.3.

(ii) By Proposition 5.5 we have that $S(t)D(A) \subset D(A)$, $AS(t)x = S(t)Ax$ and $S(t)x = \int_0^t S(s)Axds + t^{n+1}Cx/(n+1)!$ for all $x \in D(A)$ and $t \geq 0$, which shows that $X_1 = \overline{D(A)}$ is invariant under $S(\cdot)$, and $S(\cdot)x$ is strongly differentiable and $\frac{d}{dt}S(t)x \in X_1$ when $x \in D(A)$. Then (6.5) together with the sequential completeness of X guarantees that $S(\cdot)x$ is continuously differentiable for all $x \in X_1, S_1(t) := \frac{d}{dt}S(t)|_{X_1} \in L(X_1)$, and $\{e^{-wt}S_1(t); t \geq 0\}$ is equicontinuous.

(6.4) implies that X is invariant under C, and the definition $R(\lambda) = (\lambda - A)^{-1}C$ ensures that X_1 is also invariant under $R(\cdot)$. Let $C_1 := C|_{X_1}$ and $R_1(\cdot) = R(\cdot)|_{X_1}$. Then as the restriction of the C-pseudoresolvent $R(\cdot)$ to X_1, $R_1(\cdot)$ is a C_1-pseudoresolvent. Since now $R_1(\lambda)x = \int_0^\infty \lambda^{n+1}e^{-\lambda t}S(t)xdt = \int_0^\infty \lambda^n e^{-\lambda t}S_1(t)xdt$ for all $x \in X_1$, and since Corollary 5.6 shows that $S_1(0) = C$ for $n = 0$ and 0 for $n > 0$, it follows from Lemma 4.1 that $S_1(\cdot)$ is an n-times integrated C_1-semigroup on X_1.

It remains to show that A is the generator of $S_1(\cdot)$. First, let $x \in D(A_1)$. Then $x \in D(A)$ and $A_1x = Ax \in X_1$. Then (5.6) implies that $S_1(t)x = S(t)x \in D(A)$ and $AS_1(t)x = AS(t)x = S(t)Ax = S_1(t)A_1x \in X_1$. Hence one has $S_1(t)x \in D(A_1)$ and $A_1S_1(t)x = AS_1(t)x = S_1(t)A_1x$ for $x \in D(A_1)$ and $t \geq 0$. Next, using (5.7) we have that for $x \in X_1$, $\int_0^t S_1(s)xds \in D(A)$ and $A\int_0^t S_1(s)xds = S_1(t)x - t^nC_1x/n! \in X_1$ so that $\int_0^t S_1(s)xds \in D(A_1)$ and $A_1\int_0^t S_1(s)xds = S_1(t)x - t^nC_1x/n!$. Thus A_1 is a subgenerator of $S_1(\cdot)$. Finally, by routine computation one can deduce $C_1^{-1}A_1C_1 = A_1$ from the assumption (6.4). Hence A_1 is the generator of $S_1(\cdot)$.

Corollary 6.3. *Let $C \in L(X)$ be an injective operator. Then a densely defined closed operator A is the generator of a nondegenerate n-times integrated C-semigroup if and only if it satisfies conditions* (6.1) – (6.4).

Remarks. (i) When the condition (6.4) is replaced by $A \subset C^{-1}AC$, i. e. $Cx \in D(A)$ and $ACx = C\,Ax$ for $x \in D(A)$, then in Theorem 6.2 (and also in Corollary 6.3) the generators of $S(\cdot)$ and $S_1(\cdot)$ are $C^{-1}AC$ and $C_1^{-1}A_1C_1$, respectively. When $n = 0$, (i) and (ii) of Theorem 6.2 reduce to Theorem 1 of Miyardera [18] and Theorem 2.1 of Tanaka & Miyadera [28], respectively, and

Corollary 6.3 subsurnes Theorem 10 and 11 of Davies and Pang [3].

(ii) When $C = I$, (i) of Theorem 6.2 reduces to Theorem 4.1 of Arendt [1], meanwhile, (ii) of Theorem 6.2 and Corollary 6.3 reduce respectively to his Corollary 4.2 and Theorem 4.3 in the same paper.

(iii) When $n = 0$ and $C = I$, Corollary 6.3 reduces to the classical Hille-Phillips-Yosida theorem (cf. [8], [10], [22], [32]).

