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SOME E-OPTIMAL REGULAR GRAPH DESIGNS

Chwen-Ming Chang

Abstract. In the list of optimal regular graph designs (RGDs) provided by J. A. John and Mitchell, the E-optimality of some designs is still unknown. We have found the answers for some parameter sets including type of $(v,\ v,\ 2)$, $(v,\ v,\ v-2)$. These E-optimal regular graph designs added by BIBDs are again E-optimal under certain domain.

1. Introduction

Let $\mathcal{D}_{v,b,k}$ be the class of incomplete block designs which are arrangements of v treatments into b blocks of k experimental units each. The subset of equireplicate designs, in which each treatment appears the same number of times, r, will be denoted by $\mathcal{D}^*_{v,b,k}$.

Among the designs, balanced incomplete block designs (BIBDs) are proved to be A, D and E-optimal [7]. The candidate that is closest to the BIBD is the regular graph design (RGD) of which every pair of treatments appears in the same block λ or $\lambda+1$ times [4]. J. A. John and Mitchell perform a systematic search to find the A, D and E-optimal design in the set of RGDs within the range of $v \leq 12,\ r \leq 10$ and $v \leq b$ [3, 4]. There are 209 sets of parameters in their list. Among them, it was proved that most of the E-optimal designs in the set of RGD are really optimal in $\mathcal{D}_{v,b,k}$ or $\mathcal{D}^*_{v,b,k}$. However, still 33 sets of parameters were completely unknown [1]. Let $(v,\ b,\ k)$ represent a design in $\mathcal{D}_{v,b,k}$. We have solved some of the parameter sets including

$$(7,7,2)(8,8,2)(9,9,2)(10,10,2)(11,11,2)(12,12,2)$$
$$(7,7,5)(8,8,6)(9,9,7)(10,10,8)(11,11,9)(12,12,10).$$

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The cyclic designs $(v, v, 2)$ for $v > 6$ are proved to be E-optimal in $\mathcal{D}^*_{v,b,k}$ but not E-optimal in $\mathcal{D}_{v,b,k}$. We have found unequireplicate designs which have better E-values than the cyclic designs. The E-values of these unequireplicate designs are derived. For $v = 6$, we found that cyclic (6, 6, 2) is E-optimal in $\mathcal{D}_{6,6,2}$ by enumerating all possible designs. However, it is not the only E-optimal design, some of the unequireplicate designs are also E-optimal.

In the complementary space, we show that cyclic $(v,\ v,\ v-2)$ with $v \geq 5$ are E-optimal for the binary designs in $\mathcal{D}_{v,b,k}$. By adding BIBDs to the E-optimal RGD, we obtain new designs that are the E-optimal designs in the set of RGD. This is confirmed by the following designs in J. A. John and Mitchell's list:

$$(6, 21, 2)(7, 28, 2)(8, 36, 2)(9, 45, 2).$$

It is also proved that they are not E-optimal in $\mathcal{D}_{v,b,k}$. As we take the complementary space, we obtain the following designs that are E-optimal in the set of RGDs.

$$(6, 21, 4)(7, 28, 5)(8, 36, 6)(9, 45, 7).$$

It is shown that these designs have better E-values than any unequireplicate designs in the $(v,\ b,\ k)$ parameter sets.

We now introduce notations which will be used throughout this paper. For each design d, let $\mathbf{C}$ be its information matrix and μ_d be the smallest positive eigenvalue of $\mathbf{C}$. Based on [6, 7], an E-optimal design is the design d^* such that

$$\mu_{d^*} = \max_{d \in \mathcal{D}} \mu_d.$$

If d is not equireplicate, we let $r = [bk/v]$ be the integral part of bk/v.

2. The Optimal $(v,\ v,\ 2)$ Designs

Before we state two theorems about the properties of optimal $(v,\ v,\ 2)$ designs, we give the definition of a connected design below.

A design can be converted to a *graph* by two steps.

