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A POINTWISE BOUND FOR ROTATION-INVARIANT HOLOMORPHIC FUNCTIONS THAT ARE SQUARE INTEGRABLE WITH RESPECT TO A GAUSSIAN MEASURE

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Abstract. We consider the subspace of Segal-Bargmann space which is invariant under the action of the special orthogonal group. We establish a pointwise bound for a function in this space which is polynomially better than the pointwise bound for a function in the Segal-Bargmann space.

1. Introduction

The Segal-Bargmann space $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$ is the space of holomorphic functions on $\mathbb{C}^d$ that are square-integrable with respect to the Gaussian measure $\mu_t(z)\,dz = (\pi t)^{-d}e^{-|z|^2/t}\,dz$, where $|z|^2 = |z_1|^2 + \cdots + |z_d|^2$. Here t is a fixed positive real number. See [1, 5, 7, 8, 10, 11, 15], for details about the importance of this space.

Various generalizations of the Segal-Bargmann space have been considered. An important part of the study of such generalizations is to obtain sharp pointwise bounds on the functions. (See, for example, [2, 4, 9, 12, 14]). Such bounds amount to estimates for the reproducing kernel on the diagonal.

In this paper, we consider the subspace of the standard Segal-Bargmann space that is invariant under the special orthogonal group. The goal of the paper is to compare two bounds for functions in this space, a simple bound obtained by minimizing the standard bounds in the full Segal-Bargmann space over the orbits of the group, and a sharp bound obtained by directly estimating the reproducing kernel for the subspace. We show that the sharp bounds are polynomially better than the simple bounds, with the difference between the two growing larger and larger as the dimension d goes to infinity.

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This analysis is motivated in part by a comparison of [3] and [9]. In [3], Driver obtains (among other things) bounds for a *generalized* Segal-Bargmann space by representing it as the subspace of a certain infinite-dimensional *standard* Segal-Bargmann space that is invariant under a certain group action. (See also [7, 16, 13]). Meanwhile, in [9], Hall obtains sharp bounds for the relevant generalized Segal-Bargmann space by directly estimating the reproducing kernel. The difference between the two bounds is significant; the sharper bounds of [9] are essential, for example, in the analysis in [14].

It is well-known that for any function $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, we have the pointwise bound

$$|F(z)|^2 \leq e^{|z|^2/t} \|F\|^2_{L^2(\mathbb{C}^d,\mu_t)} \qquad (z \in \mathbb{C}^d). \tag{1}$$

Now suppose that F is invariant under the action of $SO(d)$, and therefore, by analytic continuation, under the action of $SO(d, \mathbb{C})$. By minimizing (1) on each orbit, for any $SO(d)$-invariant function F in the Segal-Bargmann space, we obtain the preliminary estimate

$$|F(z)|^2 \leq e^{|(z,z)|/t} \|F\|^2_{L^2(\mathbb{C}^d,\mu_t)} \qquad (z \in \mathbb{C}^d), \tag{2}$$

where $(z, z) = z_1^2 + \cdots + z_d^2$. Since $|(z, z)| \leq |z|^2$, this is already an improvement over the pointwise bound in (1).

The $SO(d)$ invariance means that F is determined by its values on $\{(z, 0, ..., 0)\} \simeq \mathbb{C}^1$. (By holomorphicity, F is determined by its values on $\mathbb{R}^d$, then any point in $\mathbb{R}^d$ can be rotated into $\mathbb{R}^1$.) Conversely, any even holomorphic function on $\mathbb{C}^1$ has an extension to an $SO(d)$-invariant function on $\mathbb{C}^d$. Then the space of $SO(d)$-invariant functions in the Segal-Bargmann space over $\mathbb{C}^d$ can be expressed as an L^2-space of holomorphic functions on $\mathbb{C}^1$, with some non-Gaussian measure. By estimating the reproducing kernel for this space, we obtain a sharp bound for an $SO(d)$-invariant function F in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, which will be polynomially better than (2). This bound is described in the following theorem.

