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POSITIVE OPERATORS AND INTEGRAL REPRESENTATION

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Abstract. In this paper, we give a new and useful criterion for the positivity of generalized functions and study positive operators on test function space of entire functions on the dual space of a nuclear space with a certain exponential growth condition. This new criterion is used to prove that every positive operator has an integral representation given by positive Borel measure, which can be characterized by integrability conditions. Moreover, this new criterion of positivity can be easily applied to operators such as the identity, the translation, the multiplication, and the convolution operators. This enable us to obtain characterization and integral representation of the associated measure. We also apply the above results to study regularity property of the solution of some quantum stochastic differential equations.

1. Introduction

The main purpose of this paper is to introduce a new and useful criterion of positivity of generalized functions and operators. This enable us to prove an integral representation of such distributions and operators. In this section, we will quickly summarize some known results needed in this paper. In the second section, we reformulate the usual definition of positive generalized functions in two infinite dimensional variables. Then we prove an integral representation of such generalized functions by means of positive Borel measure. This result yields an equivalent natural definition of positive generalized functions. The advantage of this new criterion is that we can recognize positivity of generalized functions by their action on a small class of test functions.

In the third section, we define positive operators in $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\varphi(M')^*)$ in terms of the positivity of their kernels. Then we prove in Theorem 3.2 that every

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positive operator has an integral representation given by a Borel measure. The Borel measures associated with positive generalized functions and positive operators are characterized by integrability conditions of the Fernique type. For related work, see the book [8] and the papers [9, 10].

In the last section, we will give some examples to demonstrate the advantage of our new criterion. In particular, we give a necessary condition for the positivity of the translation operators. Then we study necessary and sufficient conditions for the positivity of the multiplication and convolution operators. We will provide a characterization of the associated measure in terms of the Laplace transform. In particular, we study the identity operator of $\mathcal{F}_\theta(N')$ with our setup and recover the results previously obtained by Obata [12]. Finally, we apply our results to prove the regularity property of solutions of certain linear quantum stochastic differential equations in Equation (4.3).

Now, we assemble a general framework which is necessary for this paper. For detail, see [1-3, 7, 16].

Let N and M be two complex nuclear Fréchet spaces with topologies defined by the families of increasing Hilbertian norms $\{|\cdot|_p;\ p\in\mathbb{N}\}$ and $\{\|\cdot\|_q;\ q\in\mathbb{N}\}$), respectively. For $p,q\in\mathbb{N}$, we denote by N_p (respectively, M_q) the completion of N (respectively, M) with respect to the norm $|\cdot|_p$ (respectively, $\|\cdot\|_q$). Let

$$N = \underset{p\to\infty}{\text{proj-lim}}\, N_p, \quad M = \underset{p\to\infty}{\text{proj-lim}}\, M_q.$$

Denote by N_{-p} (respectively, M_{-q}) the topological dual of the space N_p (respectively, M_q). Denote by $|\cdot|_{-p}$ (respectively, $\|\cdot\|_{-q}$) the associated norms to N_{-p} (respectively, M_{-q}). Then by general duality theory, the strong dual space N' (respectively, M') can be obtained as

$$N' = \underset{p\to\infty}{\text{ind-lim}}\, N_{-p}, \quad M' = \underset{p\to\infty}{\text{ind-lim}}\, M_{-q}.$$

Note that due to the nuclearity of N (respectively, M), the strong and the inductive limit topology of N' (respectively, M') coincide.

Let $M_p\oplus N_p$ be the Hilbert space direct sum. Then the direct sum $M\oplus N$ is given by $M\oplus N = \underset{p\to\infty}{\text{proj-lim}}\, M_p\oplus N_p$. Similarly, we have $(M\oplus N)' = M'\oplus N' = \underset{p\to\infty}{\text{ind-lim}}\, M_{-p}\oplus N_{-p}$.

We recall the definition of Young function which will be used in the sequel

Definition 1.1. A *Young function* is a continuous, convex, and increasing function defined on $\mathbb{R}_+$ and satisfies the two conditions: $\theta(0)=0$ and $\lim_{x\to\infty}\theta(x)/x = +\infty$.

We fix a pair of Young functions (θ, φ) and define the following space of entire functions of two variables:

$$\begin{aligned}&\mathrm{Exp}\big[N_{-p} \oplus M_{-p}, (\theta, \varphi), (m_1, m_2)\big]\\&\quad = \big\{f \in H(N_{-p} \oplus M_{-p})\,;\, \|f\|_{(\theta,\varphi),(m_1,m_2)} < \infty\big\},\end{aligned}$$

where $H(N_{-p} \oplus M_{-p})$ is the space of entire functions on $N_{-p} \oplus M_{-p}$ (see [7] for detail.) Define

$$\|f\|_{(\theta,\varphi),(m_1,m_2)} := \sup_{(z_1,z_2)\in N_{-p}\times M_{-p}} |f(z_1 \oplus z_2)|e^{-\theta(m_1|z_1|_{-p})-\varphi(m_2\|z_2\|_{-p})},$$

where $p \in \mathbb{N}$ and $m_1 > 0, m_2 > 0$. Then

$$\big\{\mathrm{Exp}\big[N_{-p} \oplus M_{-p}, (\theta, \varphi), (m_1, m_2)\big]\,;\, p \in \mathbb{N}, m_1 > 0, m_2 > 0\big\}$$

becomes a projective system of Banach spaces. Put

$$\mathcal{F}_{(\theta,\varphi)}(N' \oplus M') = \operatorname*{proj\text{-}lim}_{p\to\infty;m_1,m_2\downarrow 0} \mathrm{Exp}\big[N_{-p} \oplus M_{-p}, (\theta, \varphi), (m_1, m_2)\big],$$

which is called the space of entire functions on $M' \oplus N'$ with (θ, φ)-exponential growth of minimal type. Similarly,

$$\big\{\mathrm{Exp}\big[N_p \oplus M_p, (\theta, \varphi), (m_1, m_2)\big] \,:\, p \in \mathbb{N}, m_1 > 0, m_2 > 0\big\}$$

becomes an inductive system of Banach spaces. The space of entire functions on $N \oplus M$ with (θ, φ)-exponential growth of finite type is defined by

$$\mathcal{G}_{(\theta,\varphi)}(N \oplus M) = \operatorname*{ind\text{-}lim}_{p\to\infty;m_1,m_2\to\infty} \mathrm{Exp}\big[N_p \oplus M_p, (\theta, \varphi), (m_1, m_2)\big].$$

Denote by $\mathcal{F}_{(\theta,\varphi)}(N'\oplus M')^*$ the strong dual of the test function space $\mathcal{F}_{(\theta,\varphi)}(N'\oplus M')$. For any $(\xi, \eta) \in N \times M$, we consider the exponential function $e_{(\xi,\eta)} : N' \oplus M' \to \mathbb{C}$ defined by

$$e_{(\xi,\eta)}(z_1, z_2) := \exp(\langle z_1, \xi\rangle + \langle z_2, \eta\rangle).$$

