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# PARTIALLY ORDERED NORMED lINEAR SPACES WITH WEAK FATOU PROPERTY

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**Abstract.** Let $E$ be a Riesz space with lattice ordered norm $\|\cdot\|$. Amemiya proved that $E$ is complete under this norm if $E$ has weak Fatou property for monotone sequence (: $E$ is monotone complete) with respect to the norm $\|\cdot\|$. This is a generalization of the Riesz-Fisher's and the Nakano's theorem. In the cases of non normed Riesz space or non lattice ordered norm, this theorem is not true in general. We shall investigate in this paper a necessary and sufficient condition for Amemiya's theorem to be valid in a partially ordered normed linear space.

## 1. Partially Ordered Normed Linear Spaces

Let $E$ be a linear space with real coefficient. Let us consider a convex cone $P$ in $E$ satisfying

(a) $\quad P$ generates $E$, i.e. $E = P - P$,

and

(b) $\quad P \cap (-P) = \{0\}$.

If we define $x \geq y$ (we can write $y \leq x$ in the same meaning) $\Longleftrightarrow x - y \in P$, then the relation $\geq$ satisfies the following properties:

(1) $x \geq y$ and $y \geq x \Longrightarrow x = y$.

(2) $x \geq y, \quad y \geq z \Longrightarrow x \geq z$.

(3) $x \geq y \Longrightarrow x + z \geq y + z$ for all $z \in E$.

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(4) $x \geq 0$ and non-negative scalar $\alpha$ imply $\alpha x \geq 0$.

(5) For every $x \in E$ there exist $x_1, x_2 \in E$, $x_1, x_2 \geq 0$ with $x = x_1 - x_2$.

If there is a relation $\geq$ in $E$ satisfying (1), (2), (3), (4), and (5), then $P = \{x; x \geq 0\}$ is a convex cone satisfying properties (a) and (b). If there is a convex cone with (a) and (b) in $E$ or there is an order relation with (1), (2), (3), (4), (5) in $E$, then $E$ is called a *partially ordered linear sapce* and $P$ is called an *order* in $E$.

If there exists the least upper bound for any two elements $x, y \in E$, then $E$ is called a *Riesz space* or *vector lattice*. In general, a partially ordered linear space $E$ is not necessarily a Riesz space or vector lattice, i.e. it is not necessary to have the least upper bound for two elements $x, y \in E$.

Let us consider a norm in $E$. A norm $\| \cdot \|$ is a non-negative functional defined on $E$ as usual with the following properties:

(1) $\| x \| = 0 \Leftrightarrow x = 0$

(2) $\| \alpha x \| = |\alpha| \, \| x \|$

(3) $\| x + y \| \leq \| x \| + \| y \|$

for $x, y \in E$ and real scalar $\alpha$.

A norm is defined by an absorbing symmetric convex set $U$ in $E$ which separates a nonzero element $x \neq 0$ and 0, i.e. if $x \neq 0$, then there exists a nonzero real number $\alpha$ such that $\alpha U$ does not contain $x$.

It is well known that $\| x \| = \inf\{\alpha \geq 0; x \in \alpha U\}$, where $U = \{x; \| x\| \leq 1\}$ and $U$ is an absorbing symmetric convex subset in $E$.

The relation between an order $P$ and a norm in $E$ is quite interesting and also complicated if $E$ is infinite-dimensional. If $E$ is a Riesz space and a norm $\| \cdot \|$ of $E$ is lattice order preserving, i.e. $|x| \leq |y|$ for $x, y$ in $E$ implies $\| x \| \leq \| y \|$, where $|x| = \sup\{x, -x\}$ = least upper bound of $x$ and $-x$, then $E$ is called a *normed Riesz space* and the norm $\| \cdot \|$ is called a *lattice ordered norm* or *Riesz norm*.

Almost 40 years ago, I. Amemiya proved that if $E$ is a normed Riesz space and its norm $\| \cdot \|$ has the weak Fatou property for monotone sequence, then the norm $\| \cdot \|$ is complete, i.e. $E$ is a Banach space. This theorem is considered a generalization of the Riesz-Fisher's and Nakano's theorem.