Theorem 1. *There exists a constant C, depending only on d and t, such that for each $SO(d)$-invariant function F in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$, we have*

$$|F(z)|^2 \leq \frac{C\, e^{|(z,z)|/t}}{1 + |(z, z)|^{(d-1)/2}} \|F\|^2_{L(\mathbb{C}^d,\mu_t)} \qquad (z \in \mathbb{C}^d).$$

2. $SO(d, \mathbb{C})$-Invariant Measure on a Complex Sphere

Denote by $SO(d, \mathbb{C})$ the set of $d \times d$ complex orthogonal matrices with determinant one. Elements of $SO(d, \mathbb{C})$ preserve the bilinear form $(\cdot, \cdot)$ on $\mathbb{C}^d$ defined by

$$(z, \xi) = z_1\xi_1 + z_2\xi_2 + \cdots + z_d\xi_d$$

for any $z, \xi \in \mathbb{C}^d$. For each $w \in \mathbb{C}$, we define

$$S_w = \{z \in \mathbb{C}^d \mid (z, z) = w^2\}.$$

In particular, $S_0 = \{z \in \mathbb{C}^d \mid (z, z) = 0\}$. Using the nondegeneracy of the form $(\cdot, \cdot)$, it is not hard to show that $SO(d, \mathbb{C})$ acts transitively on S_w for all $w \in \mathbb{C} - \{0\}$. Moreover, let

$$S = \{z \in \mathbb{C}^d \mid (z, z) \in (-\infty, 0]\}.$$

By the Implicit Function Theorem, $S_0 - \{0\}$ and $S - S_0$ are submanifolds of $\mathbb{C}^d$ with dimensions less than the dimension of $\mathbb{C}^d$. This implies that S has Lebesgue measure zero.

Denote by $\mathbb{H}^+ = \{z \in \mathbb{C} \mid \Re(z) > 0\}$ the open right-half plane of $\mathbb{C}$. Define $\Psi \colon \mathbb{C}^d - S \to \mathbb{H}^+ \times S_1$ by

$$\Psi(z) = (w, z')$$

where $w = |(z, z)|^{1/2} e^{i\frac{\theta}{2}}$, θ is the principal value of $\arg(z, z)$, $\theta \in (-\pi, \pi)$, and $z' = \frac{z}{w}$. It is easy to verify that Ψ is a continuous bijective map whose inverse is $\Psi^{-1}(w, z') = wz'$. We can think of this map as a "complex polar form" of an element in $\mathbb{C}^d$ that is not in S. Let m be Lebesgue measure on $\mathbb{C}^d$ and m_* the Borel measure on $\mathbb{H}^+ \times S_1$ such that $m_*(E) = m(\Psi^{-1}(E))$. The next theorem shows that the pushed-forward measure m_* on $\mathbb{H}^+ \times S_1$ can be written as a product measure $m_* = \rho \times \alpha$, where ρ is a measure on $\mathbb{H}^+$ defined by

$$\rho(A) = \int_A |w|^{2d-2} dw$$

and α is an $SO(d, \mathbb{C})$-invariant Borel measure on S_1.

Theorem 2. *There is an $SO(d, \mathbb{C})$-invariant Borel measure α on S_1 such that $m_* = \rho \times \alpha$. If f is a Borel function on $\mathbb{C}^d$ such that $f \geq 0$ or $f \in L^1(\mathbb{C}^d, m)$, then*

$$\int_{\mathbb{C}^d} f(z)\, dz = \int_{\mathbb{C}} \int_{S_1} f(wz')\, d\alpha(z')\, |w|^{2d-2}\, dw, \tag{3}$$

where dw denotes the two-dimensional Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$.