Obviously, we have $e_{(\xi,\eta)}(z_1, z_2) = e_{\xi\oplus\eta}(z_1 \oplus z_2) = (e_\xi \otimes e_\eta)(z_1, z_2)$. Moreover, it is easily to see that $e_{(\xi,\eta)} \in \mathcal{F}_{(\theta,\varphi)}(N' \oplus M')$. The Laplace transform of a generalized function $\Phi \in \mathcal{F}_{(\theta,\varphi)}((N \oplus M)')$ is defined by

$$\mathcal{L}(\Phi)(\xi, \eta) = \langle\!\langle \Phi, e_{\xi\oplus\eta}\rangle\!\rangle, \quad \xi \in N\ \eta \in M.$$

We recall from [14] that there is a unique topological isomorphism

$$\mathcal{F}_{(\theta,\varphi)}((N \oplus M)') \cong \mathcal{F}_\theta(N')\widehat{\otimes}\mathcal{F}_\varphi(M'), \tag{1.1}$$

which extends the correspondence $e_{\xi\oplus\eta} \leftrightarrow e_\xi \otimes e_\eta$.

Denote by X (respectively, Y) the real Fréchet nuclear space whose complexification is N (i.e. , $N = X + iX$) (respectively, $M = Y + iY$). Let X' (respectively, Y') be the strong dual of X (respectively, Y). We recall the next theorem from [7].

Theorem 1.2. *Let M and N be complex nuclear Fréchet spaces and let θ and φ be two Young functions. Then the Laplace transform is a topological isomorphism:*

$$\mathcal{L} : \mathcal{F}_{(\theta,\varphi)}(N' \oplus M')^* \to \mathcal{G}_{(\theta^*,\varphi^*)}(N \oplus M),$$

where θ^* and φ^* are the conjugate functions respectively of θ and φ and they are given by

$$\theta^*(x) = \sup_{t\geqslant 0}(tx - \theta(t)), \quad \varphi^*(x) = \sup_{t\geqslant 0}(tx - \varphi(t)), \quad x \geqslant 0.$$

If $N = M$ and $\varphi = \theta$, we write simply $\mathcal{F}_\theta(N' \oplus N') = \mathcal{F}_{(\theta,\theta)}(N' \oplus N')$. Denote by $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta(N')^*)$ the space of all linear continuous operators from $\mathcal{F}_\theta(N')$ into $\mathcal{F}_\theta(N')^*$. From the nuclearity of the space $\mathcal{F}_\theta(N')$, we have by the Schwartz- Grothendieck kernel theorem

$$\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\varphi(M')^*) \cong \mathcal{F}_\theta(N')^* \widehat{\otimes} \mathcal{F}_\varphi(M')^* \cong \mathcal{F}_{(\theta,\varphi)}(N' \oplus M')^*. \tag{1.2}$$

Since the kernel Ξ^K of an operator $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\varphi(M')^*)$ is an element of $\mathcal{F}_{(\theta,\varphi)}(N' \oplus M')^*$, the symbol $\widehat{\Xi}$ of Ξ is by definition the Laplace transform of Ξ^K. Hence we have the following relationship:

$$\widehat{\Xi}(\xi \oplus \eta) = \langle\langle \Xi^K, e_\xi \otimes e_\eta \rangle\rangle = \mathcal{L}(\Xi^K)(\xi \oplus \eta), \quad \xi \in N \ \eta \in M. \tag{1.3}$$

We can then use Theorem 1.2 to get the following corollary:

Corollary 1.3. *A function $\Theta : N \otimes N \to \mathbb{C}$ is the symbol of some $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta(N')^*)$ if and only if $\Theta \in \mathcal{G}_{\theta^*}(N \oplus N)$.*

We will assume that the Young functions θ and φ satisfy the additional growth conditions:

$$\lim_{x\mapsto\infty} \frac{\theta(x)}{x^2} < \infty, \quad \lim_{x\mapsto\infty} \frac{\varphi(x)}{x^2} < \infty, \tag{1.4}$$

then we have the following Gel'fand triples (see [3, 14] for detail):

$$\mathcal{F}_\theta(N') \ \subset \ L^2(X', \gamma) \ \subset \ \mathcal{F}_\theta(N')^* \tag{1.4}$$

$$\mathcal{F}_{\theta,\varphi}(N' \oplus M') \ \subset \ L^2(X' \times Y', \gamma_1 \otimes \gamma_2) \ \subset \ \mathcal{F}_{\theta,\varphi}(N' \oplus M')^*, \tag{1.5}$$

where $\gamma_i,\ i \in \{1, 2\}$, are respectively the standard Gaussian measures on the strong dual spaces of X and Y (see [5]).

We remark that if $M = \{0\}$, then $\mathcal{F}_{\theta,\varphi}(N' \oplus M') = \mathcal{F}_\theta(N')$. Therefore, the results we obtain for the space $\mathcal{F}_{\theta,\varphi}(N' \oplus M')$ of two variable test functions are also valid for $\mathcal{F}_\theta(N')$.

2. Positive Generalized Functions

In this section, we will use the involution defined in Equation (2.1) to obtain a new criterion for positive generalized function in $\mathcal{F}_{(\theta,\varphi)}(N' \oplus M')^*$. Then we give an integral representation of such generalized functions. First recall from [14] that $\mathcal{F}_{(\theta,\varphi)}(N' \oplus M')^*$ is a nuclear algebra with the involution $*$ defined for any $f \in \mathcal{F}_{(\theta,\varphi)}(N' \oplus M')^*$ by

$$f^*(z, w) := \overline{f(\bar{z}, \bar{w})}, \quad z \in N' \ w \in M'. \tag{2.1}$$

Note that the isomorphism in Equation (1.1) implies that $f = f_1 \otimes f_2 \in \mathcal{F}_{(\theta,\varphi)}(N' \oplus M')$ for any $f_1 \in \mathcal{F}_\theta(N')$ and $f_2 \in \mathcal{F}_\varphi(M')$. Let $\mathcal{A}$ denote the subset of $\mathcal{F}_{(\theta,\varphi)}(N' \oplus M')$ given by

$$\mathcal{A} = \left\{ g = ff^*\,,\ f = f_1 \otimes f_2,\ f_1 \in \mathcal{F}_\theta(N')\ f_2 \in \mathcal{F}_\varphi(M') \right\}.$$

Definition 2.1. A generalized function $\Phi \in \mathcal{F}_{(\theta,\varphi)}(N' \oplus M')^*$ is said to be $\mathcal{A}$-positive if $\langle\langle \Phi, g \rangle\rangle \geqslant 0$ for any $g \in \mathcal{A}$.

