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DERIVATIONS COCENTRALIZING MULTILINEAR POLYNOMIALS

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Abstract. Let R be a prime ring with center $\mathcal{Z}$ and let $f(X_1, ..., X_n)$ be a multilinear polynomial which is not central-valued on R. Suppose that d and δ are derivations on R such that $d(f(x_1, ..., x_n))f(x_1, ..., x_n) - f(x_1, ..., x_n)\,\delta(f(x_1, ..., x_n)) \in \mathcal{Z}$ for all $x_1, ..., x_n$ in some nonzero ideal of R. Then either $d = \delta = 0$ or $\delta = -d$ and $f(X_1, ..., X_n)^2$ is central-valued on R, except when char $R = 2$ and R satisfies the standard identity s_4 in 4 variables.

Throughout this note K will denote a commutative ring with unity and R will denote a prime K-algebra with center $\mathcal{Z}$. By d and δ we always mean derivations on R. For $x, y \in R$, let $[x, y] = xy - yx$.

A well-known result proved by Posner [17] states that if $[d(x), x] \in \mathcal{Z}$ for all $x \in R$, then either $d = 0$ or R is commutative. In [12], P. H. Lee and T. K. Lee generalized Posner's theorem by showing that if char $R \neq 2$ and $[d(x), x] \in \mathcal{Z}$ for all x in some Lie ideal L of R, then either $d = 0$ or L is contained in $\mathcal{Z}$. As to the case when char $R = 2$, Lanski [11] obtained the same conclusion except when R satisfies the standard identity s_4 in 4 variables. Note that a noncentral Lie ideal of R contains all the commutators $[x_1, x_2]$ for x_1, x_2 in some nonzero ideal of R except when char $R = 2$ and R satisfies s_4. So it is natural to consider the situation when $[d([x_1, x_2]), [x_1, x_2]] \in \mathcal{Z}$ for x_1, x_2 in some nonzero ideal of R. In a recent paper [13], a full generalization in this vein was proved by Lee and Lee that if $[d(f(x_1, ..., x_n)), f(x_1, ..., x_n)] \in \mathcal{Z}$ for all $x_1, ..., x_n$ in some nonzero ideal of R, where $f(X_1, ..., X_n)$ is a multilinear polynomial, then either $d = 0$ or $f(X_1, ..., X_n)$ is central-valued on R, except when char $R = 2$ and R satisfies s_4.

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On the other hand, Bresar [2] showed that if $d(x)x - x\delta(x) \in \mathcal{Z}$ for all $x \in R$, then either $d = \delta = 0$ or R is commutative. Recently we [14] proved that if $d(x)x - x\delta(x) \in \mathcal{Z}$ for all x in some noncentral Lie ideal of R, then either $d = \delta = 0$ or R satisfies s_4. In the present note, we shall extend these results to the case when $d(f(x_1, ..., x_n))f(x_1, ..., x_n) - f(x_1, ..., x_n)\delta(f(x_1, ..., x_n)) \in \mathcal{Z}$ for all x_i in some nonzero ideal of R, where $f(X_1, ..., X_n)$ is a multilinear polynomial.

First we dispose of the simplest case when R is the matrix ring $M_m(F)$ over a field F and d, δ are inner derivations on R.

Lemma 1. *Let F be a field and $R = M_m(F)$, the $m \times m$ matrix algebra over F. Suppose that $a, b \in R$ and that $f(X_1, ..., X_n)$ is a multilinear polynomial over F such that*

$$[a, f(x_1, ..., x_n)]f(x_1, ..., x_n) - f(x_1, ..., x_n)[b, f(x_1, ..., x_n)] \in \mathcal{Z}$$

for all $x_i \in R$. Then either $a + b \in \mathcal{Z}$ or $f(X_1, ..., X_n)$ is central-valued on R.