1. Each treatment is a point.
2. If a pair of treatments appear in the same block we draw a line joining those two points.

A and B are connected if we can start at point A and proceed to point B following existing line segments on the graph. A design is called a connected design if every pair of the treatments is connected.

Theorem 2.1 *For $v \geq 4$, a connected equireplicate design with $(v,\ b,\ k) = (v,\ v,\ 2)$ is cyclic and unique. Moreover, it is an RGD.*

Proof. Here is a cyclic $(v,\ v,\ 2)$ which has the initial block $(1,\ 2)$:

$$(1,\ 2)(2,\ 3)\cdots(v-1,\ v)(v,\ 1)$$

We call it d_0. It can be easily checked that d_0 is an RGD. Let d_1 be any connected equireplicate design of $(v,\ v,\ 2)$. Since $r = 2$ and $k = 2$, any block in d_1 must have (two) different treatments; otherwise, the design will be disconnected.

Next we want to show that d_1 is equivalent to d_0. Pick one of the treatments in the first block from d_1, relabel it as treatment l, relabel the other one, say treatment j, as treatment 2. Relabel the other block that contains treatment j as block 2. In block 2, change treatment j as treatment 2 and change the other treatment, say treatment p, as treatment 3. Let this relabeling process continue. Suppose that in the middle of relabeling,

$$S_{new} = \{1,\ 2, \cdots,\ i\}, \quad 2 \leq i < v$$

is the set which contains the treatments that have been assigned so far. Let $S_{old} = \{i{+}1, \cdots,\ v\}$ be the set of the treatments which have not been relabeled yet. At the current stage, the first i blocks are $(1,\ 2)\cdots(i,\ t)$.

We now prove that $t \in S_{old},\ t \notin S_{new}$. By contradiction, suppose $t \in S_{new}$, i.e., $1 \leq t < i$, then treatment t has already occurred twice in the first i blocks. However, $r = 2$ which means that treatment t is disconnected from the rest of the treatments, i.e. disconnected from $\{i+1, \cdots,\ v\}$. Therefore the design cannot be connected.

It is clear now that, after relabeling, d_1 and d_0 are identical. □

Theorem 2.2 *The cyclic design $d = (v,\ v,\ 2)$ with $v \geq 4$ has $\mu_d = 1 - \cos(2\pi/v)$. For $v > 6,\ \mu_d < 0.5$.*

Proof. Let $\theta_i,\ 1 \leq i < v$, be the eigenvalues of the concordance matrix $\mathbf{NN}'$ of d. According to [5], if the first row of $\mathbf{NN}'$ has the elements $a_0, \cdots, a_{v-1}$, then θ_i is given by

$$\theta_i = a_0 + a_1\omega_i + a_2\omega_i^2 + \cdots + a_{v-1}\omega_i^{v-1},$$

where

$$\omega_i = e^{2\pi ij/v}, \quad i = 1,\ 2, \cdots,\ v;\ j^2 = -1.$$

We have proved that the connected cyclic $(v,\ v,\ 2)$ is unique in Theorem 2.1. Without loss of generality, assume that the initial block is $(1,\ 2)$. Then the $(a_0, \cdots, a_{v-1})$ is $(2,\ 1,\ 0, \cdots,\ 0,\ 1)$. This implies

$$\begin{aligned}\theta_i &= 2 + \omega_i + \omega_i^{v-1} \\ &= 2 + 2\cos\left(\frac{2\pi i}{v}\right) \qquad \text{for } i = 1, \cdots,\ v.\end{aligned}$$

Then $\theta_v = 4$ is corresponding to eigenvalue 0 of $\mathbf{C}$ and

$$\begin{aligned}\mu_d &= 2 - \frac{1}{2}\max_{1\le i\le v-1}\theta_i \\ &= 2 - \left(1 + \max_{1\le i\le v-1}\cos\frac{2\pi i}{v}\right) \\ &= 2 - \left(1 + \cos\frac{2\pi}{v}\right) \\ &= 1 - \cos\frac{2\pi}{v}.\end{aligned}$$