Recall from ([14, 16, 17, 19]) that a generalized function $\Phi \in \mathcal{F}_{(\theta,\varphi)}(N' \oplus M')^*$ is positive in the usual sense, if it satisfies the following condition

$$\langle\langle \Phi, f \rangle\rangle \geqslant 0, \quad \forall\, f \in \mathcal{F}_{(\theta,\varphi)}(N' \oplus M')_+.$$

where $\mathcal{F}_{(\theta,\varphi)}(N' \oplus M')_+$ denotes the following set

$$\left\{ f \in \mathcal{F}_{\theta,\varphi}(N' \oplus M')\,,\ f((x + i0) \oplus (y + i0)) \geqslant 0,\, \forall (x, y) \in X' \times Y' \right\}.$$

We also recall the notation $\mathcal{F}_{(\theta,\varphi)}(N' \oplus M')^*_+$ for the set of usual positive generalized functions. Denote by $\mathcal{F}_{(\theta,\varphi)}(N' \oplus M')^*_{+,\mathcal{A}}$ the set of $\mathcal{A}$-positive generalized functions given by Definition 2.1.

Lemma 2.2. *Any positive generalized function* $\Phi \in \mathcal{F}_{(\theta,\varphi)}(N' \oplus M')^*$ *is also* $\mathcal{A}$*-positive.*

Proof. Let $\Phi \in \mathcal{F}_{(\theta,\varphi)}(N' \oplus M')^*_+$ and consider $f = f_1 \otimes f_2$ with $f_1 \in \mathcal{F}_\theta(N')$ and $f_2 \in \mathcal{F}_\varphi(M')$. Then we have $\langle\langle \Phi, ff^* \rangle\rangle \geqslant 0$ because

$$(ff^*)((x + i0) \oplus (y + i0)) = |f((x + i0) \oplus (y + i0))|^2 \geqslant 0$$

for all $(x, y) \in X' \times Y'$. This proves the lemma. ∎

Theorem 2.3. *For any $\mathcal{A}$-positive generalized function $\Phi \in \mathcal{F}_{(\theta,\varphi)}(N' \oplus M')^*$, there exist a unique Borel measure μ_Φ on $X' \oplus Y'$, positive on nonempty open sets, such that for all $f \in \mathcal{F}_{(\theta,\varphi)}(N' \oplus M')$,*

$$\langle\langle \Phi, f \rangle\rangle = \int_{X' \oplus Y'} f((x + i0) \oplus (y + i0))\, d\mu_\Phi(x \oplus y). \tag{2.2}$$

Moreover, the Fourier transform of μ_Φ is given by

$$\mathcal{F}_{\mu_\Phi}(\xi \oplus \eta) = \langle\langle \Phi, e_{i(\xi \oplus \eta)} \rangle\rangle, \quad (\xi, \eta) \in X \times Y.$$

Remark 2.4. Such a Borel measure μ_Φ is often called a *Hida measure* in white noise theory, see the book [8].

Proof. Let $\Phi \in \mathcal{F}_{(\theta,\varphi)}(N' \oplus M')^*_{+,\mathcal{A}}$. The Fourier transform C_Φ of Φ is defined for each $(\xi, \eta) \in X \times Y$ by

$$C_\Phi(\xi, \eta) := \langle\langle \Phi, e_{i(\xi \oplus \eta)} \rangle\rangle = (\mathcal{L}\Phi)(i\xi \oplus i\eta).$$

By Theorem 1.2, C_Φ is a continuous function and $C_\Phi(0, 0)$ is a finite number. Moreover, C_Φ is positive definite. In fact, for any $\alpha_j \in \mathbb{C}$, $\xi_j \in X$, $\eta_j \in Y$, $1 \leq j \leq n$, we have

$$\begin{aligned}
&\sum_{1 \leq i,j \leq n} \alpha_i \overline{\alpha_j} C_\Phi((\xi_i, \eta_i) - (\xi_j, \eta_j)) \\
&\quad = \sum_{1 \leq i,j \leq n} \alpha_i \overline{\alpha_j} \langle\langle \Phi, e_{i(\xi_i + \eta_i)} e_{-i(\xi_j + \eta_j)} \rangle\rangle \\
&\quad = \left\langle\left\langle \Phi, \Big(\sum_{1 \leq i \leq n} \alpha_i e_{i(\xi_i + \eta_i)} \Big) \Big(\sum_{1 \leq j \leq n} \alpha_j e_{i(\xi_j + \eta_j)} \Big)^* \right\rangle\right\rangle \\
&\quad \geqslant 0.
\end{aligned}$$

Thus by the Bochner-Minlos theorem (see [5, 8, 11]), there exist a unique positive Borel measure μ_Φ on $X' \oplus Y'$ such that for all $(\xi, \eta) \in X \times Y$,

$$\langle\langle \Phi, e_{i(\xi \oplus \eta)} \rangle\rangle = \int_{X' \oplus Y'} e^{i(\langle x, \xi \rangle + \langle y, \eta \rangle)}\, d\mu_\Phi(x \oplus y). \tag{2.3}$$

Next we need to extend Equation (2.3) to $f \in \mathcal{F}_{(\theta,\varphi)}(N' \oplus M')$. It is clear that Equation (2.3) is satisfied on the linear span of the exponential functions $\{e_{i(\xi,\eta)};\ (\xi, \eta) \in$

$X \times Y\}$, which is dense in $\mathcal{F}_{(\theta,\varphi)}(N' \oplus M')$. Let $f \in \mathcal{F}_{(\theta,\varphi)}(N' \oplus M')$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{E}$ converging to f in the topology of $\mathcal{F}_{(\theta,\varphi)}(N' \oplus M')$. Then

$$\langle\langle \Phi, (f_n - f_m)(f_n - f_m)^* \rangle\rangle = \int_{X'\oplus Y'} |(f_n - f_m)(x \oplus y)|^2 \, d\mu_\Phi(x \oplus y). \tag{2.4}$$

Since $(f_n)_{n\in\mathbb{N}}$ converges to f, it is a Cauchy sequence in $\mathcal{E}$. Then by the continuity of Φ and the multiplication operator on $\mathcal{F}_{(\theta,\varphi)}(N' \oplus M')$, we can take the limit in Equation (2.4) to show that the sequence $(f_n)_{n\in\mathbb{N}}$ is also Cauchy in $L^2(X' \oplus Y', \mu_\Phi)$. Let l denote the limit of $(f_n)_{n\in\mathbb{N}}$ with respect the norm in $L^2(X' \oplus Y', \mu_\Phi)$. Because the topology of $\mathcal{F}_{(\theta,\varphi)}(N' \oplus M')$ is finer than the topology of $L^2(X' \oplus Y', \mu_\Phi)$, we conclude that $l = f$, μ_Φ-a.e. Finally, we use the Lebesgue dominated convergence theorem and the fact that μ_Φ is supported by $X_{-p} \oplus Y_{-p}$ for some $p > 0$ to conclude that

$$\begin{aligned}
\langle\langle \Phi, f \rangle\rangle &= \lim_{n\mapsto\infty} \langle\langle \Phi, f_n \rangle\rangle \\
&= \lim_{n\mapsto\infty} \int_{X'\oplus Y'} f_n(x \oplus y) \, d\mu_\Phi(x \oplus y) \\
&= \int_{X'\oplus Y'} l(x \oplus y) d\mu_\Phi(x \oplus y) \\
&= \int_{X'\oplus Y'} f(x \oplus y) \, d\mu_\Phi(x \oplus y).
\end{aligned}$$

Thus the proof is complete. ∎

Recall that by a positive Borel measure, we mean a Borel measure which is positive on nonempty open sets.