We shall discuss this problem again in the case that $E$ is not necessarily a normed Riesz space with respect to $P$ or the norm is not a Riesz norm.

## 2. Weak Fatou Property

Let $E$ be a partially ordered linear space with an order $P$. A norm $\| \cdot \|$ on

$E$ is said to be an *ordered norm* (w.r.t. $P$) if $0 \leq x \leq y$, i.e. $x, y, y-x \in P$ imply $\| x \| \leq \| y \|$. In this section we shall assume that every norm on a partially ordered linear space $E$ is an ordered norm.

A norm $\| \cdot \|$ on $E$ is said to have the *weak Fatou property for monotone sequence* if $0 \leq a_n \uparrow_n$ (*i.e.* $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$) *and* $\sup \| a_n \| < +\infty$ imply that $\{a_1, a_2, \cdots a_n, \cdots\}$ has the least upper bound, which is denoted by $\sup a_n$ or $\cup_n a_n$.

$E$ is called *Archimedean* if $\inf (1/n)x =$ the greatest lower bound of $\{x, (1/2)x, \cdots (1/n)x, \cdots\} = 0$ for all $x \in E$ with $x \geq 0$, i.e. for all $x$ with $x \in P$.

**Lemma 1.** *Let E be a partially ordered normed linear space with the weak Fatou property for monotone sequence and let the norm is ordered. If* $0 \leq a_n \uparrow$ (*i.e.* $0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$) *and* $a_n \leq a$ *for* $n = 1, 2, \cdots$, *then there exists sup* $a_n$, *i.e. the least upper bound for* $a_n, n = 1, 2. \cdots$

**Lemma 2** *Let E be the same as in Lemma 1 above. If* $a_1 \geq a_2 \geq \cdots \geq a_n \cdots \geq a \geq 0$, *then there exists inf* $a_n$, *i.e. the greatest lower bound for* $a_n$, $n = 1, 2, \cdots$.

Proofs of Lemma 1 and Lemma 2 are easily deduced from the weak Fatou property for monotone sequence of $E$. So it is omitted.

**Lemma 3.** *Let E be the same as in Lemma 1 above. Then E is Archimedean.*

*Proof.* By Lemma 2, there exists $b = \inf_n (1/n)\, a$ for every $a \geq 0$. We know $b \geq 0$ and $a \geq nb$ for all $n = 1, 2, \cdots$. But it means that $\| a \| \geq n \| b \|$ for all $n$, i.e. it must be $b = 0$, otherwise $\| a \| = +\infty$.

The next lemma is essentially due to I. Amemiya.

**Lemma 4.** *Let E be a partially ordered normed linear space with weak Fatou property for monotone sequence and let the norm of E be ordered. Then, there exists a positive real number* $\Lambda$ *such that*

$$0 \leq a_n \uparrow a \ \text{ always implies } \sup_n \| a_n \| \geq \Lambda \| a \|.$$

*Proof.* If the conclusion is false, then we can find a sequence $a_n \, (n = 1, 2, \cdots)$ with $0 \leq a_{n,m} \uparrow_m a_n$,

$$\| a_n \| \geq n$$

and

$$\sup_m \| a_{n,m} \| \leq 1/2^n.$$

Putting

$$b_n = a_{1,n} + a_{1,n} + \cdots + a_{n,n} \quad \text{for} \quad n = 1,\, 2,\, \cdots,$$

we find that the sequence $\{b_n \geq 0\}$ is increasing and $\| b_n \| \leq 1$ for all $n = 1,\, 2,\, \cdots$. Hence $b = \sup b_n$ exists by the weak Fatou property of the norm.

But $b \geq a_{n,m}$ for all $n$ and $m$ imply

$$b \geq \sup_m a_{n,m} = a_n$$

and hence $\| b \| \geq \| a_n \| \geq n$. But it means that $\| b \| = +\infty$ and this is a contradiction. Q.E.D.