Proof. If $m = 1$, there is nothing to prove; so we assume that $m \geq 2$ and proceed to show that $a + b \in \mathcal{Z}$ if $f(X_1, ..., X_n)$ is not central-valued on R. For simplicity, we write $f(x_1, ..., x_n) = f(x)$ for $x = (x_1, ..., x_n) \in R^n = R \times \cdots \times R$ (n times). Then the hypothesis can be written as $[a, f(x)]f(x) - f(x)[b, f(x)] = af(x)^2 - f(x)(a + b)f(x) + f(x)^2 b \in \mathcal{Z}$ for all $x \in R^n$. Since $f(X_1, ..., X_n)$ is assumed to be noncentral on R, by [6, Lemma 1] and [15, Lemma 2] there exists a sequence of matrices $r = (r_1, ..., r_n)$ in R such that $f(r) = f(r_1, ..., r_n) = \alpha e_{st} \neq 0$ where $\alpha \in F$, $s \neq t$ and e_{st} is the matrix with 1 as the (s, t)-entry and $0'$s elsewhere. Thus $af(r)^2 - f(r)(a+b)f(r) + f(r)^2 b = -\alpha^2 e_{st}(a + b)e_{st} = -\alpha^2(a + b)_{ts} e_{st} \in \mathcal{Z}$, where $(a + b)_{ts}$ is the (t, s)-entry of $a + b$. Hence, $(a + b)_{ts} = 0$. For distinct h, k, let σ be a permutation in the symmetric group S_m such that $\sigma(t) = h$ and $\sigma(s) = k$, and let ψ be the F-automorphism on R defined by

$$\left(\sum_{i,j} \xi_{ij} e_{ij}\right)^{\psi} = \sum_{i,j} \xi_{ij} e_{\sigma(i),\sigma(j)}.$$

Then $f(r^{\psi}) = f(r_1^{\psi}, ..., r_n^{\psi}) = f(r)^{\psi} = \alpha e_{kh} \neq 0$ and we have as above $(a + b)_{hk} = 0$ for $h \neq k$. Thus $a+b$ is a diagonal matrix. For any F-automorphism θ of R, a^{θ} and b^{θ} enjoy the same property as a and b do, namely, $[a^{\theta}, f(x)]f(x) - f(x)[b^{\theta}, f(x)] \in \mathcal{Z}$ for all $x \in R^n$. Hence, $(a + b)^{\theta} = a^{\theta} + b^{\theta}$ must be also diagonal. Write $a + b = \sum_{i=1}^{m} \alpha_i e_{ii}$; then for each $j \neq 1$, we have

$$(1 + e_{1j})(a + b)(1 - e_{1j}) = \sum_{i=1}^{m} \alpha_i e_{ii} + (\alpha_j - \alpha_1)e_{1j}$$

diagonal. Therefore, $\alpha_j = \alpha_1$ and so $a+b$ is a scalar matrix.

We are now ready to prove the main theorem.

Theorem 1. *Let R be a prime K-algebra with center $\mathcal{Z}$ and let $f(X_1, ..., X_n)$ be a multilinear polynomial over K which is not central-valued on R. Suppose that d and δ are derivations on R such that*

$$d(f(x_1, ..., x_n))f(x_1, ..., x_n) - f(x_1, ..., x_n)\delta(f(x_1, ..., x_n)) \in \mathcal{Z}$$

for all x_i in some nonzero ideal I of R. Then either $d = \delta = 0$ or $\delta = -d$ and $f(X_1, ..., X_n)^2$ is central-valued on R, except when char $R = 2$ and R satisfies s_4.

Proof. First note that if $\delta = -d$, then $d(f(x_1, ..., x_n)^2) \in \mathcal{Z}$ for all $x_i \in I$. Let A be the additive subgroup generated by all the elements of the form $f(x_1, ..., x_n)^2$ with $x_i \in I$. By a theorem due to Chuang [3], either $f(X_1, ..., X_n)^2$ is central-valued on R or A contains a noncentral Lie ideal L of R, except when $R = M_2(GF(2))$, the ring of 2×2 matrices over the field of 2 elements. If $L \subseteq A$, then $d(L) \subseteq \mathcal{Z}$ and it follows from [1, Lemma 6] and [8, Lemma 2] that $d = 0$ unless char $R = 2$ and R satisfies s_4. So it suffices to show that either $d = \delta = 0$ or $\delta = -d$ on condition that either char $R \neq 2$ or R does not satisfy s_4.