Corollary 2.5. *Every positive generalized function* Φ *has an integral representation given by Equation* (2.2). *Moreover, we have*

$$\mathcal{F}_{\theta,\varphi}(N' \oplus M')^*_+ = \mathcal{F}_{\theta,\varphi}(N' \oplus M')^*_{+,\mathcal{A}}. \tag{2.5}$$

Proof. The first part of the corollary is a direct consequence of Lemma 2.2 and Theorem 2.3. To prove the equality (2.5), it is sufficient to prove the following inclusion in view of Lemma 2.2

$$\mathcal{F}_{\theta,\varphi}(N' \oplus M')^*_{+,\mathcal{A}} \subseteq \mathcal{F}_{\theta,\varphi}(N' \oplus M')^*_+.$$

Let $\Phi \in \mathcal{F}_{\theta,\varphi}(N' \oplus M')^*_{+,\mathcal{A}}$. Then by Theorem 2.3, Φ has an integral representation given by

$$\langle\langle \Phi, f \rangle\rangle = \int_{X'\oplus Y'} f((x + i0) \oplus (y + i0)) \, d\mu_\phi(x \oplus y), \quad f \in \mathcal{F}_{(\theta,\varphi)}(N' \oplus M').$$

Let $f \in \mathcal{F}_{(\theta,\varphi)}(N' \oplus M')_+$. Then by definition, $f((x+i0)\oplus(y+i0)) \geqslant 0$ for all $x \in X'$, $y \in Y'$. Since μ_Φ is a positive Borel measure, we see that

$$\langle\langle \Phi, f\rangle\rangle = \int_{X'\oplus Y'} f((x+i0)\oplus(y+i0))\, d\mu_\phi(x\oplus y) \geqslant 0.$$

Hence $\Phi \in \mathcal{F}_{\theta,\varphi}(N' \oplus M')^*_+$. ■

We point out that Corollary 2.5 gives a new criterion of positive generalized functions. This new criterion leads in a natural way to the standard definition of positive generalized functions. Moreover, it enables us to recognize positive generalized functions by their action on a smaller set of test functions. In fact, let $\mathcal{B}$ be the subset of $\mathcal{F}_{\theta,\varphi}(N' \oplus M')$ defined by

$$\mathcal{B} = \big\{ f f^* \,;\, f \in \mathcal{F}_{\theta,\varphi}(N' \oplus M')\big\}.$$

Then we have

$$\mathcal{A} \;\subseteq\; \mathcal{B} \;\subseteq\; \mathcal{F}_{\theta,\varphi}(N' \oplus M')_+. \tag{2.6}$$

For the next corollary, we introduce the notation

$$\mathcal{F}_{\theta,\varphi}(N' \oplus M')^*_{+,\mathcal{B}} = \big\{ \phi \in \mathcal{F}_{\theta,\varphi}(N' \oplus M')^* \,;\, \langle\langle \phi, h\rangle\rangle \geqslant 0,\ \forall h \in \mathcal{B}\big\}.$$

Corollary 2.6. *The set of $\mathcal{B}$-positive generalized functions coincides with the set of positive generalized functions, i.e.,*

$$\mathcal{F}_{\theta,\varphi}(N' \oplus M')^*_+ = \mathcal{F}_{\theta,\varphi}(N' \oplus M')^*_{+,\mathcal{B}}.$$

Proof. Equation (2.6) yields the following inclusions:

$$\mathcal{F}_{\theta,\varphi}(N' \oplus M')^*_+ \subseteq \mathcal{F}_{\theta,\varphi}(N' \oplus M')^*_{+,\mathcal{B}} \subseteq \mathcal{F}_{\theta,\varphi}(N' \oplus M')^*_{+,\mathcal{A}}.$$

Then this corollary follows from Equation (2.5). ■

By using Equation (1.4), we can obtain the following triple:

$$\mathcal{F}_{\theta,\varphi}(N' \oplus M') \;\subset\; L^2(X' \times Y', \gamma_1 \otimes \gamma_2) \;\subset\; \mathcal{F}_{\theta,\varphi}(N' \oplus M')^*, \tag{2.7}$$

which implies that every $\Phi \in \mathcal{F}_{\theta,\varphi}(N' \oplus M')^*$ can be interpreted as a Gaussian distribution. Let $\Phi \in \mathcal{F}_{\theta,\varphi}(N' \oplus M')^*_+$ and let μ_Φ the associated measure given by Equation (2.2). Then Φ can be interpreted as the generalized Radon Nikodym derivative of the measure μ_Φ with respect the standard Gaussian measure $\gamma_1 \otimes \gamma_2$ and we can use the notation

$$\Phi = \frac{d\mu_\Phi}{d(\gamma_1 \otimes \gamma_2)}. \tag{2.8}$$

Next we will give a characterization of those Borel measure specified in Theorem 2.3.

Theorem 2.7. *Let μ a finite measure on $X' \oplus Y'$ equipped with the Borel σ-algebra $\mathcal{B}(X' \oplus Y')$. Then μ represents a positive generalized function $\Phi \in \mathcal{F}_{\theta,\varphi}(N' \oplus M')^*_+$ if and only if it satisfies the two following conditions:*

(1) *There exist $q > 0$ such that μ is supported by $X_{-q} \oplus Y_{-q}$.*

(2) *There exists $m_1, m_2 > 0$ such that*

$$\int_{X_{-q}\oplus Y_{-q}} e^{\theta(m_1|x|_{-q})+\varphi(m_2\|y\|_{-q})}\, d\mu(x \oplus y) < \infty. \tag{2.8}$$

To prove this theorem, we need the following two lemmas, which can be proved by similar arguments as those for one variable case in [16].

Lemma 2.8. *Let μ be a measure which represents a positive generalized function Φ. Then there exist $m'_1 > 0$, $m'_2 > 0$ and $p, q \in \mathbb{N}$ satisfying $q > p$ such that for any $(\xi, \eta) \in X_q \times Y_q$ and for any $n, l \in \mathbb{N}$, we have:*

$$\begin{aligned} &\int_{X_{-q}\oplus Y_{-q}} \langle x^{\otimes n}, \xi^{\otimes n}\rangle^2 \langle y^{\otimes l}, \eta^{\otimes l}\rangle^2\, d\mu(x \oplus y) \\ &\le \| \mathcal{L}(\Phi) \|_{\theta^*,\varphi^*,-p,-p,m'_1,m'_2} (2n)!(2l)! m'^{2n}_1 m'^{2l}_2 \theta^*_{2n} \varphi^*_{2l} |\xi|^{2n}_p \|\eta\|^{2l}_p. \end{aligned} \tag{2.10}$$