## 3. Completeness of the Norm

A norm $\| \cdot \|$ of a partially ordered linear space $E$ is called *well-situated* to an order $P$ in $E$ if there exists a positive constant $\alpha > 0$ such that, for all $x \in E$ with $\| x \| \leq 1$, we can find $x_1 \geq 0$ and $x_2 \geq 0$ with $x = x_1 - x_2$ and with $\| x_1 \| \leq \alpha$ and $\| x_2 \| \leq \alpha$. It is equivalent to that there exists a positive constant $\alpha > 0$ such that for all $x \in E$ we can find $x_1 \geq 0$ and $x_2 \geq 0$ with $x = x_1 - x_2$ and with $\| x_1 \| \leq \alpha \| x \|$ and $\| x_2 \| \leq \alpha \| x \|$.

**Theorem 1.** *Let $E$ be a partially ordered normed linear space with an order $P$ satisfying the weak Fatou property for monotone sequence and let the norm is a well-situated ordered norm. Then the norm $\| \cdot \|$ is complete. Conversely, if $E$ is complete by the norm and $P$ is closed, then the norm is well-situated.*

So, let $P$ be a closed order. Then $E$ is complete by the norm if and only if the norm is well-situated.

*Proof.* We shall prove that $E$ is complete if the order $P$ has the weak Fatou property for monotone sequence and the norm is a well-situated ordered norm. For a Cauchy sequence we can select a subsequence $a_n, n = 1,\, 2,\, \cdots$ with

$$\| a_n - a_m \| \leq 1/2^n \quad \text{for } m \geq n.$$

Since the norm is well-situated, we find a positive number $\alpha$ and positive elements $b_n,\ c_n \in E$ such that

$$a_n - a_{n+1} = b_n - c_n; \quad b_n, c_n \geq 0$$

with

$$\| b_n \| \text{ and } \| c_n \| \leq \alpha \| a_n - a_{n+1} \|.$$

We can find positive elements $b$ and $c \in E$ such that

$$b = \sup_n \sum_{i=1}^{n} b_i \quad \text{and} \quad c = \sup_n \sum_{i=1}^{n} c_i.$$

Hence we have

$$b - \sum_{i=1}^{n} b_i = \sup_m (b_n + b_{n+1} + \cdots + b_m)$$

and

$$\begin{aligned} \| b - \sum_{i=1}^{n} b_i \| &= \| \sup_m (b_n + b_{n+1} + \cdots b_m) \| \\ &\leq (1/\Lambda) \sup_m \| b_n + b_{n+1} + \cdots + b_m \| \\ &\leq (\alpha/\Lambda)(1/2^{n-1}), \end{aligned}$$

since $\|b_n\|$ and $\|c_n\| \leq \alpha/2^n$. Hence $\sum_{i=1}^{n} b_i$ converges to $b$ by norm as $n \to \infty$. Similarly, we have $\sum_{i=1}^{n} c_i$ converges to $c$ by norm as $n \to \infty$.

Since

$$\begin{aligned} a_1 - a_{n+1} &= a_1 - a_2 + a_2 - a_3 + \cdots + a_n - a_{n+1} \\ &= b_1 - c_1 + b_2 - c_2 + \cdots + b_n - c_n, \end{aligned}$$

we have that $a_1 - a_{n+1}$ converges to $b - c$ by norm and hence $a_{n+1}$ (also $a_n$) converges to $a_1 - b + c$ by norm.

If a subsequence of a Cauchy sequence is convergent, then the Cauchy sequence is convergent, hence the norm is complete.

We shall show the proof of the last half of the theorem. This fact is first proved by $T$. Ando [2] whose proof relies on Klee's theorem. Here we shall show by more elementary way.

Let us assume that $E$ is a Banach space and $P$ is closed. Let $U = \{x; \|x\| \leq 1\}$ be the closed unit ball of $E$. Since Banach space $E$ has 2nd category property and

$$E = \bigcup_{n=1}^{\infty} \{P \cap nU - P \cap nU\},$$

there exists a number $m$ such that $\{P \cap mU - P \cap mU\}^-$, the closure of $\{P \cap mU - P \cap mU\}$, contains a non-empty open ball $V$.