Since $\cos(2\pi/v) > 0.5$ for $v > 6$, $\mu_d < 0.5$ for $v \geq 7$. $\square$

The bound in Theorem 2.2 [1] can be calculated.

$$\begin{aligned}&\max[\{r(k-1)-\lambda-2\}v/\{(v-2)k\}, \{r(k-1)+\lambda-1\}/k,\\ &\{(r-1)(k-1)\}v/\{(v-1)k\}]\\ &= \max[1/2, v/\{2(v-1)\}]\\ &= v/\{2(v-1)\}.\end{aligned}$$

The bound equals 0.667 and 0.625 for $v = 4$ and $v = 5$, respectively. Now, $1-\cos(2\pi/4) = 1$ and $1-\cos(2\pi/5) = 0.691$ which exceed the bound, therefore the cyclic (4, 4, 2) and (5, 5, 2) are optimal in $\mathcal{D}$ according to Theorem 2.2 [1]. However, $\cos(2\pi/v) \geq 1/2$ and $v/(v-l) > 1$ for $v \geq 6$, hence

$$1 - \cos(2\pi/v) \leq 1/2 < v/\{2(v-1)\}$$

for $v > 6$. In this situation, the E-value is below the bound specified by Theorem 2.2 [1], so we do not know whether the cyclic designs are E-optimal in $\mathcal{D}_{v,v,2}$ when v is greater than or equal to 6. Indeed three counterexamples are found. For $v \geq 4$,

- Design 1: (1, 2) (1, 2) (1, 3) $\cdots$ (1, v).
- Design 2: (1, 2) (1, 3) $\cdots$ (1, v) (2, 3).
- Design 3: (1, 1) (1, 2) (1, 3) $\cdots$ $(1, v)$. Note that this is a nonbinary design.

We can prove that for each design d in our counterexamples, μ_d is equal to 0.5. In fact, all the eigenvalues may be derived.

Theorem 2.3 *The eigenvalues of the information matrix of Design* 1 *are* 0 *with multiplicity* 1, 0.5 *with multiplicity* $v-3$ *as well as* $[(v+3) \pm \sqrt{v^2-2v+9}]/4$.

Proof. See the Appendix A.

From the above result, the value of E-criterion for Design 1 is $\mu_d = 0.5$.

Using similar deriving technique, we find all eigenvalues of the information matrices of Design 2 and 3. We now state two theorems in the following and omit their proofs.

Theorem 2.4 *The eigenvalues of the information matrix of Design* 2 *are* 0 *with multiplicity* 1, 0.5 *with multiplicity* $v-3$, 1.5 *with multiplicity* 1 *and* $v/2$ *with multiplicity* 1.

Therefore, we have $\mu_d = 0.5$ for Design 2.

Theorem 2.5 *The eigenvalues of the information matrix of Design* 3 *are* 0 *with multiplicity* 1, 0.5 *with multiplicity* $v-2$*, and* $v/2$ *with multiplicity* 1.

Hence, for Design 3 we also have $\mu_d = 0.5$.

We can verify that Designs 1-3 are E-optimal designs in $\mathcal{D}_{v,v,2}$ for $v = 6$. By counting all the cases, we obtain 28 different designs for (6, 6, 2). The largest value of μ_d among those designs is 0.5. Although we can not prove that Designs 1-3 are E-optimal in $\mathcal{D}_{v,v,2}$ for $v > 6$, we can show that they are *close* for large value of v.

Theorem 2.6 *For* $(v,\ v,\ 2)$ *designs, as* v *becomes large enough, the Designs* $1-3$ *are close to E-optimal.*

Proof. We have shown in Theorem 2.1 that the $(v,\ v,\ 2)$ equireplicate design is cyclic and has $\mu_d < 0.5$ for $v \geq 7$. Moreover, our Designs 1-3 have $\mu_d = 0.5$. Therefore for $v \geq 7$ the E-optimal design in $\mathcal{D}_{v,v,2}$ must be a design without equal replication.