Lemma 2.9. *Let μ be a measure which represents a positive generalized function Φ. Then there exist $m'_1 > 0$, $m'_2 > 0$ and $p, q \in \mathbb{N}$ satisfying $q > p$ such that for any $n, l \in \mathbb{N}$, we have:*

$$\begin{aligned} &\textstyle\int_{X_{-q}\oplus Y_{-q}} |x|^n_{-q} \|y\|^l_{-q}\, d\mu(x \oplus y) \\ &\le C \left(\sqrt{e} m'^2_1 \| i_{p,q} \|_{\mathrm{HS}}\right)^n \left(\sqrt{e} m'^2_2 \| i_{p,q} \|_{\mathrm{HS}}\right)^l, \end{aligned} \tag{2.11}$$

where $C = \left\{\|\mathcal{L}(\Phi)\|_{\theta^*,\varphi^*,-p,-p,m_1,m_2}(2n)!(2l!)\theta^*_{2n}\varphi^*_{2l}\right\}^{1/2}$ and $\| i_{p,q} \|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm of the injection

$$i_{p,q} : N_q \oplus M_q \to N_p \oplus M_p.$$

Proof of Theorem 2.7. To prove the sufficiency part, let μ be supported by $X_{-q} \oplus Y_{-q}$ for some $q > 0$ and there exist $m_1, m_2 > 0$ such that

$$\int_{X_{-q}\oplus Y_{-q}} e^{\theta(m_1|x|_{-q})+\varphi(m_2\|y\|_{-q})}\, d\mu(x\oplus y) < \infty.$$

Then the linear functional Φ_μ defined by

$$\langle\langle \Phi_\mu, f\rangle\rangle := \int_{X'\oplus Y'} f((x+i0)\oplus(y+i0))\, d\mu(x\oplus y)$$

is continuous on $\mathcal{F}_{\theta,\varphi}(N'\oplus M')$. Hence Φ_μ is positive generalized function.

Conversely, suppose that μ satisfies

$$\langle\langle \Phi, f\rangle\rangle = \int_{X'\oplus Y'} f((x+i0)\oplus(y+i0))\, d\mu_\Phi(x\oplus y)$$

for a generalized function $\Phi \in \mathcal{F}_{\theta,\varphi}(N'\oplus M')^*_+$. Since there exists $p \in \mathbb{N}$ such that

$$C_\Phi(\xi,\eta) = \langle\langle \Phi, e_{i(\xi,\eta)}\rangle\rangle \in \mathrm{Exp}\big[N_p\oplus M_p, \theta^*, \varphi^*, m'_1, m'_2\big]. \tag{2.12}$$

The characteristic function C_Φ is continuous with respect to the family of norms $\{\|\cdot\|_{(\theta,\varphi),(m_1,m_2)}\}$. Then the Bochner-Minlos theorem which asserts the existence of μ, also ensures the existence of $q > p$ such that μ is supported by $X_{-q}\oplus Y_{-q}$. Moreover, the injection

$$i_{p,q} : N_q\oplus M_q \to N_p\oplus M_p$$

is a Hilbert-Schmidt operator. For $n \in \mathbb{N}$, let

$$\theta^*_n = \inf_{r>0}\frac{e^{\theta^*(r)}}{r^n}, \quad \varphi^*_n = \inf_{r>0}\frac{e^{\varphi^*(r)}}{r^n}.$$

To complete the proof, it is enough to find $m_1 > 0$, $m_2 > 0$ such that

$$\int_{X_{-q}\oplus Y_{-q}} e^{\theta(m_1|x|_{-q})+\varphi(m_2\|y\|_{-q})}\, d\mu(x\oplus y) < \infty.$$

Since the Young functions θ and φ are convex, we have $(\theta^*)^* = \theta$ and $(\varphi^*)^* = \varphi$. So it is enough to find $m_1 > 0$, $m_2 > 0$ such that

$$\int_{X_{-q}\oplus Y_{-q}} \sup_{t\geq 0}\left\{e^{tm_1|x|_{-q}-\theta^*(t)}\right\} \sup_{t'\geq 0}\left\{e^{t'm_2\|y\|_{-q}-\varphi^*(t')}\right\} d\mu(x\oplus y) < \infty.$$

In fact, for any $m_1 > 0$, $m_2 > 0$, we have

$$e^{tm_1|x|_{-q}-\theta^*(t)} = e^{-\theta^*(t)} \sum_{n\geq 0} \frac{(tm_1)^n}{n!} |x|_{-q}^n,$$
$$e^{t'm_2\|y\|_{-q}-\varphi^*(t')} = e^{-\varphi^*(t)} \sum_{l\geq 0} \frac{(tm_2)^l}{l!} \|y\|_{-q}^l.$$

Note that $t^n e^{-\theta^*(t)}\theta_n^* \leqslant 1$ for $t > 0$ and $n \in \mathbb{N}$, and $t^l e^{-\varphi^*(t)}\varphi_l^* \leqslant 1$ for $t > 0$ and $l \in \mathbb{N}$. Therefore,

$$\sup_{t\geq 0} \left\{ e^{tm_1|x|_{-q}-\theta^*(t)} \right\} \leq \sum_{n\in\mathbb{N}} \frac{(m_1|x|_{-q})^n}{n!\theta_n^*},$$
$$\sup_{t'\geq 0} \left\{ e^{t'm_2\|y\|_{-q}-\varphi^*(t')} \right\} \leq \sum_{l\in\mathbb{N}} \frac{(m_2\|y\|_{-q})^l}{l!\varphi_l^*}.$$

By Lemma 2.9 and the inequalities $\theta_{2n}^* \leqslant 2^n\theta_n^*$ and $\varphi_{2l}^* \leqslant 2^l\varphi_l^*$ for all $n, l \in \mathbb{N}$, we obtain

$$\int_{X_{-q}\oplus Y_{-q}} \sup_{t\geq 0} \left\{ e^{tm_1|x|_{-q}-\theta^*(t)} \right\} \sup_{t'\geq 0} \left\{ e^{t'm_2\|y\|_{-q}-\varphi^*(t')} \right\} d\mu(x\oplus y)$$
$$\leq L \sum_{n,l\in\mathbb{N}} \frac{\sqrt{(2n)!(2l)!}}{n!l!} \tag{2.13}$$
$$\left(2\sqrt{e}m_1 m'_1 \parallel i_{q,p} \parallel_{\mathrm{HS}}\right)^n \left(2\sqrt{e}m_2 m'_2 \parallel i_{q,p} \parallel_{\mathrm{HS}}\right)^l,$$

where $L = \{\parallel \mathcal{L}(\Phi) \parallel_{\theta^*,\varphi^*,-p,-p,m'_1,m'_2}\}^{1/2}$. Since $\frac{\sqrt{(2n)!(2l)!}}{n!l!} \simeq \frac{2^n 2^l}{(\pi n)^{\frac{1}{4}}(\pi l)^{\frac{1}{4}}}$ as $n,\, l \to \infty$, it follows for $m_1 > 0, m_2 > 0$ such that $4m'_1\sqrt{e}m_1 \parallel i_{q,p} \parallel_{\mathrm{HS}} < 1$ and $4m'_2\sqrt{e}m_2 \parallel i_{q,p} \parallel_{\mathrm{HS}} < 1$, the series in Equation (2.13) converges to K. Hence we get

$$\int_{X_{-q}\oplus Y_{-q}} \sup_{t\geq 0}\{e^{tm_1|x|_{-q}-\theta^*(t)}\}\{\sup_{t'\geq 0} e^{t'm_2\|y\|_{-q}-\varphi^*(t')}\}\, d\mu(x\oplus y)$$
$$\leq K\sqrt{\parallel \mathcal{L}(\Phi) \parallel_{\theta^*,\varphi^*,-p,-p,m'_1,m'_2}}.$$

