TAIWANESE JOURNAL OF MATHEMATICS
Vol. 1, No. 1, pp. 65-73, March 1997

PRODUCT PROCESSES IN VARYING ENVIRONMENTS

Mokhtar H. Konsowa

Abstract. A particular type of branching processes in varying environments is considered. It is assumed that all individuals of the same generation produce, given that the preceding generation is not extinct, randomly and independently of the past generations the same number of children. We show that the number of children in the nth generation normed by its expectation converges almost surely to a limit whose expectation is 0 or 1. We give a sufficient condition for convergence in quadratic mean to a limit whose mean is one. A nonclassical norming sequence of constants is defined so that the almost sure limit is finite greater than zero with probability 1. We also show, under certain circumstances, that the almost sure limit has infinite mean.

1. Introduction

In this paper, we consider branching processes such that all individuals of the nth generation produce, given nonextinction, randomly and independently of the past generations same number of children according to a law of distribution that depends on n. Then we have a type of branching processes which is different from a well known branching processes in varying environments (BPIVE) in which each individual of the nth generation produces, randomly and independently of the individuals of the same and of the past generations a number of children according to a certain law of distribution that depends on n. BPIVE have been studied extensively by many authors. See, for instance, [1], [4], [5] and [6]. We call the type we consider here product processes in varying environments (PPIVE). We were led to this type of processes originally in a derivation of some limit theorems while determining the type of

Received January 21, 1996.
Communicated by Y.–J. Lee.
1991 *Mathematics Subject Classification*: Primary 6067; Secondary 6030.
Key words and phrases: Branching processes, varying environments, nonextinction, martingale, limit theorems.

being the random walks on spherically symmetric random trees transient or recurrent. See [7]. Let $Z_0 = 1$ and for $n \geq 1$ let Z_n denote the number of children in the nth generation. If d_n denotes the number of children of each individual of the nth generation, then

$$Z_{n+1} = d_n Z_n = \prod_{j=0}^{j=n} d_j \,; \quad n = 0\,,1\,,2\,,3\,,...$$

Let $M_n = E(Z_n)$ and $W_n = Z_n/M_n$. The sequence $\{W_n\}$ is a nonnegative martingale, hence it converges almost surely (a.s.) to a limit W. We show, provided $M_n \to \infty$, that $E(W) = 0$ or 1. We give a sufficient condition for the quadratic mean convergence to a limit with mean 1. It could happen that the a.s. limit W is identical zero, in which case, we can not say much about Z_n for large n. In that case the sequence $\{M_n\}$ is not the right norming sequence. A nonclassical norming sequence of constants $\{C_n\}$ is defined such that Z_n/C_n converges a.s. to a non-zero limit, provided that $\sum_n p(d_n \neq 1)$ is finite. If, in addition, $M_n \to \infty$ and $C_n/M_n \to 0$, then

$$E(\lim_n Z_n/C_n) = E(\sup_n Z_n/M_n) = \infty.$$

In the next section we cast the light on some differences between BPIVE and PPIVE.

2. Some Differences Between BPIVE and PPIVE

Theorem 1 (Lyons 1992). *For BPIVE, let d_{nk} denote the number of children of the kth individual of the nth generation and assume that the doubly indexed sequence $\{d_{nk}\}$ is uniformly bounded. Then the a.s. limit of Z_n/M_n is positive a.s.; given nonextinction.*

In the case of PPIVE, the following example violates Theorem 1.

Example 1 [7]. Let $q_n = \min(c/n, 1)$, where $c > 0$. Define d_n such that

$$d_n = \begin{cases} 1 & \text{with prob.} \quad 1 - q_n, \\ 2 & \text{with prob.} \quad q_n. \end{cases}$$

Then $Z_n/M_n \to 0$ a.s. as $n \to \infty$. This can be investigated by showing that

$$\frac{E\sqrt[4]{Z_n}}{\sqrt[4]{EZ_n}} \to 0 \quad \text{as} \quad n \to \infty$$

and the result follows from Fatou's lemma.