7. The Abstract Cauchy Problem of Order One

We consider the abstract Cauchy problem:

$$(P(x,f)) \qquad \begin{cases} u'(t) = Au(t) + f(t), 0 \le t \le b; \\ u(0) = x \in X, \end{cases}$$

where A is assumed to be the generator of a nondegenerate n-times integrated C-semigroup $S(\cdot)$, and $f \in C([0,b],X)$. A solution of $P(x,f)$ is a function $u \in C_1([0,b],D(A)) := \{u;$ u is continuously differentiable and $u(0,b]) \subset D(A)\}$ such that $P(x,f)$ holds. Let $v_x : [0,b] \to X$ be the function given by

$$v_x(t) = S(t)x + \int_0^t S(s)f(t-s)ds, \quad 0 \le t \le b. \tag{7.1}$$

Theorem 7.1. *For given $x \in E$, $P(x,f)$ has a solution if and only if $v_x^{(n)}([0,b]) \subset R(C)$ and $C^{-1}v_x^{(n)} \in C^1([0,b],X)$. In this case, the function $u_x = C^{-1}v_x^{(n)}$ is the unique solution of $P(x,f)$.*

Proof. First, we suppose u is a solution of $P(x,f)$. Let $t \in [0,b]$ and $w(s) := S(t-s)u(s)$ for $s \in [0,t]$. Since $u(s) \in D(A)$, we differentiate w and use (5.6) and Corollary 5.6 to get

$$\begin{aligned} w'(s) &= -(1-\delta_{0,n})\left[\frac{1}{(n-1)!}(t-s)^{n-1}Cu(s)\right] - S(t-s)Au(s) + S(t-s)u'(s) \\ &= -(1-\delta_{0,n})\left[\frac{1}{(n-1)!}(t-s)^{n-1}Cu(s)\right] + S(t-s)f(s)(s \in [0,t]), \end{aligned}$$

$$\begin{aligned} S(t)x - \delta_{0,n}Cu(t) &= -\int_0^t w'(s)ds \\ &= (1-\delta_{0,n})\left[\frac{1}{(n-1)!}C\int_0^t (t-s)^{n-1}u(s)ds\right] - \int_0^t S(t-s)f(s)ds \\ &= (1-\delta_{0,n})\left[\frac{1}{(n-1)!}C\int_0^t (t-s)^{n-1}u(s)ds\right] - \int_0^t S(s)f(t-s)ds, \end{aligned}$$

where $\delta_{0,n}$, is the Kronecker delta. Consequently,

$$u_x(t) = \begin{cases} Cu(t) & \text{if } n = 0; \\ \frac{1}{(n-1)!} C \int_0^t (t-s)^{n-1} u(s)ds & \text{if } n \geq 1. \end{cases}$$

This shows that $v_x^{(n)}(t) = Cu(t) \in R(C)$ for $t \in [0,b]$ and $C^{-1}v_x^{(n)} = u \in C^1([0,b], D(A))$.

Conversely, by Fubini's theorem we have

$$\begin{aligned} \int_0^t \int_0^r S(s)f(r-s)dsdr &= \int_0^t \int_s^t S(s)f(r-s)drds \\ &= \int_0^t \int_0^{t-s} S(s)f(r-s)drds \\ &= \int_0^t \int_0^{t-r} S(s)f(r-s)drds \end{aligned}$$

Hence $\int_0^t v_x(r)dr = \int_0^t S(r)xdr + \int_0^t \int_0^{t-r} S(s)f(r)dsdr$ for $t \in [0,b]$. Using (5.7) we obtain

$$\begin{aligned} A\int_0^t v_x(r)dr &= S(t)x - \frac{t^n}{n!}Cx + \int_0^t [S(t-r)f(r) - \frac{1}{n!}(t-r)^n Cf(r)]dr \\ &= v_x(t) - \frac{t^n}{n!}Cx - \frac{1}{n!}C\int_0^t (t-r)^n f(r)dr \end{aligned} \tag{7.2}$$

for $t \in [0,b]$. If $C^{-1}v_x^{(n)} \in C^1([0,b], X)$, then $v_x \in C^{n+1}([0,b], X)$. Since A is closed, we may differentiate (7.2) $n+1$ times and obtain that $v_x^{(k-1)}(t) \in D(A)$ and