Proof. Since S has Lebesgue measure zero, (3) is equivalent to

$$\int_{\mathbb{C}^d - S} f(z)\, dz = \int_{\mathbb{C}} \int_{S_1} f(wz')\, d\alpha(z')\, |w|^{2d-2}\, dw. \tag{4}$$

First, we need to construct α. If E is a Borel set in S_1, let E_1 be the set in $\mathbb{C}^d$ given by

$$E_1 = \{wz' \mid w \in \mathbb{H}^+, |w| < 1, z' \in E\}.$$

If (4) is to hold when $f = \chi_{E_1}$, we must have

$$m(E_1) = \frac{1}{2}\int_{D_1}\int_E d\alpha(z')\,|w|^{2d-2}\,dw = \frac{\pi}{2d}\,\alpha(E).$$

Hence, for any Borel set E in S_1, we define

$$\alpha(E) = \frac{2d}{\pi}\,m(E_1).$$

Since the map $E \mapsto E_1$ takes Borel sets to Borel sets and commutes with unions, intersections and complements, α is a Borel measure on S_1. If E is a Borel set in S_1 and $A \in SO(d,\mathbb{C})$ then

$$\alpha(AE) = \frac{2d}{\pi}\,m((AE)_1) = \frac{2d}{\pi}\,m(A(E)_1) = \frac{2d}{\pi}\,\det(A)m(E_1) = \alpha(E),$$

where $\det(A)$ is the determinant of A over $\mathbb{R}$, which is 1. Hence α is $SO(d,\mathbb{C})$-invariant. Following a similar argument to the real polar coordinates formula (see, e.g., [6, Theorem 2.49]) we can show that $m_* = \rho\times\alpha$ on all Borel sets. Hence equation (4) holds when f is a characteristic function of a Borel set and it follows for general f by the usual linearity and approximation argument. ∎

The measure α in Theorem 2 is uniquely determined and can be given explicitly. There is a diffeomorphism between the tangent bundle $T(S^{d-1})$ of the real unit sphere S^{d-1} and the complex unit sphere S_1 given by

$$\mathbf{a}(\mathbf{x},\mathbf{p}) = \cosh(p)\,\mathbf{x} + \frac{i}{p}\sinh(p)\,\mathbf{p} \quad \text{for any } \mathbf{x}\in S^{d-1} \text{ and } \mathbf{x}\cdot\mathbf{p} = 0$$

where $p = |\mathbf{p}|$. See [15] for more details. Using these coordinates, we can write the measure α explicitly as follows:

Lemma 3. *The measure α is given by*

$$\alpha(z) = a_0\Big(\frac{\sinh 2p}{2p}\Big)^{d-2} 2^{d-1}\,d\mathbf{p}\,d\mathbf{x}.$$

Here $z = \mathbf{a}(\mathbf{x},\mathbf{p})$, a_0 is a constant, $d\mathbf{x}$ is the surface area measure on $S^{\,d-1}$ and $d\mathbf{p}$ is Lebesgue measure on $\mathbb{R}^d$.

Proof. The measure α and the measure $\big(\frac{\sinh 2p}{2p}\big)^{d-2}2^{d-1}\,d\mathbf{p}\,d\mathbf{x}$ are both $SO(d,\mathbb{C})$-invariant(Lemma 3 of [15]) and finite on compact sets. Thus, by Theorem 8.36 of [17], these two measures must agree up to a constant. ∎

3. Pointwise Bound for a Function in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{\mathcal{O}}$

Denote by $\mathcal{H}(\mathbb{C}^d)^{\mathcal{O}}$ the space of $SO(d,\mathbb{C})$-invariant holomorphic functions on $\mathbb{C}^d$, i.e., the space of holomorphic functions f for which $f(Az) = f(z)$ for all $z \in \mathbb{C}^d$ and $A \in SO(d,\mathbb{C})$. In this section, we will establish a pointwise bound for a function in the space $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{\mathcal{O}} := \mathcal{H}(\mathbb{C}^d)^{\mathcal{O}} \cap L^2(\mathbb{C}^d, \mu_t)$.

By minimizing over each orbit, we obtain the following pointwise bound:

Proposition 4. *For any $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{\mathcal{O}}$ and for any $z \in \mathbb{C}^d$*

$$|F(z)|^2 \leq e^{|(z,z)|/t} \|F\|^2_{L^2(\mathbb{C}^d,\mu_t)}. \tag{5}$$

Proof. Note that $|(z,z)| = |(Az, Az)| \leq |Az|^2$ for any $z \in \mathbb{C}^d$ and $A \in SO(d,\mathbb{C})$. If $z \notin S_0$, we have that $(\sqrt{(z,z)}, 0, \ldots, 0) \in \{Az \mid A \in SO(d,\mathbb{C})\}$, because $SO(d,\mathbb{C})$ acts transitively on S_w where $w = \sqrt{(z,z)}$, and thus

$$|(z,z)| = \inf \left\{ |Az|^2 : A \in SO(d,\mathbb{C}) \right\}.$$

But S_0^c is dense in $\mathbb{C}^d$, so this equation is also true for all $z \in \mathbb{C}^d$. This immediately gives (5). ∎

