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A NOTE ON THE ADMISSIBILITY OF P-VALUE FOR THE ONE-SIDED HYPOTHESIS TEST IN THE NEGATIVE BINOMIAL MODEL

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Abstract. Let X be a random variable with negative binomial density

$$f(x|\theta) = \frac{\Gamma(x+r)}{\Gamma(x+1)\Gamma(r)}\theta^x(1-\theta)^r,$$

where $x = 0, 1, 2, \cdots, 0 < \theta < 1,\ r > 0$. For the hypothesis testing problem

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0$$

based on observing X= x, where θ_0 is specified, we consider it as an estimation problem within a decision-theoretic framework. We prove the admissibility of estimator $p(x) = P_{\theta_0}(X \geq x)$, the p-value, for estimating the accuracy of the test, $\mathbf{1}_{(0,\theta_0)}(\theta)$, under the squared error loss.

1. Introduction

For a random vector X with the parameter space Θ, Hwang, Casella, Robert, Wells, and Farrell (1992) from decision-theorectic approach did the hypothesis testing problem

$$H_0 : \theta \in \Theta_0 \quad versus \quad H_1 : \theta \in \Theta_0^c$$

based on observing X= x, where Θ_0 is a specified subset of Θ in the following framework. We want to know the viability of the set specified by H_0 by estimating the parameter $\mathbf{1}_{\Theta_0}(\theta)$ (where $\mathbf{1}_A(\cdot)$ denotes the indicator of set A), and we consider the parameter $\mathbf{1}_{\Theta_0}(\theta)$ to measure the accuracy of the test. In

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one word we do the hypothesis testing problem by estimating the parameter $\mathbf{1}_{\Theta_0}(\theta)$, the accuracy of the test, and the performance of an estimator, say $\delta(x)$, is evaluated by some loss function $L(\theta,\delta) = d(\mathbf{1}_{\Theta_0}(\theta) - \delta(x))$, where $d(t)$ is minimum at $t = 0$, nondecreasing for $t > 0$ and nonincreasing for $t < 0$. In this note we do a hypothesis testing problem in the same decision-theoretic framework.

The definition of the p-value of a hypothesis test follows Lehmann's (1986) through the note, and we consider the admissibility of the p-value for the problem of estimating the accuracy of the one-sided testing problem under the squared error loss,

$$H_0 : \theta \leq \theta_0 \quad versus \quad H_1 : \theta > \theta_0 \tag{1.1}$$

where θ_0 is a specified point. There are many criticisms raised at the p-value to be as a measure of evidence against the null hypotheses for the hypothesis testing problem (see, *e.g.*, Lindley (1957); Berger and Sellke (1987)). Specifically it is generally inadmissible for estimating the accuracy in a two-sided hypothesis testing problem (Hwang, *et al.*, (1992)). However for the one-sided hypothesis testing problem of some random varibles with location parameter, symmetric density and monotone likelihood ratio, the p-value can be reconciled with the infimum of the Bayesian measure of evidence against the null hypotheses (Casella and Berger (1987)). Moreover for the problem (1.1) considered here, the p-value is admissble in many modles. For example if the randon variable X is from the model of normal $N(\theta,1)$, or binomial $B(n,\theta)$, or Poisson $P(\theta)$, Hwang, *et al.*, (1992) had proved the p-value is admissible, and in this note for X from the negative binomial model, $NB(r,\theta)$, we prove the p-value, $p(x) = P_{\theta_0}(X \geq x)$, does also have the admissibility property for estimating the accuracy of the problem (1.1).

2. Result

Let X be a random variable with negative binomial density

$$f(x|\theta) = \frac{\Gamma(x+r)}{\Gamma(x+1)\Gamma(r)}\theta^x(1-\theta)^r, \tag{2.1}$$

where $x = 0, 1, 2, \cdots, 0 < \theta < 1,\ r > 0$. For the problem of estimating $\mathbf{1}_{(0,\theta_0)}(\theta)$, the accuracy of the one-sided hypothesis test (1.1), with loss function

$$L(\theta,\delta) = \Big(\delta(x) - \mathbf{1}_{(0,\theta_0)}(\theta)\Big)^2, \tag{2.2}$$

where $\delta(x)$ is an estimator of $\mathbf{1}_{(0,\theta_0)}(\theta)$, we are going to prove that estimator $p(x) = P_{\theta_0}(X \geq x),\ x$ the observed value, is a generalized Bayes estimator with finite Bayes risk, and thus admissible.

Theorem. *Let X be a random variable with the density* (2.1)*. For the problem of estimating the accuracy of the one-sided hypothesis test* (1.1)*, $\mathbf{1}_{(0,\theta_0)}(\theta)$, with the loss* (2.2)*, the p-value $p(x) = P_{\theta_0}(X \geq x)$ is an admissible estimator.*