Let r_d be the smallest number of replicates in all treatments. Since $r = 2$ for an equireplicate $(v,\ v,\ 2)$ design, $r_d = 1$ if the design is not equireplicate. Hence

$$\mu_d \leq \frac{v}{2(v-1)} = 0.5\ \frac{v}{v-1}.$$

Moreover, $\lim\limits_{v\to\infty} v/(v-l) = 1$. So for large v, 0.5 is the upper bound of μ_d in $(v,\ v,\ 2)$ according to Theorem 3.2 [2]. □

3. Extending the Optimal $(v,\ v,\ 2)$ to the Complementary Space

The result in the previous section can be extended and hence we can solve the E-optimality of more parameter sets.

We first introduce the complementary design [5]. Let d be a binary design with parameters v and b. A design d_c can be obtained from d by replacing the treatments that appear in any block by the treatments that do not appear in that block. The d_c is called the *complementary* design of d. Let d be a proper, binary and equireplicate design and μ be a positive eigenvalue of the information matrix of d. Then, for each μ, we have $[k/(v-k)]\mu + (b-r) - kr/(v-k)$ as an eigenvalue of the information matrix of d_c.

We prove the following theorem.

Theorem 3.1 *The cyclic designs of* $(v,\ v,\ v-2)$ *for* $v \geq 5$ *are E-optimal for the binary designs in* $\mathcal{D}_{v,v,v-2}$.

Proof. We carry out the proof in two steps. First, we show that the cyclic $(v,\ v,\ v-2)$ is unique in $\mathcal{D}^*_{v,v,v-2}$. Second, we show that the cyclic $(v,\ v,\ v-2)$ has larger E-value than any unequireplicate design in $\mathcal{D}_{v,v,v-2}$.

Let d be the cyclic $(v,\ v,\ 2)$ which has the initial block $(1,\ 2)$ and d_c be its complementary design. It can be verified that every cyclic $(v,\ v,\ v-2)$ is isomorphic to d_c by the following two arguments:

- A design is cyclic if and only if its complementary design is cyclic.
- Design d is the unique cyclic design in $\mathcal{D}^*_{v,v,2}$.

Next we derive the E-value for d_c and compare it with the E-values of unequireplicate designs. Let μ_d and μ_{d_c} be the smallest nonzero eigenvalues of the information matrices of d and d_c respectively. Then

$$\mu_{d_c} = \frac{k}{v-k}\mu_d + (b-r) - \frac{kr}{v-k}\,.$$

Since $\mu_d = 1 - \cos(2\pi/v)$ is always positive, μ_{d_c} is greater than $(b-r) - [kr/(v-k)]$. In our case $b = v$ and $r = k = 2$, so that

$$\mu_{d_c} > (v-2) - [4/(v-2)]. \tag{1}$$

Let μ_1 be the smallest nonzero eigenvalue of any block design with unequal replication in $\mathcal{D}_{v,v,v-2}$. Let r_1 be the smallest number of replicates in all treatments. Then $r_1 < v-2$. By Theorem 3.2 [2],

$$\mu_1 \leq \frac{[(v-2)-1]v\ r_1}{(v-2)(v-1)} \leq \frac{(v-3)^2 v}{(v-2)(v-1)}\,. \tag{2}$$

It can be proved algebraically that

$$(v-2) - \frac{4}{v-2} \geq \frac{(v-3)^2 v}{(v-2)(v-1)} \quad \text{if and only if } v \geq 5.$$

By the transition property of the inequality on Equation l and Equation 2, we have $\mu_1 < \mu_{d_c}$ if and only if $v \geq 5$. Therefore any block design with unequal replicates is inferior to our d_c by the E-criterion. This proves that our d_c is the E-optimal design for the binary designs in $\mathcal{D}_{v,v,v-2}$. □

4. Adding BIBDs to E-optimal Regular Graph Designs

Sometimes when a BIBD is added to an optimal design, the new design is optimal too. Suppose two designs have the same number of treatments and the same size of block. We can obtain a new design d_+ by putting the two designs together.