In the case of BPIVE, let $\phi_n^*(s)$ denote the probability generating function of d_{nk} and $f_n^*(s)$ is that of Z_n. Then ϕ_n^* and f_n^* are related by the well known result:

$$f_{n+1}^*(s) = f_n^*(\phi_n^*(s)), \qquad s \in [0,1].$$

For the PPIVE, let $\phi_n(s)$ and $f_n(s)$ denote the probability generating function of d_n and Z_n, respectively. It can easily be shown that

$$f_{n+1}(s) \geq f_n(\phi_n(s)) \tag{1}$$

and we may have strict inequality as the following example shows.

Example 2. Let $d_0 = d_1 = 1$ a.s. For $n \geq 2$, let

$$d_n = \begin{cases} 1 & \text{with prob.} \quad 1/n, \\ 2 & \text{with prob.} \quad 1 - 1/n. \end{cases}$$

Then

$$Z_4 = \begin{cases} 1 & \text{with prob.} \quad 1/6, \\ 2 & \text{with prob.} \quad 1/2, \\ 4 & \text{with prob.} \quad 1/3. \end{cases}$$

Simple calculations show that $f_4(1/2) = 11/48$ and $f_3(\phi_3(1/2)) = 2/9$.

3. The Asymptotic Behaviour of Z_n and Z_n/M_n

It is well known for classical Galton–Watson Processes that there is no stability for population sizes; that is, for $k \geq 1$,

$$\lim_n p(Z_n = k) = 0.$$

The violation of this behaviour in case of PPIVE is displayed in the following:

Theorem 1. *Assume that for every n, $p(d_n = 0) < 1$ and that $Z_n \to Z_\infty$ a.s. as $n \to \infty$. Then*

$$p(0 < Z_\infty < \infty) = 1 \ \textit{if and only if} \ \sum_n p(d_n \neq 1) < \infty.$$

Proof. If $p(0 < Z_\infty < \infty) = 1$, then

$$p(Z_\infty = 0) = p(Z_\infty = \infty) = 0. \tag{2}$$

Obviously $Z_\infty > 0$ a.s. implies that $d_n \geq 1$ a.s. for every n. Hence, $Z_\infty < \infty$ a.s. assures that $p(d_n \neq 1 i.o.) = 0$. Consequently, $d_n = 1$ eventually a.s. Thus, $\sum_n p(d_n \neq 1) < \infty$. Conversely, suppose that $\sum_n p(d_n \neq 1) < \infty$. It follows from Borel–Cantelli lemma that $p(d_n \neq 1 i.o.) = 0$; that is, d_n eventually equals 1. Consequently, $p(0 < Z_\infty < \infty) = 1$.

The following proposition is comparable to Proposition 3.10 of Lyons [8].

Proposition 2. *If the sequence* $\{d_n\}$ *is uniformly bounded, then* $\lim_n M_n = \infty$ *if and only if* $Z_n \to \infty$ *a.s.; given nonextinction.*

Proof. Given nonextinction, it is obvious that we have

$$p(Z_\infty = Z_n | Z_n) = \prod_{k \geq n} p(d_k = 1) = \prod_{k \geq n} [1 - p(d_k \geq 2)]$$

and that

$$E(d_k) = 1 + E(d_k - 1\,; d_k \geq 2).$$

If $\{d_n\}$ is uniformly bounded by A, then

$$1 + p(d_k \geq 2) \leq E(d_k) \leq 1 + (A-1)p(d_k \geq 2).$$

Consequently, $\prod_k E(d_k) = \infty$ if and only if $\sum_k p(d_k \geq 2) = \infty$, if and only if $p(Z_\infty = Z_n | Z_n) = 0$; that is, $\lim_n M_n = \infty$ if and only if $p(Z_n \to \infty) = 1$.

