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BOOK REVIEW

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B. S. Mordukhovich
Variational Analysis and Generalized Differentiation, I. Basic Theory.
Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], A Series of Comprehensive Studies in Mathematics Vol. 330, Springer-Verlag, Berlin, 2006, xxii+579 pp.

Variational Analysis and Generalized Differentiation, II. Applications,
Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], A Series of Comprehensive Studies in Mathematics Vol. 331, Springer-Verlag, Berlin, 2006, xxii+610 pp.

The first Volume "Basic Theory" has four chapters: 1. Generalized Differentiation in Banach Spaces, 2. Extremal Principle in Variational Analysis, 3. Full Calculus in Asplund Spaces, 4. Characterizations of Well-Posedness and Sensitivity Analysis; the second Volume "Applications" also has four chapters: 5. Constrained Optimization and Equilibria, 6. Optimal Control of Evolution Systems in Banach Spaces, 7. Optimal Control of Distributed Systems, 8. Applications to Economics. The Volumes present a comprehensive and deep theory of generalized differentiation based on the *geometric dual-space approach* and the *extremal principle* proposed by the author. The theory has been developed very successfully in the last three decades by the efforts of the author, his collaborators, and of many other researchers around the world. Note that the present complete infinite-dimensional version of the theory has been established quite recently. This self-contained two-volume book can be considered as a wonderful continuation of the preceding book of the author [B. S. Mordukhovich, *Approximation Methods in Problems of Optimization and Control*, Nauka, Moscow, 1988, 360 pp., (in Russian); MR0945143 (89m:49001)].

Chapter 1 introduces the basic concepts of this generalized differentiation theory: the (generally nonconvex) dual objects called the *basic/limiting normal cone* to

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a subset in a Banach space, the *normal coderivative* of a multifunction between Banach spaces, the *basic/limiting subdifferential* of a extended-real-valued function defined on a Banach space. As the author stresses in the book, *there is no tangent objects in the primal spaces corresponding to these dual objects*. In fact, in whole of the book, all the results are obtained without using tangent constructions (like tangent cones to sets, tangent cones to graphs of multifunctions, tangent cones to epigraphs of functions). The closely related notions of Fréchet/prenormal cone, Fréchet/prenormal coderivative and Fréchet subdifferential, are also described and studied in details. In order to have an idea about the dual objects used in this book, let us consider some basic definitions. Given a subset Ω of a real Banach space X, one writes $x \xrightarrow{\Omega} \bar{x}$ if and only if $x \to \bar{x}$ and $x \in \Omega$. For any $x \in \Omega$ and $\varepsilon \geq 0$, the set of $\varepsilon-$normals to Ω at x is defined by

$$\widehat{N}_\varepsilon(x;\Omega) := \left\{ x^* \in X^* \;\Big|\; \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \leq \varepsilon \right\}.$$

If $\varepsilon = 0$, one writes $\widehat{N}(x;\Omega)$ instead of $\widehat{N}_0(x;\Omega)$ and one calls it *the Fréchet normal cone* or *the prenormal cone* to Ω at x. Elements of $\widehat{N}(x;\Omega)$ are called the Fréchet normals. If $x \notin \Omega$, one puts $\widehat{N}_\varepsilon(x;\Omega) = \emptyset$ for all $\varepsilon \geq 0$. *The basic normal cone* or *the limiting normal cone* (we would call it *the Mordukhovich normal cone*) to Ω at $\bar{x} \in \Omega$ is defined by

$$N(\bar{x};\Omega) := \limsup_{x \to \bar{x},\, \varepsilon \downarrow 0} \widehat{N}_\varepsilon(x;\Omega)$$

(Limsup denotes the sequential Painlevé-Kuratowski upper limit of a sequence of sets); that is $x^* \in N(\bar{x};\Omega)$ iff there exist sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\Omega} \bar{x}$, and $x_k^* \xrightarrow{\text{weakly}^*} x^*$ such that $x_k^* \in \widehat{N}_{\varepsilon_k}(x_k;\Omega)$ for all k. If $\bar{x} \notin \Omega$, one puts $N(\bar{x};\Omega) = \emptyset$. *A crucial fact is that this normal cone stores the most essential information about the structure of* Ω *around* $\bar{x}$ *in the dual code.* Given a multifunction $F : X \rightrightarrows Y$ between Banach spaces and a point $z = (x, y) \in \operatorname{gph} F := \{(x, y) \in X \times Y \,|\, y \in F(x)\}$, one defines *the Fréchet/prenormal coderivative* $\widehat{D}^*F(z) : Y^* \rightrightarrows X^*$ and *the normal coderivative* (we would call it *the Mordukhovich coderivative*) $D_N^*F(z) : Y^* \rightrightarrows X^*$ of F at z, respectively, by the formulas

$$\widehat{D}^*F(z)(y^*) := \{x^* \in X^* \,|\, (x^*, -y^*) \in \widehat{N}(z; \operatorname{gph} F)\} \qquad (\forall y^* \in Y^*)$$

and

$$D_N^*F(z)(y^*) := \{x^* \in X^* \,|\, (x^*, -y^*) \in N(z; \operatorname{gph} F)\} \qquad (\forall y^* \in Y^*).$$

Let $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ be an extended-real-valued function with the epigraph $\operatorname{epi} \varphi := \{(x, \alpha) \in X \times \mathbb{R} \,|\, \alpha \geq \varphi(x)\}$. Let $\bar{x} \in X$ be such that $\varphi(\bar{x}) \in \mathbb{R}$. *The*