Theorem 4.1 *If $d_1 = (v, b_1, k)$ is E-optimal in the set of RGDs and $v \geq 4$, $d_2 = (v, b_2, k)$ is a BIBD, then $d_+ = (v, b = b_1 + b_2, k)$ is also an RGD and is E-optimal in the set of RGD.*

Proof. Let $\mathbf{N}_i$ be the incidence matrix for design $d_i, i = 1, 2$. Suppose the off-diagonal elements of $\mathbf{N}_1\mathbf{N}_1'$ take exactly two values, λ_1 or $\lambda_1 + 1$. Suppose each treatment pair appears λ_2 times for d_2. Then for design d_+, each treatment pair appears either $\lambda_1 + \lambda_2$ or $\lambda_1 + \lambda_2 + 1$ times and so d_+ is an RGD. Let d be an arbitrary RGD in $\mathcal{D}^*_{v,b,k}$. The concordance matrix of d can be decomposed to $\mathbf{N}_2\mathbf{N}_2' + \mathbf{M}$, where $\mathbf{M}$ is also a concordance matrix. Let $\mathbf{C}$ and $\mathbf{C}_2$ be the information matrices of d and d_2 respectively. Let $\mathbf{C}_m$ be the information matrix associated with $\mathbf{M}$. Then $\mathbf{C} = \mathbf{C}_2 + \mathbf{C}_m$. Since d_2 is a BIBD,

$$\mathbf{C}_2 = \lambda_2 \left(\frac{v}{k}\mathbf{I} - \frac{1}{k}\mathbf{J} \right).$$

If μ_m is a positive eigenvalue of $\mathbf{C}_m$ with vector $\mathbf{x}_m$, then $\mathbf{1}'\mathbf{x}_m = 0$ and we have

$$\begin{aligned}
\mathbf{C}\mathbf{x}_m &= (\mathbf{C}_2 + \mathbf{C}_m)\mathbf{x}_m \\
&= \lambda_2 \left(\frac{v}{k}\mathbf{I} - \frac{1}{k}\mathbf{J} \right) \mathbf{x}_m + \mathbf{C}_m\mathbf{x}_m \\
&= \frac{\lambda_2 v}{k}\mathbf{x}_m + \mu_m \mathbf{x}_m \\
&= \left(\mu_m + \frac{\lambda_2 v}{k} \right) \mathbf{x}_m.
\end{aligned}$$

So $\mu = \mu_m + \lambda_2 v/k$ is an eigenvalue of $\mathbf{C}$. Since d_1 is E-optimal, $\mu_{d_1} \geq \min \mu_m$ and so

$$\begin{aligned} \mu_{d_+} &= \mu_{d_1} + \lambda_2 v/k \\ &\geq \min \mu_m + \lambda_2 v/k \\ &\geq \min \mu. \end{aligned} \tag{3}$$

This implies that d_+ is E-optimal in the set of RGD. □

The above proof is confirmed by the E-optimal RGDs for the parameter sets

- (6, 6, 2) + (6, 15, 2) = (6, 21, 2),
- (7, 7, 2) + (7, 21, 2) = (7, 28, 2),
- (8, 8, 2) + (8, 28, 2) = (8, 36, 2),
- (9, 9, 2) + (9, 36, 2) = (9, 45, 2)

in J. A. John and Mitchell's list. However, they are not E-optimal in $\mathcal{D}$ for $v \geq 7$. If we add our Designs 1-3 to the BIBDs, the new designs have larger values for E. For example, when $v = 7$, the above (7, 28, 2) design is

(1, 2) (2, 3) (3, 4) (4, 5) (5, 6) (6, 7) (7, 1)
(1, 2) (1, 3) (1, 4) (1, 5) (1, 6) (1, 7) (2, 3)
(2, 4) (2, 5) (2, 6) (2, 7) (3, 4) (3, 5) (3, 6)
(3, 7) (4, 5) (4, 6) (4, 7) (5, 6) (5, 7) (6, 7).