The following example shows that in Proposition 2 the condition that $\{d_n\}$ is uniformly bounded can not be excluded.

Example 3. Let

$$d_n = \begin{cases} 1 & \text{with prob.} \quad 1 - 1/n^2, \\ n & \text{with prob.} \quad 1/n^2. \end{cases}$$

Then $\sum_n p(d_n = n) < \infty$; that is $d_n = 1$ eventually a.s. Therefore, there exists Z such that $Z_\infty \leq Z$ a.s. However, $\lim_n M_n = \infty$.

In the rest of this section, we use W_n to denote Z_n/M_n. It can easily be shown that $\{W_n\}$ is a nonnegative martingale. Hence the following proposition follows from the martingale convergence theorem and Fatou's lemma.

Proposition 3. *The sequence $\{W_n\}$ converges a.s. to a limit W such that $E(W) \leq 1$.*

Proposition 4. *If $\{d_n\}$ is uniformly bounded and $Z_\infty < \infty$ a.s., then $E(W) = 1$; given nonextinction.*

Proof. It follows from Proposition 2 that $\lim_n M_n < \infty$. Hence,

$$E(W) = E(\lim_n Z_n/M_n) = E(\lim_n Z_n)/\lim_n M_n$$
$$= \lim_n M_n/\lim_n M_n = 1.$$

The equality before the last one follows from the monotonicity of Z_n.

For the next theorem, we need the following lemma of Cohn [1].

Lemma 5. *Let $\{X_n\}$ be a sequence of real valued random variables converging to a limit X such that $p(0 < X < \infty) > 0$. If for some real α and all $\delta > 0$*

$$p(X \in (\alpha - \delta, \alpha + \delta)) > 0,$$

then there exists a sequence of real numbers $\{\alpha_n\}$ converging to α such that for all $\gamma > 0$

$$\lim_n p(X \in (\alpha - \gamma, \alpha + \gamma) | X_n = \alpha_n) = 1\,.$$

The technique used by Cohn and Hering [2] is employed to prove the following result:

Theorem 6. $E(W) = 0$ *or 1, provided that* $M_n \to \infty$.

Proof. Assume that $E(W) > 0$. If $Z_{n,k}$ denotes the number of the kth generation offspring of an individual of the nth generation, then $Z_{n+k} = Z_n \cdot Z_{n,k}$ and $EZ_{n,k} = Ed_n \cdots Ed_{n+k-1}$. Hence, we have

$$Z_{n+k}/M_{n+k} = Z_n(Z_{n,k}/Ed_0 \cdots Ed_{n-1}Ed_n \cdots Ed_{n+k-1})$$
$$= M_n^{-1} Z_n(Z_{n,k}/Ed_n \cdots Ed_{n+k-1})$$
$$= M_n^{-1} Z_n(Z_{n,k}/EZ_{n,k}).$$

Therefore,

$$W = \lim_k Z_{n+k}/M_{n+k} = M_n^{-1} Z_n \lim_k Z_{n,k}/EZ_{n,k}$$
$$= M_n^{-1} Z_n \tilde{W}_n \quad \text{a. s.}$$

Since $\tilde{W}_n$ is independent of Z_n, it follows that

$$E(W) = E(\tilde{W}_n). \tag{3}$$

Since $E(W) > 0$, Lemma 5 guarantees the existence of a sequence $\{a_n\}$ such that $a_n \to a > 0$ and

$$M_n^{-1} a_n M_n \tilde{W}_n \to a \qquad \text{in probability,}$$

and consequently,

$$\tilde{W}_n \to 1 \qquad \text{in probability.}$$

Then there exists a subsequence $\{\tilde{W}_{n_k}\}$ such that

$$\tilde{W}_{n_k} \to 1 \qquad \text{a. s.}$$

It follows from (3) and Fatou's lemma that $E(W) \geq 1$. Hence the result follows from Proposition 3.