This completes the proof of the theorem. ■

3. Positive Operators

We now use the results in the previous section to study positive operators in $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\varphi(M')^*)$ and give an integral representation for such operators. The case $N = M$ and $\varphi = \theta$ corresponds to the white noise operators studied in [7]. For every $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\varphi(M')^*)$, the associated kernel, denoted by $\Xi^K \in (\mathcal{F}_\theta(N')\widehat{\otimes}\mathcal{F}_\varphi(M'))^*$, satisfies the following equality:

$$\langle\langle \Xi f, g\rangle\rangle = \langle\langle \Xi^K, f \otimes g\rangle\rangle, \quad f \in \mathcal{F}_\theta(N')\ g \in \mathcal{F}_\varphi(M'). \tag{3.1}$$

The symbol of $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\varphi(M')^*)$ is defined by:

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi^K, e_\xi \otimes e_\eta\rangle\rangle = \mathcal{L}(\Xi^K)(\xi, \eta), \quad \xi \in N\ \eta \in M.$$

Definition 3.1. An operator $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\varphi(M')^*)$ is positive if its kernel Ξ^K is an element of $\mathcal{F}_{\theta,\varphi}(N' \oplus M')^*_+$.

Theorem 3.2. *For any positive operator $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\varphi(M')^*)$, there exists a unique positive Borel measure μ_Ξ on $X' \oplus Y'$ such that for all $f \in \mathcal{F}_{\theta,\varphi}(N' \oplus M')$,*

$$\langle\langle \Xi^K, f\rangle\rangle = \int_{X'\oplus Y'} f((x + i0) \oplus (y + i0))\, d\mu_\Xi(x \oplus y).$$

The Laplace transform of μ_Ξ is given by

$$\mathcal{L}(\mu_\Xi)(\xi, \eta) = \langle\langle \Xi^K, e_\xi \otimes e_\eta\rangle\rangle = \widehat{\Xi}(\xi, \eta), \quad \xi \in N,\ \eta \in M.$$

Moreover, μ_Ξ is characterized by the following integrability conditions:

(1) *There exists $q > 0$ such that μ_Ξ is supported by $X_{-q} \oplus Y_{-q}$.*

(2) *There exist $m_1, m_2 > 0$ such that*

$$\int_{X_{-q}\oplus Y_{-q}} e^{\theta(m_1|x|_{-q})+\varphi(m_2\|y\|_{-q})}\, d\mu_\Xi(x \oplus y) < \infty. \tag{3.2}$$

Since $\Xi^K \in \mathcal{F}_{\theta,\varphi}(N' \oplus M')^*_+$, we can apply Theorem 2.3 to get a unique positive Borel measure μ_Ξ on $X' \oplus Y'$ such that

$$\langle\langle \Xi^K, f\rangle\rangle = \int_{X'\oplus Y'} f((x + i0) \oplus (y + i0))\, d\mu_\Xi(x \oplus y)$$

for all $f \in \mathcal{F}_{\theta,\varphi}(N' \oplus M')$. Thus the characterization of μ_Ξ is a consequence of Theorem 2.7. ∎

The next corollary can be easily deduced from Theorem 3.2 and Equations (2.8) and (3.1).

Corollary 3.3. *Let $\Xi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\varphi(M')^*)$ be a positive operator. Then for any $f \in \mathcal{F}_\theta(N')$, $g \in \mathcal{F}_\varphi(M')$, we have*

$$\langle \Xi f, g\rangle = \langle \Xi^K, f \otimes g\rangle = \int_{X' \oplus Y'} f(x + i0)g(y + i0)\, d\mu_\Xi(x \oplus y).$$

Moreover, the Gaussian distribution Ξ^K can be interpreted as a generalized Radon Nikodym derivative of the measure μ_Ξ with respect the standard Gaussian measure $\gamma_1 \otimes \gamma_2$, namely,

$$\frac{d\mu_\Xi}{d(\gamma_1 \otimes \gamma_2)} = \Xi^K. \tag{3.3}$$

4. Examples and Applications

We will give several examples to show the advantage of our new criterion of the positivity of generalized functions. At the same time, we will study the positivity of some operators. We need to point out that in some examples, the available theorems in the literature cannot be easily applied because the explicit expression of kernels might be hard to derive.

4.1 Multiplication operators

Let $\Phi \in \mathcal{F}_\theta(N')^*$ and let $M_\Phi \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta(N')^*)$ be the multiplication operator by Φ defined in [2] by

$$\langle\langle M_\Phi f, h\rangle\rangle = \langle\langle \Phi, fh\rangle\rangle, \quad f,\, h \in \mathcal{F}_\theta(N').$$

Moreover, the multiplication operator M_g by test function $g \in \mathcal{F}_\theta(N')$ is defined by $M_g(f) = gf,\ f \in \mathcal{F}_\theta(N')$. Using the Gel'fand triple (1.5), we can consider the operator M_g as an element of $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta(N')^*)$.

Proposition 4.1. (1) *The multiplication operator M_Φ is positive if and only if $\Phi \in (\mathcal{F}_\theta(N'))^*_+$. In this case, M_Φ has an integral representation given by Theorem 3.2 and the Laplace transform of the associated measure μ_{M_Φ} is given by*

$$\mathcal{L}(\mu_{M_\Phi})(\xi, \eta) = \mathcal{L}(\Phi)(\xi, \eta), \quad \xi, \eta \in N.$$

(2) *The multiplication operator M_g by a test function g is positive if and only if $g \in \mathcal{F}_\theta(N')_+$. In this case, M_g has an integral representation given by Theorem 3.2 and the Laplace transform of the associated measure μ_{M_g} is given by*

$$\langle\langle M_g^K, e_\xi \otimes e_\eta\rangle\rangle = \mathcal{L}(\mu_{M_g})(\xi, \eta).$$

In particular, for every $\tau \in X$ *and* $g = e_\tau \in \mathcal{F}_\theta(N')_+$, *we have*

$$\mathcal{L}(\mu_{M_g})(\xi, \eta) = e^{\langle \tau + \xi, \eta \rangle}, \quad \xi, \eta \in N.$$