Since

$$\alpha\{P\cap mU - P\cap mU\} = P\cap \alpha mU - P\cap \alpha mU$$

for each positive number $\alpha$ and

$$\{P\cap 2mU - P\cap 2mU\} = \{P\cap mU - P\cap mU\} - \{P\cap mU - P\cap mU\},$$

we have

$$\begin{aligned}\{P\cap 2mU - P\cap 2mU\}^- &= \{P\cap mU - P\cap mU\}^- - \{P\cap mU - P\cap mU\}^- \\ &\supset V - V \supset \beta U\end{aligned}$$

for some $\beta > 0$. Hence, there exists a positive number $\gamma > 0$ such that

$$\{P\cap U - P\cap U\}^- \supset \gamma U.$$

So, for each $x \in \gamma U$, we can choose $a_n$, $b_n \in P\cap U (n = 1, 2, \cdots)$ with

$$\| x - (a_1 - b_1) - \cdots - (a_n - b_n) \| \leq \gamma/2^n$$

and

$$\| a_n \|, \ \| b_n \| \leq 1/2^{n-1}.$$

Hence $\sum\limits_{n=1}^{\infty} a_n = a$, $\sum\limits_{n=1}^{\infty} b_n = b$ are convergent, $a, b \in U$ and

$$x = a - b.$$

So, we find that $(\gamma/2)U \subset P\cap U - P\cap U$. This means that the norm $\|\cdot\|$ is well situated. Q.E.D

We shall consider the case when $P$ is not closed.

A norm $\|\cdot\|$ of $E$ is called *weakly well-situated* if there exists a positive number $\alpha > 0$ such that for every $x \in E$ and for every positive number $\epsilon > 0$ there exist $x_1$, $x_2 \in P$ with

$$\| x_1 \|, \ \| x_2 \| \leq \alpha \| x \|$$

and

$$\|x - (x_1 - x_2)\| < \epsilon.$$

**Theorem 2.** *Let $E$ be a partially ordered normed linear space whose norm has the weak Fatou property for monotone sequence and is ordered. The norm of $E$ is complete if and only if it is weakly well-situated.*

The proof of Theorem 2 is almost the same as the proof of Theorem 1 and so it is omitted.

## 4. Remarks and Examples

We shall show some examples of partially ordered normed linear spaces which are not measurable function spaces on some measure spaces, in which the Riesz-Fisher's theorem is valid. Mesurable function spaces are usually lattice ordered and norms in these spaces are usually ordered norms.

1. Let $H$ be a Hilbert space. The totality of bounded self-adjoint operators on $H$ is denoted by $S(H)$. Let $P$ be a set of all positive-definite operators. Then $P$ is a positive cone in $S(H)$. Furthermore, $P - P = S(H)$. Hence, $S(H)$ is an partially ordered linear space.

$S(H)$ is not lattice ordered, since usually two operators $A$ and $B$ have no least upper bound if they are not commutative. By Theorem 2, any ordered norm with the weak Fatou property for monotone sequence and weakly well-situated is complete.

2. Let $E$ be the n-dimensional Euclidean space $R^n$. Let $P$ be the positive cone determined by lexicographic order. $P$ is not closed if $n \geq 2$. Since $E$ is not Archimedean by the order $P$, any norm in $E$ does not have the weak Fatou property for monotone sequence by Lemma 3. But any norm in $E$ is complete since $E$ is finite dimensional. In this case, it is not true that $0 \leq a \leq b$ imply $\| a \| \leq \| b \|$.

3. In 2-dimensional Euclidean space $E = \mathbf{R^2}$, every closed order $P$ in $E$ is lattice ordered. But every non closed $P$ is not lattice ordered if $P^-$ is also an order in $E$. On the contrary, there is a closed order $P$ in 3-dimensional Euclidean space $E = \mathbf{R^3}$ which is not lattice ordered. We shall show such example.