$$Av_x^{(k-1)}(t) = v_x^{(k)}(t) - \frac{1}{(n-k)!}t^{n-k}Cx - \frac{1}{(n-k)!}C\int_0^t (t-r)^{n-k}f(r)dr \tag{7.3}$$

for $k = 1, 2, \cdots, n$ and $Cu_x(t) = v_x^{(n)}(t) \in D(A)$ and $ACu_x(t) = Av_x^{(n)}(t) = v_x^{(n+1)}(t) - Cf(t) = Cu_x'(t) - Cf(t) \in R(C)$. By (6.4) we have $u_x(t) \in D(A)$ and $u_x'(t) = Au_x(t) + f(t)$. Moreover, if $n = 0$, then $S(0) = C$ and so $u_x(0) = C^{-1}vx(0) = x$. If $n > 0$, then $S(0) = 0$ and so $v_x(0) = 0$. We see from (7.3) that $v_x^{(k)}(0) = 0$ for $k < n$ and $v_x^{(n)}(0) = Av_x^{(n-1)}(0) + Cx = Cx$, i.e. $u_x(0) = x$. This completes the proof.

Remark. When $C = I$, Theorem 7.1 reduces to Proposition 5.1 and Theorem 5.2 of Arendt [1].

By Proposition 5.5, we know that $S(t)x = \int_0^t S(s)Axds + t^nCx/n!$ for $x \in D(A)$. Repeated substitutions show that for $x \in D(A^k), k \geq 1$,

$$
\begin{aligned}
S(t)x &= \int_0^t S(s)Axds + t^n Cx/n! \\
&= \int_0^t \left[\int_0^s S(r)A^2xdr + s^n CAx/n!\right] ds + t^n Cx/n! \\
&= \int_0^t (t-s)S(s)A^2xds + \sum_{j=0}^{1} t^{n+j}CA^jx/(n+j)! \qquad (7.4)\\
&= \cdots \\
&= \int_0^t (t-s)^{k-1}S(s)A^kx/(k-1)!ds + \sum_{j=0}^{k-1} t^{n+j}CA^jx/(n+j)!
\end{aligned}
$$

Now suppose $x \in C(D(A^{n+1}))$. Then $x = Cy$ for some $y \in D(A^{n+1})$. By (6.4) and induction we have that $Cy \in D(A^j)$ and $A^jCy = CA^jy$ for $0 \le j \le n+1$. In particular, we have $x \in D(A^{(n+1)})$ so that (7.4) holds for $k = n$. Hence

$$
\begin{aligned}
(d^n/dt^n)S(t)x &= S(t)A^nx + \sum_{j=0}^{n-1} \frac{1}{j!} t^j CA^jx \\
&= C\left[S(t)A^ny + \sum_{j=0}^{n-1} \frac{t^j}{j!} CA^jy\right] \in C(D(A))
\end{aligned}
$$

and $C^{-1}D^nS(\cdot)x \in C^1([0,b], D(A))$. Thus in order that $P(x,f)$ be solvable for all $x \in C(D(A^{n+1}))$ it is necessary and sufficient that the function

$$
g(t) := \int_0^t S(t-s)f(s)ds = \int_0^t S(s)f(t-s)ds,\ 0 \le t \le b, \qquad (7.5)
$$

satisfies $g^{(n)}([0,b]) \subset R(C)$ and $C^{-1}g^{(n)} \in C^1([0,b], X)$. We give two sufficient conditions on f in the following.

Lemma 7.2. *If $f^{(n+1)}([0,b]) \subset R(C), C^{-1}f^{(n+1)} \in C([0,b], X), f^{(k)}(0) \in D(A^{n-k})$, and $A^{n-k}f^{(k)}(0) \in R(C)$ for all $0 \le k \le n$, then $g^{(n)}([0,b]) \subset R(C)$ and $C^{-1}g^{(n)} \in C^1([0,b], D(A))$, and*

$$
\begin{aligned}
C^{-1}g^{(n)}(t) = &\sum_{j=1}^{n} \frac{1}{j!} t^j \sum_{k=0}^{j-1} A^{j-k-1} f^{(k)}(0) \\
&+ \int_0^t S(s) \sum_{k=1}^{n} C^{-1}A^{n-k}f^{(k)}(0)ds \qquad (7.6)\\
&+ \int_0^t \int_0^{t-s} S(r)C^{-1}f^{(n+1)}(s)drds, t \in [0,b].
\end{aligned}
$$