The E-value of the cyclic (7, 7, 2) is 0.377, so the design has the E-value equal to $0.377 + 1 \times 7/2 = 3.877$ according to Equation 3. When we add our Design 1 to the BIBD (7, 21, 2), we obtain the following design:

(1, 2) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6) (1, 7)
(1, 2) (1, 3) (1, 4) (1, 5) (1, 6) (1, 7) (2, 3)
(2, 4) (2, 5) (2, 6) (2, 7) (3, 4) (3, 5) (3, 6)
(3, 7) (4, 5) (4, 6) (4, 7) (5, 6) (5, 7) (6, 7).

and the E-value of the whole design equal to $0.5 + 1 \times 7/2 = 4.0$.

Since the complementary design of a BIBD is a BIBD too, and a complementary design of an RGD is also an RGD, the following designs are E-optimal in the set of RGDs:

- (6, 6, 4) + (6, 15, 4) = (6, 21, 4),
- (7, 7, 5) + (7, 21, 5) = (7, 28, 5),
- (8, 8, 6) + (8, 28, 6) = (8, 36, 6),
- (9, 9, 7) + (9, 36, 7) = (9, 45, 7).

These designs are again confirmed in J. A. John and Mitchell's list.

Furthermore, we can show that the above designs have better values of E than any design without equal replication.

Theorem 4.2 *Let d_1 be the cyclic $(v, v, v-2)$ with $v \geq 5$ and d_2 be the BIBD of parameters $(v, v(v-1)/2, v-2)$. Then $d_+ = (v, b = v(v+1)/2, k = v-2)$ has a larger value of E than any unequireplicate design in $\mathcal{D}_{v,b,k}$.*

Proof. Let d be any block design without equal replication in $\mathcal{D}_{v,b,k}$. Let r_d be the smallest number of replicates in all treatments. Then we have

$$r_d < bk/v = [v(v+1)/2](v-2)/v = (v+1)(v-2)/2,$$

i.e.,

$$r_d \leq \frac{(v+1)(v-2)}{2} - 1.$$

By Theorem 3.2 [2],

$$\begin{aligned} \mu_d &\leq \frac{(v-2-1)v\ r_d}{(v-2)(v-1)} \\ &\leq \frac{(v-3)v}{(v-2)(v-1)} \left\{ \frac{(v+1)(v-2)}{2} - 1 \right\} \\ &= \frac{v(v-3)(v^2-v-4)}{2(v-2)(v-1)}. \end{aligned}$$

Since d_2 is a BIBD, all pairs of treatments appear exactly λ times with $\lambda = (v-2)(v-3)/2$ and we have $\lambda v/k = \lambda v/(v-2) = v(v-3)/2$. Moreover, by Equation 1 $\mu_{d_1} > (v-2) - [4/(v-2)]$. Therefore

$$\mu_{d_+} > (v-2) - \frac{4}{v-2} + \frac{v(v-3)}{2}.$$

It can be calculated algebraically that

$$(v-2) - \frac{4}{v-2} + \frac{v(v-3)}{2} \geq \frac{v(v-3)(v^2-v-4)}{2(v-2)(v-1)} \text{ and only if } v \geq 5.$$

This implies that $\mu_{d_+} \geq \mu_d$ and hence d_+ has better E value than d. □

Unfortunately, we still can not prove the E-optimality of design d_+ in $\mathcal{D}^*_{v,b,k}$. If it is E-optimal in $\mathcal{D}^*_{v,b,k}$, then the above theorem will conclude that d_+ is indeed E-optimal in $\mathcal{D}^*_{v,b,k}$.