The following theorem gives a sufficient condition for the quadratic mean convergence. But first we introduce a straightforward lemma. Let $var(d_j) = \sigma_j^2$.

Lemma 7. $var(W_n) = \prod_{j=1}^{j=n} \left[\frac{\sigma_j^2}{(Ed_j)^2} + 1 \right] - 1.$

Theorem 8. *If* $\lim_n var(W_n) < \infty$, *then* $W_n \to W$ *in quadratic mean such that* $E(W) = 1$ *and*

$$var(W) = \prod_{j=1}^{j=\infty} \left[\frac{\sigma_j^2}{(Ed_j)^2} + 1 \right] - 1. \tag{4}$$

Proof. The assumption that $\lim_n var(W_n) < \infty$ implies that W_n converges in quadratic mean to a r. v. Y such that $E(Y^2) < \infty$. See Theorem 4.1 of Doob [3]. Since $W_n \to W$ a.s., then $Y = W$ a.s. That $\lim_n E(W_n^2) < \infty$ follows from the assumption that $\lim_n var(W_n^2) < \infty$ and the fact that $E(W_n) = 1$. It follows also from the same result of Doob that $E(W) = 1$. Minkowski's inequality assures that

$$(var(W_n))^{1/2} - (E(W_n - W)^2)^{1/2} \leq (var(W))^{1/2} \leq (var(W_n))^{1/2} + (E(W_n - W)^2)^{1/2}.$$

Taking the limit as $n \to \infty$ we have

$$var(W) = \lim_n var(W_n)$$

and equation (4) follows from Lemma 7.

4. Nonclassical Norming Sequence of Constants

We have seen that we could have zero as the a.s. limit of Z_n/M_n. In which case, we can not tell much about Z_n for large n. That is, M_n is not the right normalization for Z_n. We will employ the notion of cumulant generating function (cgf) to obtain a normalizing sequence $\{C_n\}$ such that Z_n/C_n has a non–zero a.s. limit. The cgf of a random variable X is $-\log E(\bar{e}^{sx})$. Then the cgf is strictly increasing and continuous. Therefore its inverse does exist. The cgf of Z_n/C_n is $-\log f_n(e^{-s/C_n})$ and its inverse is $-\log f_n^{-1}(e^{-s})C_n$ for $s \in [0, -\log q]$ where $q = p(Z_n \to 0)$. Let $h_n(s) = -\log f_n^{-1}(e^{-s})$ and $C_n = 1/h_n(s_0)$ for $s_0 \in (0, -\log q)$.

We are now ready for the following theorem.

Theorem 9. *If $W_n' = Z_n/C_n$, then there exists a random variable W' such that $W_n' \to W'$ a.s.*

Proof. We first show that $Y_n = e^{-W_n'}$ is a submartingale and then the result follows from the fact that $\sup_n |Y_n| < \infty$ and the martingale convergence theorem.

$$\begin{aligned} E(Y_{n+1}|Y_n) &= E(e^{-h_{n+1}(s_0)Z_{n+1}}|Z_n) = E(e^{-h_{n+1}(s_0)d_n Z_n}|Z_n) \\ &= E[(e^{-h_{n+1}(s_0)})^{d_n Z_n}] = E[(f_{n+1}^{-1}(e^{-s_0}))^{d_n Z_n}] \\ &\geq [E(f_{n+1}^{-1}(e^{-s_0}))^{d_n}]^{Z_n} = [\phi_n(f_{n+1}^{-1}(e^{-s_0}))]^{Z_n} \\ &\geq [\phi_n(\phi_n^{-1}((f_n^{-1}(e^{-s_0}))))]^{Z_n} = [f_n^{-1}(e^{-s_0})]^{z_n} \\ &= [e^{\log f_n^{-1}(e^{-s_0})}]^{Z_n} = e^{-h_n(s_0)Z_n} = e^{-W_n'}. \end{aligned}$$

The first inequality above follows from convexity, while the second one follows from (1) and the fact that $f_{n+1}^{-1}(e^{-s})$ is strictly decreasing.