Assume first that both d and δ are Q-inner, that is, $d(x) = ad_a(x) = [a, x]$ and $\delta(x) = ad_b(x) = [b, x]$ for all $x \in R$, where a and b are elements in the *symmetric quotient ring* Q of R [9]. Then

$$\begin{aligned} g(x_1, ..., x_{n+1}) = \ & [[a, f(x_1, ..., x_n)]f(x_1, ..., x_n) \\ & -f(x_1, ..., x_n)[b, f(x_1, ..., x_n)], x_{n+1}] = 0 \end{aligned}$$

for all $x_i \in I$. By [4, Theorem 2], this generalized polynomial identity (GPI) $g(X_1, ..., X_{n+1})$ is also satisfied by Q. In case the center C of Q is infinite, we have $g(x_1, ..., x_{n+1}) = 0$ for all $x_i \in Q\bigotimes_C \bar{C}$ where $\bar{C}$ is the algebraic closure of C. Since both Q and $Q\bigotimes_C \bar{C}$ are prime and centrally closed [5, Theorems 2.5 and 3.5] we may replace R by Q or $Q\bigotimes_C \bar{C}$ according as C is finite or infinite respectively. Thus we may assume further that $a, b \in R$ and R is centrally closed over C which is either finite or algebraically closed and $g(x_1, ..., x_{n+1}) = 0$ for all $x_i \in R$.

Suppose that $d \neq 0$ or $\delta \neq 0$. Then $a \notin C$ or $b \notin C$ and so the GPI $g(X_1, ..., X_{n+1})$ is nontrivial. By Martindale's theorem [16], R is then a primitive ring having nonzero socle H with C as the associated division ring. In light of Jacobson's theorem [7, p.75], R is isomorphic to a dense ring of linear transformations of some vector space V over C, and H consists of the linear transformations in R of finite rank. Assume first that V is finite-dimensional over C. Then the density of R on ${}_CV$ implies that $R \cong M_m(C)$ with $m = \dim_C V$. By Lemma 1, we have $a + b \in C$ and so $\delta = -d$. Assume next that V is infinite-dimensional over C. Suppose that $a + b$ is not central in R; then it does not centralize the nonzero ideal H of R, so $(a + b)h_0 \neq h_0(a + b)$ for some $h_0 \in H$. Also, $f(X_1, ..., X_n)$ is not central-valued on H, for otherwise R would satisfy the polynomial identity $[f(X_1, ..., X_n), X_{n+1}]$, contrary to the infinite-dimensionality of ${}_CV$. So $[f(h_1, ..., h_n), h_{n+1}] \neq 0$ for some $h_1, ..., h_{n+1} \in H$. By Litoff's theorem [11, p.280], there is an idempotent $e \in H$ such that $(a + b)h_0$, $h_0(a + b)$, h_0, $h_1, ..., h_{n+1}$ are all in eRe. Note that we have $eRe \cong M_m(C)$ with $m = \dim_C Ve$. Since R satisfies the GPI $eg(eX_1e, ..., eX_{n+1}e)e$, the subring eRe satisfies the GPI

$$\begin{aligned} g_e(X_1, ..., X_{n+1}) = \ & [[eae, f(X_1, ..., X_n)]f(X_1, ..., X_n) \\ & -f(X_1, ..., X_n)[ebe, f(X_1, ..., X_n)], X_{n+1}]. \end{aligned}$$

By Lemma 1 again, $eae + ebe$ is central in eRe because $f(X_1, ..., X_n)$ is not central-valued on eRe. Thus $(a + b)h_0 = e(a + b)h_0 = e(a + b)eh_0 = h_0e(a + b)e = h_0(a + b)e = h_0(a + b)$, a contradiction. Hence, $a + b$ is central in R and so $\delta = -d$.

Now assume that d and δ are not both Q-inner. Suppose first that d and δ are C-dependent modulo Q-inner derivations, say, $\delta = \lambda d + ad_a$ where $\lambda \in C$ and $a \in Q$. Then d cannot be Q-inner and $d(f(x))f(x) - \lambda f(x)d(f(x)) - f(x)[a, f(x)] \in \mathcal{Z}$ for all $x \in I^n$. Recall that d can be extended uniquely to a derivaion $\bar{d}$ on Q [9]. We denote by $f^d(X_1, ..., X_n)$ the polynomial obtained from $f(X_1, ..., X_n)$ by replacing each coefficient α with $\bar{d}(\alpha \cdot 1)$. Since

$$\begin{aligned} \left(f^d(x)\right. & \left. + \sum_{i=1}^{n} f(x_1, ..., d(x_i), ..., x_n)\right) f(x) \\ & -\lambda f(x)\left(f^d(x) + \sum_{i=1}^{n} f(x_1, ..., d(x_i), ..., x_n)\right) - f(x)[a, f(x)] \in \mathcal{Z} \end{aligned}$$