Proof. Choose an improper prior π on the parameter space $\Theta = (0, 1)$ with

$$d\pi = \frac{1}{\theta(1-\theta)}\,\mathbf{1}_{(0,1)}(\theta)d\theta, \tag{2.3}$$

where $d\theta$ is the Lebesgue measure on R. First note that for any estimator $\delta(X)$, the posterior Bayes risk at $x = 0$ is finite if and only if $\delta(x) = 1$ at $x = 0$. Since the estimator p-value $p(x) = 1$ at $x = 0$, then the p-value achieves the minimum posterior Bayes risk at $x = 0$. For $x \geq 1$, the posterior density is

$$g(\theta|x) = \frac{f(x|\theta)\pi(\theta)}{\int_0^1 f(x|\theta)\pi(\theta)d\theta} = \frac{\Gamma(x+r)}{\Gamma(x)\Gamma(r)}\,\theta^{x-1}(1-\theta)^{r-1}, \tag{2.4}$$

and the Bayes estimator, say B(x), is the conditional expection of $\mathbf{1}_{(0,\theta_0)}(\theta)$, $E_{\theta|x}$ $\mathbf{1}_{(0,\theta_0)}(\theta)$. By using the technique of changing variable in the integration and the binomial theorem, calculations give us

$$\begin{aligned}
B(x) &= E_{\theta|x}\mathbf{1}_{(0,\theta_0)}(\theta) \\
&= \int_0^{\theta_0} \frac{\Gamma(x+r)}{\Gamma(x)\Gamma(r)}\,\theta^{x-1}(1-\theta)^{r-1}d\theta \\
&= 1 - \frac{\Gamma(x+r)}{\Gamma(x)\Gamma(r)} \int_{\theta_0}^1 (\theta)^{x-1}(1-\theta)^{r-1}d\theta \\
&= 1 - \frac{\Gamma(x+r)}{\Gamma(x)\Gamma(r)} \int_0^{1-\theta_0} (1-\theta)^{x-1}(\theta)^{r-1}d\theta \\
&= 1 - \frac{\Gamma(x+r)}{\Gamma(x)\Gamma(r)} \int_0^1 (1-(1-\theta_0)\theta)^{x-1}(1-\theta_0)^{r-1}\theta^{r-1}(1-\theta_0)d\theta \\
&= 1 - (1-\theta_0)^r\,\frac{\Gamma(x+r)}{\Gamma(x)\Gamma(r)} \int_0^1 \Big((1-\theta)+\theta_0\theta\Big)^{x-1}(\theta)^{r-1}d\theta \\
&= 1 - (1-\theta_0)^r\,\frac{\Gamma(x+r)}{\Gamma(x)\Gamma(r)} \int_0^1 \sum_{k=0}^{x-1}\tbinom{x-1}{k}(1-\theta)^{x-1-k}\theta_0^k\theta^k(\theta)^{r-1}d\theta \\
&= 1 - \sum_{k=0}^{x-1}(1-\theta_0)^r\theta_0^k\,\frac{\Gamma(x+r)}{\Gamma(x)\Gamma(r)}\tbinom{x-1}{k}\int_0^1 (1-\theta)^{x-1-k}\theta^{k+r-1}d\theta
\end{aligned}$$

$$\begin{aligned} B(x) &= 1 - \sum_{k=0}^{x-1} (1-\theta_0)^r \theta_0^k \frac{(k+r-1)!}{k!(r-1)!} \\ &= P_{\theta_0}(X \geq x) \\ &= p(x). \end{aligned}$$

Therefore $p(x)$ is a generalized Bayes estimator and the theorem will be proved if the Bayes risk is finite. Consider the Bayed risk of $p(x)$, $r(\pi, p)$. With $g(\theta|x)$ being given in (2.4) and k_1 denoting some constant,

$$\begin{aligned} r(\pi,p) &= \int \sum_{x=0}^{\infty} \Big(\mathbf{1}_{(0,\theta_0)}(\theta) - p(x)\Big)^2 f(x|\theta) d\pi \\ &= \int_{\theta_0}^1 \theta^{-1}(1-\theta)^{r-1} d\theta + \sum_{x=1}^{\infty} \left\{ \int_0^1 \Big(\mathbf{1}_{(0,\theta_0)}(\theta) - p(x)\Big)^2 g(\theta|x) d\theta \right\} \frac{1}{x} \\ &\leq k_1 + \sum_{x=1}^{\infty} \Big(p(x) - (p(x))^2\Big) \frac{1}{x}. \end{aligned}$$

Since

$$\begin{aligned} p(x) &= \sum_{n=x}^{\infty} \frac{\Gamma(n+r)}{\Gamma(n+1)\Gamma(r)} \theta_0^n (1-\theta_0)^r \\ &= \theta_0^x \sum_{m=0}^{\infty} \frac{(m+x+r-1)\cdots(m+r)}{(m+x)\cdots(m+1)} \frac{\Gamma(m+r)}{\Gamma(m+1)\Gamma(r)} \theta_0^m (1-\theta_0)^r \\ &\leq \theta_0^x \sum_{m=0}^{\infty} k_2 f(m|\theta_0) = k_2 \theta_0^x \end{aligned}$$

where k_2 is some contant and f is the density (2.1), hence $\sum_{x=1}^{\infty} p(x)\frac{1}{x} < \infty$. This together with the fact $(p(x))^2 \leq p(x)$ imply the finite of $r(\pi, p)$.

Remark. For the hypothesis testing problem

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0$$

based on observing $X = x$, where X has the negative binomial density (2.1), the Bayesian measure of evidence given a prior distribution $\pi(\theta)$, is the probability that H_0 is true given $X = x$,

$$P_r(H_0|x) = P_r(\theta \leq \theta_0|x) = \frac{\int_0^{\theta_0} f(x|\theta) d\pi(\theta)}{\int_0^1 f(x|\theta) d\pi(\theta)},$$

where f is the density (2.1). Since $P_r(H_0|x) = E_{\theta|x}\mathbf{1}_{(0,\theta_0)}(\theta)$ and from the proof of theorem, we know that the p-value, $p(x) = P_{\theta_0}(X \geq x)$, a frequentist measure of evidence against H_0, is equal to the posterior probability of H_0 with the prior (2.3), a Bayesian measure of evidence against H_0.

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