The following theorem gives a sufficient condition for having the a.s. limit W' of Theorem 9 such that $p(0 < W' < \infty) = 1$.

Theorem 10. *If $\sum_n p(d_n \neq 1) < \infty$, then for some $s_0 \in (0, \infty)$ we have*

$$Z_n h_n(s_0) \to W' \qquad a.s.$$

such that

$$p(0 < W' < \infty) = 1 .$$

Proof. It follows from Theorem 1 that $Z_n \to Z_\infty$ a.s. such that $p(0 < Z_\infty < \infty) = 1$. The continuity theorem assures that the sequence $\{f_n(s)\}$ of probability generating functions of $\{Z_n\}$ converges to a function $f(s)$. Consequently,

$$h_n(s_0)Z_n \to -\log f^{-1}(e^{-s_0})Z_\infty = kZ_\infty \quad \text{a.s.}$$

where $0 < k < \infty$. Obviously, $p(0 < kZ_\infty < \infty) = 1$.

Example 4. If d_n is as defined in Example 3, then $Z_n/M_n \to 0$ a.s. But, $h_n(s_0)Z_n \to W'$ a.s. such that $p(0 < W' < \infty) = 1$.

The following theorem is analogous to Theorem 2.4 of Cohn and Hering [2] for BPIVE.

Theorem 11. *If $M_n \to \infty$ and there exists a sequence of normalizing constants $\{C_n\}$ such that $\lim_n C_n/M_n = 0$ and $\lim_n Z_n/C_n = \overline{W}$ a.s. with $\overline{W} < \infty$ a.s. and $p(\overline{W} > 0) > 0$, then $E\overline{W} = E \sup_n Z_n/M_n = \infty$.*

Example 5. If $\{d_n\}$ is as defined in Example 4 and $C_n = 1/h_n(s_0)$, then $E(\overline{W}) = E \sup_n Z_n/M_n = \infty$.

Conclusions

We end up this paper with the following concluding remarks:

1. We have shown that PPIVE is different from BPIVE.
2. We have given a necessary and sufficient condition under which the sequence of population sizes is finite greater than zero.
3. It is shown that the sequence of population sizes normalized by their expectations has an a.s. limit whose expectation is either 0 or 1 and quadratic mean limit whose expectation is 1.
4. A nonclassical normalizing sequence for population sizes is defined such that the a.s. limit does exist and a condition is imposed so that such limit is finite greater than 0.

References

1. H. Cohn, On a property related to convergence in probability and some applications to branching processes, *Stochastic Process. Appl.* **12** (1) (1982), 59–72.
2. H. Cohn and H. Hering, Inhomogeneous Markov branching processes: supercritical case, *Stochastic Process. Appl.* **14** (1983), 79–91.
3. J. L. Doob, Stochastic Processes, John Wiley and Sons, Inc., New York, 1953.
4. Robert T. Goettge, Limit theorems for supercritical Galton–Watson process in varying environments, *Math. Biosci.* **28** (1975), 171–190.
5. P. Jagers, Branching Processes with Biological Applications, Wiley, New York, 1975.
6. F. C. Klebaner and H. J. Schuh, A connection between the limit and the maximum random variable of a branching process in varying environments, *J. Appl. Probab.* **19** (1982), 681–684.
7. Mokhtar Konsowa and Joanna Mitro, The type problem for random walks on trees, *J. Theoret. Probab.* **4** (3) (1991), 535–550.
8. R. Lyons, Random walks, capacity, and percolations on trees, *Ann. Probab.* **20** (1992), 2043–2088.

King Saud University, Al–Mulaidaa–Al Qasseem
P.O. Box 6033, Saudi Arabia

Permanent Address:
Department of Mathematics, College of Science, Helwan University
Cairo, Egypt