for all $x = (x_1, ..., x_n) \in I^n$, we have

$$\begin{aligned} \left(f^d(x)\right. & \left. + \sum_{i=1}^{n} f(x_1, ..., y_i, ..., x_n)\right) f(x) \\ & -\lambda f(x)\left(f^d(x) + \sum_{i=1}^{n} f(x_1, ..., y_i, ..., x_n)\right) - f(x)[a, f(x)] \in \mathcal{Z} \end{aligned}$$

for all $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ in R^n by Kharchenko's theorem [10]. In particular,

$$f^d(x)f(x) - \lambda f(x)f^d(x) - f(x)[a, f(x)] \in \mathcal{Z}$$

and

$$f(x_1, ..., y_i, ..., x_n)f(x) - \lambda f(x)f(x_1, ..., y_i, ..., x_n) \in \mathcal{Z}$$

for all $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ in R^n and for each $i = 1, ..., n$. Choosing $b \in R$ with $b \notin \mathcal{Z}$, setting $y_i = [b, x_i]$ in each of the last n relations, and summing up over i, we have $[b, f(x)]f(x) - f(x)[\lambda b, f(x)] \in \mathcal{Z}$ for all $x \in R^n$. By the preceding paragraph, we have $(1 + \lambda)b \in \mathcal{Z}$ and so $\lambda = -1$. Also, by the first paragraph, $f(x)^2 \in \mathcal{Z}$ for all $x = (x_1, ..., x_n) \in R^n$. Thus, $d(f(x))f(x) + f(x)d(f(x)) \in \mathcal{Z}$ and so the hypothesis

$$d(f(x))f(x) + f(x)d(f(x)) - f(x)[a, f(x)] \in \mathcal{Z}$$

implies $f(x)[a, f(x)] \in \mathcal{Z}$ for $x \in R^n$. Again, it follows from the inner case that $a \in C$ and so $\delta = -d$ as expected. The situation when $d = \lambda\delta + ad_a$ is slmilar.

Finally, assume that d and δ are C-independent modulo Q-inner derivations. Since neither d nor δ is Q-inner, the relation

$$\left(f^d(x) \quad + \sum_{i=1}^{n} f(x_1, ..., d(x_i), ..., x_n)\right) f(x) \\ -f(x)\left(f^\delta(x) + \sum_{i=1}^{n} f(x_1, ..., \delta(x_i), ..., x_n)\right) \in \mathcal{Z}$$

for all $x = (x_1, ..., x_n) \in I^n$ yields

$$\left(f^d(x) \quad + \sum_{i=1}^{n} f(x_1, ..., y_i, ..., x_n)\right) f(x) \\ -f(x)\left(f^\delta(x) + \sum_{i=1}^{n} f(x_1, ..., z_i, ..., x_n)\right) \in \mathcal{Z}$$

for all $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$ and $z = (z_1, ..., z_n)$ in R^n. In particular, $f^d(x)f(x) - f(x)f^\delta(x) \in \mathcal{Z}$, $f(x_1, ..., y_i, ..., x_n)f(x) \in \mathcal{Z}$ and $f(x)f(x_1, ..., z_i, ..., x_n) \in \mathcal{Z}$ for all x, y, $z \in R^n$, and for each $i = 1, ..., n$. As before, choosing $b \in R$, $b \notin \mathcal{Z}$, setting $z_i = [b, x_i]$ in the last n relations and summing up over i, we obtain that $f(x)[b, f(x)] \in \mathcal{Z}$ for all $x \in R^n$, a contradiction again. This completes the proof.

It was proved in [13] that if $[d(f(x_1,...,x_n)), f(x_1,...,x_n)]_k = 0$ for all x_i in some nonzero ideal of R then either $d = 0$ or $f(X_1,...,X_n)$ is central-valued on R except when char $R = 2$ and R satisfies s_4. The case when $k = 1$ follows easily from our Theorem 1. A fact about power-central polynomial is needed for our purpose.