This simple technique yields an improvement from the Bargmann's pointwise bound (1). However, we will establish a polynomially-better bound than the bound in (5). Our strategy is to construct a non-Gaussian measure λ on $\mathbb{C}$ so that we can express $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{\mathcal{O}}$ in terms of the space $\mathcal{H}L^2(\mathbb{C}, \lambda)^e$ of holomorphic even functions on $\mathbb{C}$ that are square-integrable with respect to λ and then estimate the reproducing kernel of the latter space.

Proposition 5. *Let $\mathcal{H}(\mathbb{C})^e$ be the set of all holomorphic even functions on $\mathbb{C}$. Then for any $d \geq 2$, the map $\phi\colon \mathcal{H}(\mathbb{C}^d)^{\mathcal{O}} \to \mathcal{H}(\mathbb{C})^e$ defined by*

$$\phi(f)(x) = f(x, 0, \ldots, 0),$$

for all $f \in \mathcal{H}(\mathbb{C}^d)^{\mathcal{O}}$ and all $x \in \mathbb{C}$, is a linear isomorphism whose inverse is given by

$$\psi(g)(z) = g\left(\sqrt{(z,z)}\right)$$

for all $g \in \mathcal{H}(\mathbb{C})^e$ and all $z \in \mathbb{C}^d$.

Note that since g is even, the value of $\psi(g)(z)$ is independent of the choice of square root of (z,z). Again because g is even, $\psi(g)$ will be given by a convergent power series in integer powers of $(z,z) = z_1^2 + \cdots + z_d^2$, and therefore $\psi(g)$ will be holomorphic on $\mathbb{C}^d$.

Proof. It is clear that ϕ is a linear map and $\phi(f)$ is a holomorphic function on $\mathbb{C}$ for any $f \in \mathcal{H}(\mathbb{C}^d)^{\mathcal{O}}$. Moreover, $\phi(f)$ is even since $A(-w, 0, \ldots, 0) = (w, 0, \ldots, 0)$ for any $w \in \mathbb{C}$, where $A = \mathrm{diag}(-1, -1, 1, 1, \ldots, 1)$.

On the other hand, ψ is a linear map and $\psi(g)$ is holomorphic on $\mathbb{C}^d$ for each $g \in \mathcal{H}(\mathbb{C})^e$. Since the bilinear form is preserved under the action of the orthogonal group, $\psi(g)$ is $SO(d, \mathbb{C})$-invariant. It is straightforward to verify that $\phi \circ \psi = \mathrm{I}_{\mathcal{H}(\mathbb{C})^e}$ and $\psi \circ \phi = \mathrm{I}_{\mathcal{H}(\mathbb{C}^d)^{\mathcal{O}}}$, so the theorem is proved. ∎

Henceforth, we will choose an argument of $w \in \mathbb{C}$ so that $-\pi < \arg(w) \leq \pi$. Denote by $\mathcal{B}_d$ the Borel σ-algebra in $\mathbb{C}^d$ and by $\mathcal{B}$ the Borel σ-algebra in $\mathbb{C}$. Define $\Phi_i \colon (\mathbb{C}^d, \mathcal{B}_d, \mu_t) \to (\mathbb{C}, \mathcal{B})$, $i = 1, 2$ to be the branch of $\sqrt{(z, z)}$ with smaller and larger argument, respectively, and for each $E \in \mathcal{B}$ define

$$\lambda_i(E) = \mu_t(\Phi_i^{-1}(E)).$$

Then define $\lambda = (\lambda_1 + \lambda_2)/2$. It is easy to check that λ is a Borel measure on $\mathbb{C}$ and for any measurable function g and any $E \in \mathcal{B}$

$$\int_E g \, d\lambda \;=\; \frac{1}{2} \int_{\Phi_1^{-1}(E)} g \circ \Phi_1 \, d\mu_t + \frac{1}{2} \int_{\Phi_2^{-1}(E)} g \circ \Phi_2 \, d\mu_t.$$

It is now straightforward to verify that the restriction of ϕ to $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{\mathcal{O}}$ is a unitary map onto $\mathcal{H}L^2(\mathbb{C}, \lambda)^e$.