5. Summary

In J. A. John and Mitchell's list, the E-optimality of some designs was unknown before. Fortunately, we have found the answer for some of the parameter sets.

- We show that the cyclic $(v,\ v,\ 2)$ are E-optimal in $\mathcal{D}^*_{v,v,2}$ but not in $\mathcal{D}_{v,v,2}$ for $v > 6$.
- We show that the cyclic $(v,\ v,\ v-2)$ with $v \geq 5$ are E-optimal for the binary designs in $\mathcal{D}_{v,v,v-2}$.

More research can be done in the field of E-optimality since there are about 30 more unsolved designs. Most of them have block size 2. We also gave Designs 1-3 which are E-optimal in $\mathcal{D}_{v,v,2}$. Their E-values are shown to be close to the upper bound when v is large. We suspect that the designs are E-optimal in $\mathcal{D}_{v,v,2}$ for $v > 6$.

By adding BIBDs to the E-optimal RGD, we obtain the new designs, d_+, that are E-optimal in the set of RGD. We suspect that the d_+ obtained from the cyclic $(v, v, 2)$ and the BIBD $(v, v(v-1)/2, 2)$ is also E-optimal in $\mathcal{D}^*_{v,v(v+1)/2,2}$. Moreover, if we can prove that the d_+ obtained from the cyclic $(v, v, v-2)$ and the BIBD $(v, v(v-1)/2, v-2)$ is E-optimal in $\mathcal{D}^*_{v,v(v+1)/2,v-2}$, then our Theorem 4.2 implies that d_+ is E-optimal in $\mathcal{D}_{v,v(v+1)/2,v-2}$.

A Proof of Theorem 2.3

In the proof below, we use v as our subscript for the information matrix $\mathbf{C}$ and identity matrix $\mathbf{I}$, because they are square matrices of order v. We shall show that

$$\det[2\mathbf{C}_v - \lambda\mathbf{I}_v] = -\lambda(1-\lambda)^{v-3}(2v-(v+3)\lambda+\lambda^2). \tag{4}$$

By the method of induction,
(i) When $v = 4$, we have

$$\det[2\mathbf{C}_4 - \lambda\mathbf{I}_4] = -\lambda(1-\lambda)(8-7\lambda+\lambda^2),$$

i.e., direct calculation shows that this is true for $v = 4$.
(ii) Assume that Equation 4 is true for $v = n \geq 4$. Let

$$p(\lambda) = \det[2\mathbf{C}_n - \lambda\mathbf{I}_n] = -\lambda(1-\lambda)^{n-3}(2n - (n+3)\lambda + \lambda^2), \tag{5}$$

$$p_1(\lambda) = (2-\lambda)(1-\lambda)^{n-2}. \tag{6}$$

(iii) Consider $v = n+1$. Since the element in the first row and the first column of $2\mathbf{C}_n - \lambda\mathbf{I}_n$ is $n-\lambda$, we can also represent $p(\lambda)$ by $(n-\lambda)p_1(\lambda) + p_2(\lambda)$. Let $q(\lambda) = \det[2\mathbf{C}_{n+1} - \lambda\mathbf{I}_{n+1}]$. We can write $q(\lambda)$ as the sum of the products of the entries in the last column by their corresponding cofactors. After simplification, we obtain

$$q(\lambda) = -\lambda p_1(\lambda) + (1-\lambda)p(\lambda).$$

According to Equations 5 and 6, we obtain

$$\begin{aligned} q(\lambda) &= -\lambda(2-\lambda)(1-\lambda)^{n-2} + (1-\lambda)[-\lambda(1-\lambda)^{n-3}(2n-(n+3)\lambda+\lambda^2)] \\ &= -\lambda(1-\lambda)^{v-3}[2v-(v+3)\lambda+\lambda^2]. \end{aligned}$$

□

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Department of Business Administration, Chang Jung University,
Tainan, Taiwan
Email: cmchang@mail.cju.edu.tw