Lemma 2. *Let R be a prime K-algebra of characteristic 2 and $f(X_1,...,X_n)$ a multilinear polynomial over K. Subppose that $f(X_1,...,X_n)^{2^r}$ is central-valued on R for some r. Then $f(X_1,...,X_n)$ is central-valued on R unless R satisfies s_4.*

Proof. Since R satisfies the polynomial identity (PI) $[f(X_1,...,X_n)^{2^r}, X_{n+1}]$, the central quotient $R_{\mathcal{Z}}$ of R is a finite-dimensional central simple algebra satisfying the same PI's as R does. Without loss of generality, we may assume that $R = M_m(D)$ for some division algebra D which is finite-dimensional over its center. Suppose first that D is a field; then $m > 2$ if R does not satisfy s_4. Since char $D = 2$, the field D contains no 2^r-th roots of unity other than 1, so $f(X_1,...,X_n)$ is central-valued on R by [15, Theorem 10]. Suppose next that D is not a field; then the center $\mathcal{Z}$ must be infinite and so $R \bigotimes_{\mathcal{Z}} K \cong M_k(K)$ satisfies the same PI's as R does, where K is a maximal subfield of D and $k = (\dim_{\mathcal{Z}} R)^{1/2} > 2$ if R does not satisfy s_4. Thus $f(X_1,...,X_n)$ is central-valued on $R \bigotimes_{\mathcal{Z}} K$ as well as R.

Theorem 2. *Let R be a prime K-algebra with center $\mathcal{Z}$ and let $f(X_1,...,X_n)$ be a multilinear polynomial over K. Suppose that d is a derivation on R such that $[d(f(x_1,...,x_n)), f(x_1,...,x_n)] \in \mathcal{Z}$ for all x_i in some nonzero ideal I of R. Then either $d = 0$ or $f(X_1,...,X_n)$ is central-valued on R except when char $R = 2$ and R satisfies s_4.*

Proof. Assume that $f(X_1,...,X_n)$ is not central-valued on R and either char $R \neq 2$ or R does not satisfy s_4. By Theorem 1, either $d = 0$ or $d = -d$ and $f(X_1,...,X_n)^2$ is central-valued on R. In the later case, char $R = 2$ if $d \neq 0$, and so $f(X_1,...,X_n)$ must be central-valued on R by the preceding lemma. With this contradiction the theorem is proved.

References

1. J. Bergen, I. N. Herstein and J. W. Kerr, Lie ideals and derivations of prime rings, *J. Algebra* **71** (1981), 259–267.

2. M. Bresar, Centralizing mappings and derivations in prime rings, *J. Algebra* **156** (1993), 385–394.

3. C. L. Chuang, The additive subgroup generated by a polynomial, *Israel J. Math.* **59** (1987), 98–106.

4. C. L. Chuang, GPIs having coefficients in Utumi quotient rings, *Proc. Amer. Math. Soc.* **103** (1988), 723–728.

5. T. S. Erickson, W. S. Martindale and J. M. Osborn, Prime nonassociative algebras, *Pacific J. Math.* **60** (1975), 49–63.

6. B. Felzenszwalb and A. Giambruno, Periodic and nil polynomials in rings, *Canad. Math. Bull.* **23** (1980), 473–476.

7. N. Jacobson, Structure of Rings, Amer. Math. Soc. Colloq. Pub. **37**, Amer. Math. Soc., Providence, 1964.

8. W. F. Ke, On derivations of prime rings of characteristic 2, *Chinese J. Math.* **13** (1985), 273–290.

9. V. K. Kharchenko, Differential identities of prime rings, *Algebra i Logika* **17** (1978), 220–238.

10. V. K. Kharchenko, Differential identities of semiprime rings, *Algebra i Logika* **18** (1979), 86–119.

11. C. Lanski, Differential identities, Lie ideals, and Posner's theorems, *Pacific J. Math.* **134** (1988), 275–297.

12. P. H. Lee and T. K. Lee, Lie ideals of prime rings with derivations, *Bull. Inst. Math. Acad. Sinica* **11** (1983), 75–80.

13. P. H. Lee and T. K. Lee, Derivations with Engel conditions on multilinear polynomials, *Proc. Amer. Math. Soc.*, to appear.

14. P. H. Lee and T. L. Wong, Derivations cocentralizing Lie ideals, *Bull. Inst. Math. Acad. Sinica* **23** (1995), 1–5.

15. U. Leron, Nil and power–central polynomials in rings, *Trans. Amer. Math. Soc.* **202** (1975), 97–103.

16. W. S. Martindale, Prime rings satisfying a generalized polynomial identity, *J. Algebra* **12** (1969), 576–584.

17. E. C. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.* **8** (1957), 1093–1100.

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