Proposition 6. *The measure λ is absolutely continuous with respect to Lebesgue measure on $\mathbb{C}$ with density given by*

$$\Lambda(w) = \frac{|w|^{2d-2}}{(\pi t)^d} \int_{S_1} e^{-|wz|^2/t} d\alpha(z). \tag{6}$$

Proof. If E is a Borel set in $\mathbb{C}$, then by Theorem 2

$$\begin{aligned} \lambda(E) \quad &= \frac{1}{2} \int_{\Phi_1^{-1}(E)} \frac{e^{-|z|^2/t}}{(\pi t)^d} \, dz + \frac{1}{2} \int_{\Phi_2^{-1}(E)} \frac{e^{-|z|^2/t}}{(\pi t)^d} \, dz \\ &= \int_{\mathbb{C}} \int_{S_1} \chi_E(w) \frac{|w|^{2d-2}}{(\pi t)^d} \, e^{-|wz'|^2/t} \, d\alpha(z') \, dw \\ &= \int_E \Lambda(w) \, dw \end{aligned}$$

where Λ is given by (6). ∎

Next, we will approximate the density Λ of λ and show that on holomorphic functions, the L^2-norm with respect to λ is equivalent to the L^2-norm with respect to the measure $\beta(w)dw$, where

$$\beta(w) = \frac{e^{-|w|^2/t}}{t\pi}\,|w|^{d-1} \qquad (w \in \mathbb{C}). \tag{7}$$

Proposition 7. *There exist constants* $m, M > 0$*, depending on* d *and* t*, such that the density function* Λ *of* λ *satisfies*

$$m\,\beta(w) \leq \Lambda(w) \leq M\,\beta(w)$$

for all $w \in \mathbb{C}$ *with* $|w| \geq 1$.

Proof. From Lemma 3, for any $w \in \mathbb{C}$

$$\begin{aligned}\textstyle\int_{S_1} e^{-|wz|^2/t}\,d\alpha(z) &= a_0 \int_{S^{d-1}} \int_{\mathbf{x}\cdot\mathbf{p}=\mathbf{0}} e^{-|w\,\mathbf{a}(\mathbf{x},\mathbf{p})|^2/t} \Big(\frac{\sinh 2p}{2p}\Big)^{d-2} 2^{d-1}\,d\mathbf{p}\,d\mathbf{x} \\ &= a_d \int_0^\infty e^{-(\cosh 2p)|w|^2/t} \Big(\frac{\sinh 2p}{2p}\Big)^{d-2} 2^{d-1}\,p^{d-2}\,dp \\ &= a_d\, e^{-|w|^2/t} \int_0^\infty e^{-x|w|^2/t}\,(x^2+2x)^{(d-3)/2}\,dx,\end{aligned}$$

with $a_d = a_0\sigma(S^{d-1})\sigma(S^{d-2})$, where σ is the surface measure. The last equality follows from the change of variables $\cosh 2p = x + 1$.

Now, let us consider the case $d \geq 3$. To approximate the above integral, we expand $(x^2+2x)^{d-3}$ using the binomial theorem, apply the inequalities

$$\frac{1}{\sqrt{n}}(\sqrt{a_1} + \cdots + \sqrt{a_n}\,) \leq \sqrt{a_1 + \cdots + a_n} \leq \sqrt{a_1} + \cdots + \sqrt{a_n}$$

to $(x^2+2x)^{(d-3)/2}$ and then use the formula for the Gamma function in order to obtain

$$\frac{1}{\sqrt{d-2}} P\left(\frac{\sqrt{t}}{|w|}\right) \leq \int_0^\infty e^{-x|w|^2/t}\,(x^2+2x)^{(d-3)/2}\,dx \leq P\left(\frac{\sqrt{t}}{|w|}\right),$$

where

$$P(x) = \sum_{k=0}^{d-3} a_k^{1/2}\Gamma\left(\frac{d-1+k}{2}\right) x^{d-1+k} \quad \text{and} \quad a_k = \binom{d-3}{k} 2^{d-3-k}.$$