Let $P$ be a closed cone generated by 4 elements (0,1,0), (0,1,1), (1,1,0), (1,1,1) in $E = \mathbf{R^3}$. Then, by the order $P$, there is no least upper bound for $z = (0, 0, 1)$ and $0 = (0, 0, 0)$. Consider the set $A = \{(a, 1, 1); 0 \leq a \leq 1\}$. For $x, y \in A$, $x \neq y$, then $x$ and $y$ are not comparable by the order $P$ and for each $x$ with $x \geq 0$ and $x \geq z$ there exists $w \in A$ such that $x \geq w \geq 0$ and $w \geq z$. So, there is no least upper bound for two elements $z$ and 0 by the order $P$.

4. In 2-dimensional Euclidean space $E = \mathbf{R^2}$, every order $P$ (closed or not closed) is weak Fatou property for monotone sequence if $P^-$ is also an order in $E$. So, there are many examples of non closed order in sequence space with $\ell^2$-norm which satisfies the weak Fatou property for monotone sequence.

5. Norms in partially ordered linear spaces are not necessarily ordered norms. For example, let $P$ be a closed cone generated by two elements (1,1)

and $(0, -1)$ in $\mathbf{R^2}$. Then, the Euclidean norm ($\ell^2$-norm) in $\mathbf{R^2}$ is not an order norm.

6. In a finite dimensional space $E$, every norm is well-situated to an arbitrary order. On the contrary there is an example of not weakly well-situated norm in an infinite dimensional space. At first, we shall consider some order in 2-dimensional $\mathbf{R^2}$ with Euclidean norm.

Let $y_1 = (1, 0)$, $y_2 = (\cos\ /n, \sin\ 1/n) \in \mathbf{R^2}$ and $P_n$ be a convex cone generated by two elements $y_1$ and $y_2$. Let $\mathbf{e} = (0, 1) \in \mathbf{R^2}$.

Consider the number

$$\alpha_n = \inf\{\alpha; \|x_1\|,\ \|X_2\| \leq \alpha\|\mathbf{e}\| \text{ and } \mathbf{e} = x_1 - x_2, x_1, x_2 \in P_n\}.$$

By elementary calculus, we have

$$(^*) \qquad \alpha_n \geq 1/\sin(1/n) \to \infty \quad \text{as} \quad n \to \infty.$$

Let us consider the sequence space $\mathbf{R^2} \times \mathbf{R^2} \times \cdots$ and consider a positive convex cone $P = (P_1 \times P_2 \times \cdots) \cap \ell^2$ and put $E = P - P$, $E$ being a subspace of $\ell^2$. Then, the $\ell^2$-norm in $E$ is not well-situated to $P$ by $(^*)$ above. In this case, $E$ is not complete, i.e. $E$ is not a closed subspace of $\ell^2$ and $\ell^2$-norm in $E$ has the weak Fatou property for monotone sequence to $P$.

7. Let $E$ be a partially ordered linear space with a linear Hausdorff topology whose basis is consisting of countable semi-norms which are comparable with the order, i.e. $x \geq y \geq 0$ implies $\|x\| \geq \|y\|$ for semi-norm $\|\cdot\|$. We say that $E$ has a weak Fatou property if every increasing sequence $x_n (n = 1, 2, \cdots)$ which is topologically bounded has always $\sup_n x_n$. A linear topology defined by countable semi-norms $\|\cdot\|_n$ is called *well-situated* if there exists a positive number $\alpha$ such that for every $x \in E$, there exist $x_1, x_2 \geq 0$ with $x = x_1 - x_2$ and $\|x_1\|_n$, $\|x_2\|_n \leq \alpha\|x\|_n$ for all $n = 1, 2, \cdots$. Then we have the following fact. If a linear topology generated by countable semi-norms has a weak Fatou property and is well-situated, then the linear topology is complete. Proof of this fact is almost same as Theorem 1.

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