Proof. Using integration by parts $n+1$ times and (7.4) we obtain

$$
\begin{aligned}
g(t) &= \int_0^t S(s)f(t-s)ds = \int_0^t S(r)f(0)dr + \int_0^t \int_0^s S(r)f'(t-s)drds = \cdots \\
&= \sum_{k=0}^{n} \int_0^t (t-r)^k S(r) f^{(k)}(0)/k!dr + \int_0^t \int_0^s (s-r)^n S(r) f^{(n+1)}(t-s)/n!drds \\
&= \sum_{k=0}^{n} \int_0^t (t-r)^k \int_0^r (r-s)^{n-k-1} S(s) A^{n-k} f^{(k)}(0)/[k!(n-k-1)!]dsdr \\
&\quad + \sum_{k=0}^{n} \int_0^t (t-r)^k \sum_{j=0}^{n-k-1} r^{n+j} C A^j f^{(k)}(0)/[k!(n+j)!]dr \\
&\quad + \int_0^t \int_0^{t-s} (t-s-r)^n S(r) f^{(n+1)}(s)/n!drds.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
g^{(n)}(t) &= \sum_{k=0}^{n} \int_0^t S(s) A^{n-k} f^{(k)}(0)ds + \sum_{k=0}^{n-1} \sum_{j=k}^{n-1} t^{j+1} C A^{j-k} f^{(k)}(0)/(j+1)! \\
&\quad + \int_0^t \int_0^{t-s} S(r) f^{(n+1)}(s)drds \\
&= C \sum_{k=0}^{n} \int_0^t S(s) C^{-1} A^{n-k} f^{(k)}(0)ds + C \sum_{j=1}^{n} \frac{1}{j!} t^j \sum_{k=0}^{j-1} A^{j-k-1} f^{(k)}(0) \\
&\quad + C \int_0^t \int_0^{t-s} S(r) C^{-1} f^{(n+1)}(s)drds
\end{aligned}
$$

for $t \in [0,b]$, which shows the assertion.

Lemma 7.3. *If $f([0,b]) \subset D(A^{n+1}), A^{n+1}f([0,b]) \subset R(C), C^{-1}A^{n+1}f \in C([0,b],X)$, and $A^k f \in C([0,b],X)$ for $0 \le k \le n$, then $g^{(n)}([0,b]) \subset R(C)$, $C^{-1}g^{(n)} \in C^1([0,b],\ D(A))$, and*

$$
\begin{aligned}
C^{-1}g^{(n)}(t) = &\int_0^t \sum_{k=0}^{n} \frac{(t-s)^k}{k!} A^k f(s)ds \\
&+ \int_0^t \int_0^{t-s} S(r) C^{-1} A^{n+1} f(s)drds \quad (0 \le t \le b).
\end{aligned}
\tag{7.7}
$$

Proof. Under the hypothesis we may apply (7.4) with $k = n+1$ to obtain

$$\begin{aligned} g(t) = & \int_0^t S(t-s)f(s)ds \\ = & \int_0^t \left[\int_0^{t-s} (t-s-r)^n S(r)A^{n+1}f(s)/n!dr \right. \\ & \left. + \sum_{k=0}^{n} (t-s)^{n+k} CA^k f(s)/(n+k)!\right] ds. \end{aligned}$$

Hence

$$\begin{aligned} g^{(n)}(t) &= \int_0^t \int_0^{t-s} S(r)A^{n+1}f(s)drds + C\sum_{k=0}^{n}\int_0^t (t-s)^k A^k f(s)/k!ds \\ &= C\left\{\int_0^t \int_0^{t-s} S(r)C^{-1}A^{n+1}f(s)drds + \sum_{k-0}^{n}\int_0^t (t-s)^k A^k f(s)/k!d\right\}, \end{aligned}$$

which shows the conclusion.

The above discussion has justified the following theorem.