Proof. (1) For any $f = f_1 \otimes f_2 \in \mathcal{F}_\theta(N' \oplus N')$, the kernel M_Φ^K satisfies

$$\begin{aligned}\langle\langle M_\Phi^K, ff^* \rangle\rangle &= M_\Phi^K(f_1 f_1^* \otimes f_2 f_2^*) = \langle\langle M_\Phi(f_1 f_1^*), f_2 f_2^* \rangle\rangle \\ &= \langle\langle \Phi, f_1 f_1^* f_2 f_2^* \rangle\rangle = \langle\langle \Phi, (f_1 f_2)(f_1 f_2)^* \rangle\rangle.\end{aligned}$$

It follows that $\langle\langle M_\Phi^K, ff^* \rangle\rangle \geqslant 0$ if and only if $\Phi \in (\mathcal{F}_\theta(N'))_+^*$. In this case, by Theorem 3.2, we get

$$\begin{aligned}\mathcal{L}(\mu_{M_\Phi})(\xi, \eta) &= \langle\langle M_\Phi^K, e_\xi \otimes e_\eta \rangle\rangle = \langle\langle M_\Phi e_\xi, e_\eta \rangle\rangle \\ &= \langle\langle \Phi, e_\xi \otimes e_\eta \rangle\rangle = \mathcal{L}(\Phi)(\xi, \eta), \quad \xi, \eta \in N.\end{aligned}$$

(2) For any $f = f_1 \otimes f_2 \in \mathcal{F}_\theta(N') \otimes \mathcal{F}_\theta(N')$, the kernel M_g^K of M_g satisfies

$$\begin{aligned}\langle\langle M_g^K, ff^* \rangle\rangle &= \langle\langle M_g(f_1 f_1^*), f_2 f_2^* \rangle\rangle = \langle g f_1 f_1^*, f_2 f_2^* \rangle \\ &= \int_{X'} g(x + i0) |f_1(x + i0)|^2 |f_2(x + i0)|^2 \, d\gamma(x) \geqslant 0.\end{aligned}$$

This proves the assertion (2). ■

Corollary 4.2. *The identity map* I *of* $\mathcal{F}_\theta(N')$ *is a positive element of the space* $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta(N')^*)$ *and is associated with a measure* μ_I *such that*

$$\langle\langle I^K, f \rangle\rangle = \int_{X' \oplus X'} f((x + i0) \oplus (y + i0)) \, d\mu_I(x \oplus y), \quad f \in \mathcal{F}_\theta(N' \oplus N').$$

The Laplace transform of the measure μ_I *is given by*

$$\mathcal{L}(\mu_I)(\xi, \eta) = \langle\langle e_\xi, e_\eta \rangle\rangle = e^{\langle \xi, \eta \rangle}, \quad \xi, \eta \in N. \tag{4.1}$$

Proof. Note that $I = M_{e_0}$ and $e_0 \in \mathcal{F}_\theta(N')_+$. Hence by Proposition 4.1, I is a positive operator and we get the assertion. ■

Obata [12] has used a different method to prove that for any ξ, $\eta \in N$,

$$\widehat{I}(\xi, \eta) = \int_{N'} e^{\langle \bar{z}, \xi \rangle + \langle z, \eta \rangle} \nu'(dz) = e^{\langle \xi, \eta \rangle} = \mathcal{L}(\nu')(\xi, \eta) = \mathcal{L}(\mu_I)(\xi, \eta),$$

where $\nu' = \mu_{\frac{1}{2}} \times \mu_{\frac{1}{2}}$, and $\mu_{\frac{1}{2}}$ is the Gaussian measure on X' with variance $\frac{1}{2}$. Upon comparing this equation with Equation (4.1) and using the isomorphism property of the Laplace transform in Theorem 1.2, we immediately conclude that $\mu_I = \nu'$.

4.2 Translation operators

Let $z \in N'$, the translation operator τ_{-z} is an element of $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta(N')^*)$ defined for any $f \in \mathcal{F}_\theta(N')$ by

$$(\tau_{-z}f)(\lambda) = f(z+\lambda), \quad \lambda \in N'.$$

Proposition 4.3. *For any $z \in X'$, the translation operator τ_{-z} is a positive operator of $\mathcal{F}_\theta(N')$. In this case, τ_{-z} has an integral representation given by Theorem 3.2 and the Laplace transform of the associated measure $\mu_{\tau_{-z}}$ is*

$$\mathcal{L}(\mu_{\tau_{-z}})(\xi,\eta) \;=\; e^{\langle \xi, z+\eta\rangle}, \quad \xi,\eta \in N.$$

Proof. Let $f = f_1 \otimes f_2 \in \mathcal{F}_\theta(N' \oplus N')$. Then

$$\begin{aligned} \langle\langle \tau_{-z}^K, ff^*\rangle\rangle &= \langle\langle \tau_{-z}^K, f_1f_1^* \otimes f_2f_2^*\rangle\rangle = \langle \tau_{-z}(f_1f_1^*), f_2f_2^*\rangle \\ &= \int_{X'} (f_1f_1^*)(x+z)|f_2(x)|^2 d\gamma(x) \\ &= \int_{X'} |f_1(x+z)|^2 |f_2(x)|^2\, d\gamma(x) \geqslant 0. \end{aligned}$$

Hence the kernel τ_{-z}^K of τ_{-z} is positive. Then we apply Theorem 3.2 to get a unique positive measure such that

$$\begin{aligned} \mathcal{L}(\mu_{\tau_{-z}})(\xi,\eta) &= \langle\langle \tau_{-z}^K, e_\xi \otimes e_\eta\rangle\rangle = \langle \tau_{-z} e_\xi, e_\eta\rangle \\ &= e^{\langle \xi, z\rangle}\langle e_\xi, e_\eta\rangle = e^{\langle \xi, z+\eta\rangle}, \quad \xi\ \eta \in N. \end{aligned}$$

Thus the proposition is proved. ∎

4.3 Convolution operators

A convolution operator on the the space $\mathcal{F}_\theta(N')$ of test functions is a continuous linear operator from $\mathcal{F}_\theta(N')$ into itself which commutes with translation operators. It was proved in [1] that T is a convolution operator on $\mathcal{F}_\theta(N')$ if and only if there exists $\Phi_T \in \mathcal{F}_\theta(N')^*$ such that $Tf = \Phi_T * f$ for all $f \in \mathcal{F}_\theta(N')$. Moreover, the convolution $\Phi_1 * \Phi_2$ of $\Phi_1, \Phi_2 \in \mathcal{F}_\theta(N')^*$ is an element of $\in \mathcal{F}_\theta(N')^*$.