Fréchet subdifferential or *the presubdifferential* of φ at $\bar{x}$ is given by

$$\widehat{\partial}\varphi(\bar{x}) := \{x^* \in X^* \,\big|\, (x^*, -1) \in \widehat{N}((\bar{x}, \varphi(\bar{x})); \mathrm{epi}\,\varphi)\}.$$

The basic subdifferential or *the limiting subdifferential* (we would call it *the Mordukhovich subdifferential*) of φ at $\bar{x}$ is given by

$$\partial\varphi(\bar{x}) := \{x^* \in X^* \,\big|\, (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); \mathrm{epi}\,\varphi)\}.$$

From the above definitions it follows that, for any multifunction function and any point in its graph, the normal coderivative (respectively, the Fréchet coderivative) exists uniquely. Similar remarks are valid for subdifferentials. Thus, the main questions here are how to construct basic calculus rules (sum rules, chain rules, etc.), how to obtain key theorems (mean value theorems, open mapping theorems, inverse mapping and implicit mapping theorems, etc.), and how to apply the theory to the many classes problems of interest. A series of fundamental rules and theorems of this kind are given in this chapter in the framework of general Banach spaces. (A variety of principal applications is shown in Chapters 5–8.)

Chapter 2 is devoted to an *extremal principle*, which is a variational nonconvex counterpart of the separation theorem for convex sets. The extremal principle plays a key role in establishing basic calculus rules for the abovementioned dual constructions and in many applications.

Chapter 3 develops a full calculus for normal cones, coderivatives and subdifferentials in the framework of Asplund spaces at the same level of perfection as in finite dimensions. By definition, a Banach space X is an *Asplund space* if every continuous convex function $\varphi : U \to \mathbb{R}$ defined on an open convex set U of X is Fréchet differentiable on a dense subset of U. All reflexive Banach spaces are Asplund spaces.

Chapter 4 shows that the principles and calculus rules developed in the first three chapters can lead to *complete characterizations* of such fundamental properties of multifunctions as openness at a linear rate, metric regularity, Lipschitz-like (pseudo-Lipschitz) continuity.

Chapter 5 derives necessary optimality conditions for the following constrained optimization problems in an infinite-dimensional spaces setting: Mathematical programming problems under geometric and functional constraints, vector optimization problems, mathematical programming problems with equilibrium constraints.

In Chapter 6, the *method of discrete approximations*, due to the author of this book, is applied to optimal control problems and dynamic optimization problems of the Bolza and Mayer types. *The abstract theory here is illustrated and analyzed by as many as 14 valuable examples.*

Chapter 7 focuses on control systems with distributed parameters governed by functional-differential relations and partial differential equations.

Chapter 8 applies the techniques of variational analysis and generalized differentiation to equilibrium models of welfare economics involving nonconvex economies with infinite-dimensional commodity spaces.

The book is well written. Besides the comments given in each section, there are many useful comments at the end of each chapter. The comments provide the interested reader with an opportunity of making interesting excursions to the history of the subject via the large list of 1379 references. As an example, we would like to quote the following comments, which are given at the end of Chapter 1.

1.4.1. Motivations and Early Developments in Nonsmooth Analysis. Nonsmooth phenomena have been known for a long time in mathematics and in applied sciences. To deal with nonsmoothness, various kinds of generalized derivatives were introduced in the classical theory of real functions and in the theory of distributions; see, e.g., Bruckner [182], Saks [1186], Schwartz [1197], and Sobolev [1218]. However, those generalized derivatives, which "ignore sets of density zero," are of little help for optimization theory and variational analysis, where the main interest is in behavior of functions at *individual* points of maxima, minima, equilibria, and other optimization-related notions.

The concepts of generalized differentiability appropriate for applications to optimization were defined in *convex analysis*: first geometrically as the *normal cone* to a convex set that goes back to Minkowski [882], and then – much later – analytically as the *subdifferential* of an extended-real-valued convex function. The latter notion, inspired by the work of Fenchel [441], was explicitly introduced by Moreau [981] and Rockafellar [1140] who emphasized the *set-valuedness* of the new generalized derivative with values in dual spaces and the decisive role of subdifferential *calculus rules*. The central result in this direction, called now the Moreau-Rockafellar theorem on subdifferential sums, is based on the *separation principle* for convex sets around which the whole convex analysis actually revolves.

Convex analysis and separation theorems play a crucial role not only in studying convex sets, functions, and convex optimization problems but also in more general nonconvex settings via *convex approximations*. This idea, largely motivated by applications to optimal control, has been much explored in nonsmooth analysis and optimization starting with the early 1960s. The initial inspiration came from the

Pontryagin maximum principle and its proof given by Boltyanskii; see [124, 1102]. Note that a similar approach to abnormal problems in the calculus of variation was developed by McShane [860] whose work didn't receive a proper attention till the formulation and proof of the maximum principle; compare, e.g., Bliss [119] and Hestenes [565]. Roughly speaking, the underlying idea was to construct, by using special *needle-type* control variations, a *convex tangent cone* approximating the reachable set of system endpoints so that the optimal endpoint lies at its boundary and thus can be separated by a supporting hyperplane. Such a convex approximation approach was strongly developed and applied to new classes of extremal problems by Dubovitskii and Milyutin [369, 370] (see also the book by Girsanov [507]) and then by Gamkrelidze [496, 497], Halkin [539, 541], Hestenes [565], Neustadt [1001, 1002], Ioffe and Tikhomirov [618], and others."

This book is an encyclopedia of the theory of generalized differentiation initiated by B. S. Mordukhovich in the mid-70s. It has quickly become one of the most cited books in variational analysis, set-valued analysis, and optimization theory.

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