Theorem 7.4. *let $x \in C(D(A^{n+1}))$. If f satisfies the conditions given in Lemma* 7.2 *(resp. the condition given in Lemma* 7.3*), then $P(x, f)$ has a unique solution which is*

$$\begin{aligned} u_x(t) &= C^{-1}v_x^{(n)}(t) \\ &= C^{-1}S(t)A^n x + \sum_{j=0}^{n-1}\frac{1}{j!}t^j A^j x + C^{-1}g^{(n)}(t), \end{aligned} \tag{7.8}$$

where $C^{-1}g^{(n)}(t)$ is as expressed in (7.6) *(resp.* (7.7)*).*

Corollary 7.5. *Let A be the generator of a C-semigroup, and assume either*

(i) $\{f(0)\} \cup f'([0,b) \subset R(C)$ *and* $C^{-1}f' \in C([0,b], X)$,

or

(ii) $f \in C([0,b], D(A)), Af([0,b]) \subset R(C)$ *and* $C^{-1}Af \in ([0,b], X)$.

Then $P(x, f)$ has a unique solution for any $x \in C(D(A))$.

Corollary 7.6. *Let A be the generator of an n-times integrated C-semigroup. For every $x \in C(D(A^{n+1}))$ the unique solution of $P(x, 0)$ is*

$$u_x(t) = C^{-1}S(t)A^n x + \sum_{j=0}^{n-1}\frac{1}{j!}t^j A^j x.$$

Remarks. When $C = I$, Corollary 7.5 reduces to a theorem of Phillips [23] (see also [8, p. 84]), and Corollary 7.6 reduces to Proposition 2.6 in [17]. When $n = 0$, Corollary 7.6 reduces to Corollary 1.3 in [28].

8. Examples

We conclude the paper with examples which show that for each $n \geq 0$ there is an n-times integrated C-semigroup ($C \neq I$) whose generator does not generate any $(n-1)$-times integrated C-semigroup, We start with the case $n = 0$.

Example 8.1. For $t \geq 0$ let $S_0(t)$ be the operator on the space $C_0(R)$ defined by $S_0(t)f := \exp\{t[\chi_{[0,\infty)} - \chi_{(-\infty,0]}]\}\sin(\cdot)f(f \in C_0(R))$. Then it is easy to see that $S_0(\cdot)$ is a C-semigroup on $C_0(R)$ with C defined by $(Cf)(s) := (\sin(s))f(s)$, and the generator A_0 sends f to $(\chi_{[0,\infty)} - \chi_{(-\infty,0]})f$ for f in its domain $D(A_0) := \{f \in C_0(R);\ f(0) = 0\}$.

It is proved in [20, Proposition 2.4] that if B generates a nonholomorphic (C_0)-semigroup on a Banach space Y (for example, $B = d/ds$ generates the semigroup of left translations on $C_0(R)$), then the operator $A_1 : D(B)^{n+1} \to X_1 := Y^{n+1}$ defined by

$$A_1 := \begin{bmatrix} B & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ B & B & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & B & B & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & B & \cdot & \cdot & \cdot & B & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & B & B \end{bmatrix}$$

generates a nondegenerate n-times integrated I-semigroup $S_1(\cdot)$ on X_1 but does not generate a nondegenerate $(n-1)$-times integrated I-semigroup.

Example 8.2. Let $X := X_1 \oplus C_0(R)$ and $S(t) := S_1(t) \oplus \frac{1}{(n-1)!}\int_0^t (t - s)^{n-1} S_0(s)ds, t \geq 0$. Then $S(\cdot)$ is a nondegenerate n-times integrated C-semigroup on X with $\tilde{C} = I \oplus C$ and generator $A = A_1 \oplus A_0$. If A also generates some $(n-1)$-times integrated $\tilde{C}$-semigroup $T(\cdot)$, then from the identity $\int_0^\infty \lambda^n e^{-\lambda t} S(t)dt = (\lambda - A)^{-1}\tilde{C} = \int_0^\infty \lambda^{n-1} e^{-\lambda t} T(t)dt$ we see that $S(t) = \int_0^t T(s)ds$ for $t \geq 0$. Thus $S_1(\cdot) = S(\cdot)|_{X_1}$, is continuously differentiable on $[0,\infty)$ and $\frac{d}{dt}S_1(t) = \frac{d}{dt}S(t)|_{X_1} = T(t)|_{X_1}$ is strongly continuous on $[0,\infty)$. This means that $(d/dt)S_1(t)$ is an $(n-1)$-times integrated I-semigroup generated by A_1. This is a contradiction.

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