Proposition 4.4. *A convolution operator T acting on $\mathcal{F}_\theta(N')$ is positive if and only if the associated generalized function $\Phi_T \in \mathcal{F}_\theta(N')^*_+$. In this case, T has an integral representation and the Laplace transform of the associated measure μ_T is given by*

$$\mathcal{L}(\mu_T)(\xi,\eta) \;=\; \mathcal{L}(\Phi_T)(\xi)e^{\langle \xi,\eta\rangle}, \quad \xi,\, \eta \in N.$$

Proof. We first prove the positivity of the kernel T^K of the convolution operator T. For $f = f_1 \otimes f_2 \in \mathcal{F}_\theta(N' \oplus N')$, we have

$$\begin{aligned}\langle\langle T^K, ff^*\rangle\rangle &= \langle T(f_1 f_1^*), f_2 f_2^*\rangle = \langle \Phi_T * (f_1 f_1^*), f_2 f_2^*\rangle \\ &= \int_{X'} (\Phi_T * (f_1 f_1^*))(x)(f_2 f_2^*)(x)\, d\gamma(x) \\ &= \int_{X'} \langle\langle \Phi_T, \tau_{-x}(f_1 f_1^*)\rangle\rangle |f_2(x)|^2\, d\gamma(x).\end{aligned}$$

But by assumption, the associated generalized function $\Phi_T \in \mathcal{F}_\theta(N')^*$ is positive. Therefore,

$$\langle\langle \Phi_T, \tau_{-x}(f_1 f_1^*)\rangle\rangle = \langle\langle \Phi_T, f_1(x+.)f_1^*(x+.)\rangle\rangle \geqslant 0, \quad \forall f_1 \in \mathcal{F}_\theta(N'). \tag{4.2}$$

Moreover, by Theorem 3.2, we have

$$\begin{aligned}\mathcal{L}(\mu_T)(\xi,\eta) &= \langle\langle T^K, e_\xi \otimes e_\eta\rangle\rangle = \langle\langle T e_\xi, e_\eta\rangle\rangle \\ &= \langle\langle \Phi_T * e_\xi, e_\eta\rangle\rangle = \widehat{\Phi_T}(\xi) e^{\langle \xi,\eta\rangle}, \quad \xi,\eta \in N,\end{aligned}$$

which proves the assertion of the proposition. ∎

Corollary 4.5. *Suppose that $f \in \mathcal{F}_\theta(N')_+$ and $\Phi \in \mathcal{F}_\theta(N')^*_+$. Let M_f be the multiplication operator by f and let T_Φ be the convolution operator associated with Φ. Then the operator $L = M_f T_\Phi$ is positive.*

Proof. Let $f = f_1 \otimes f_2 \in \mathcal{F}_\theta(N' \oplus N')$. Use Equation (4.2) and the fact that $f \in \mathcal{F}_\theta(N')_+$ and $\Phi \in \mathcal{F}_\theta(N')^*_+$ to show that

$$\begin{aligned}\langle\langle L^K, ff^*\rangle\rangle &= \langle L(f_1 f_1^*), f_2 f_2^*\rangle = \langle M_f(\Phi * (f_1 f_1^*)), f_2 f_2^*\rangle \\ &= \langle f(\Phi * (f_1 f_1^*)), f_2 f_2^*\rangle \\ &= \int_{X'} f(x)(\Phi * (f_1 f_1^*))(x)(f_2 f_2^*)(x)\, d\gamma(x) \\ &= \int_{X'} f(x)\langle\langle \Phi, \tau_{-x}(f_1 f_1^*)\rangle\rangle |f_2(x)|^2\, d\gamma(x) \geqslant 0.\end{aligned}$$

Hence L is a positive operator in $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta(N'))$. ∎

4.4 Quantum Stochastic Differential Equations

As a final application in this paper, we consider a special type of quantum stochastic differential equations.

First recall some results from [2] about the convolution product of operators. Let T_1 and T_2 be two operators in $\mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta(N'))$. The convolution product of T_1 and T_2, denoted by $T_1 * T_2$, is defined by

$$T_1 * T_2 = \sigma^{-1}\big[\sigma(T_1)(\sigma T_2)\big],$$

where σ is the Wick symbol given by

$$\sigma(T_1)(\xi, \eta) = \langle\langle T_1 e_\xi, e_\eta \rangle\rangle e^{-\langle \xi, \eta \rangle}, \quad \xi, \eta \in N.$$

Let f, g, Φ, and Ψ be continuous positive stochastic processes with

$$\begin{aligned} f,\, g \ &: \ [0,T] \times \Omega \longrightarrow \mathcal{F}_\theta(N')_+, \\ \Phi,\, \Psi \ &: \ [0,T] \times \Omega \longrightarrow \mathcal{F}_\theta(N')_+^*. \end{aligned}$$

Put $L_t = M_{f(t)} T_{\Phi(t)}$ and $F_t = M_{g(t)} T_{\Psi(t)}$. Consider the following linear quantum differential equation (see [1, 11] and [7]:

$$\frac{dX_t}{dt} = L_t * X_t + F_t, \quad X_0 \in \mathcal{L}(\mathcal{F}_\theta(N'), \mathcal{F}_\theta(N')). \tag{4.3}$$

Theorem 4.6. *For any positive initial condition X_0, Equation (4.3) has a unique positive solution $X_t \in \mathcal{L}(\mathcal{F}_{(e^{\theta^*})^*}(N'), \mathcal{F}_{e^\theta}(N'))$ given by*

$$X_t = X_0 * e^{*(\int_0^t L_s ds)} + \int_0^t e^{*(\int_0^t L_u du)} * F_s \, ds.$$

Moreover, X_t has the following integral representation

$$\langle\langle X_t^K, f \rangle\rangle = \int_{X' \oplus X'} f((x + i0) \oplus (y + i0)) \, d\mu_{X_t}(x \oplus y)$$

for all $f \in \mathcal{F}_{((e^{\theta^*})^*, e^\theta)}(N' \oplus N')$. Here μ_{X_t} is the positive Borel measure on $X' \oplus X'$ uniquely determined by the following integrability conditions:

(1) *There exists $q > 0$ such that the measure μ_{X_t} is supported by $X_{-q} \oplus X_{-q}$.*

(2) *There exist $m_1, m_2 > 0$ such that*

$$\int_{X_{-q} \oplus X_{-q}} e^{(e^{\theta^*})^*(m_1 |x|_{-q}) + e^\theta(m_2 \|y\|_{-q})} \, d\mu_{X_t}(x \oplus y) < \infty. \tag{4.3}$$

Proof. We use the same technique as that in [2] to prove this theorem. Note that for $T_1 = M_{f_1} T_{\Phi_1}$ and $T_2 = M_{f_2} T_{\Phi_2}$, we have

$$T_1 * T_2 = M_{f_1 f_2} T_{\Phi_1 * \Phi_2}.$$

It follows that if $T = M_f T_\Phi$, then for every $n \in \mathbb{N}$ we have $T^{*n} = M_{f^n} T_{\Phi^{*n}}$. By Corollary 2.5 and [15], we have $\Phi_1 * \Phi_2 \in \mathcal{F}_\theta(N')^*_+$ if $\Phi_1, \Phi_2 \in \mathcal{F}_\theta(N')^*_+$. Apply the Wick symbol map σ to Equation (4.3) to show the existence and uniqueness of a solution, (see Theorem 3 in [2]). Then use Corollaries 2.5 and 4.5 to see that $e^{*(\int_0^t L_s\, ds)}$ is a positive operator. Thus we have the positivity of the solution. The integral representation and the integrability conditions can be shown by applying Theorem 3.2 for $\theta = (e^{\theta^*})^*$ and $\varphi = e^\theta$. ■

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