This shows that

$$\frac{a_d}{\sqrt{d-2}} P\left(\frac{\sqrt{t}}{|w|}\right) e^{-|w|^2/t} \;\leq\; \int_{S_1} e^{-|wz|^2/t}\,d\alpha(z) \;\leq\; a_d P\left(\frac{\sqrt{t}}{|w|}\right) e^{-|w|^2/t}.$$

It follows from (6) that

$$\frac{e^{-|w|^2/t}}{t\pi\sqrt{d-2}} Q\left(\frac{|w|}{\sqrt{t}}\right) \;\le\; \Lambda(w) \;\le\; \frac{e^{-|w|^2/t}}{t\pi} Q\left(\frac{|w|}{\sqrt{t}}\right) \tag{8}$$

where

$$Q(x) \;=\; \frac{a_d}{\pi^{d-1}} \sum_{k=0}^{d-3} a_k^{1/2} \Gamma\left(\frac{d-1+k}{2}\right) x^{d-1-k} \;=\; \sum_{k=2}^{d-1} b_k x^k.$$

From this (7) easily follows for the case $d \ge 3$.

Meanwhile in the $d = 2$ case we have

$$\begin{aligned}
\int_{S_1} e^{-|wz|^2/t}\, d\alpha(z) &= a_2\, e^{-|w|^2/t} \int_0^\infty \frac{e^{-x|w|^2/t}}{\sqrt{x^2+2x}}\, dx \\
&= \frac{a_2}{\sqrt{2}}\, e^{-|w|^2/t} \int_0^\infty e^{-u} \sqrt{\left(\frac{|w|^2}{tu} - \frac{|w|^2}{2|w|^2+tu}\right)}\, \frac{t}{|w|^2}\, du \\
&\ge \frac{a_2\sqrt{t}}{\sqrt{2}|w|}\, e^{-|w|^2/t} \left(\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\, du - \int_0^\infty \frac{e^{-u}}{\sqrt{2|w|^2/t+u}}\, du\right).
\end{aligned}$$

The function

$$\phi(r) = \int_0^\infty \frac{e^{-u}}{\sqrt{r+u}}\, du \qquad (r \ge 0)$$

is a strictly decreasing function. Hence, if we let $\delta = 2/t$ and $\varepsilon = \phi(0) - \phi(\delta)$, then $\phi(0) - \phi\left(2|w|^2/t\right) \ge \phi(0) - \phi(\delta) = \varepsilon$ for any w with $2|w|^2/t \ge \delta$. It follows that

$$\begin{aligned}
\Lambda(w) &= \frac{|w|^2}{(\pi t)^2} \int_{S_1} e^{-|wz|^2/t} d\alpha(z) \\
&\ge \frac{\varepsilon a_2}{\pi\sqrt{2t}}\, \frac{e^{-|w|^2/t}}{\pi t} |w|
\end{aligned}$$

for any $w \in \mathbb{C}$ with $|w| \ge 1$.

On the other hand,

$$\begin{aligned}
\int_{S_1} e^{-|wz|^2/t}\, d\alpha(z) &\le a_2\, e^{-|w|^2/t} \int_0^\infty \frac{e^{-x|w|^2/t}}{\sqrt{2x}}\, dx \\
&= \frac{a_2\sqrt{t\pi}}{\sqrt{2}|w|}\, e^{-|w|^2/t}.
\end{aligned}$$

Hence

$$\Lambda(w) \leq \frac{a_2}{\sqrt{2\pi t}}\frac{e^{-|w|^2/t}}{\pi t}|w|. \qquad \blacksquare$$

Corollary 8. *The norms $\|\cdot\|_{L^2(\mathbb{C},\beta)}$ and $\|\cdot\|_{L^2(\mathbb{C},\lambda)}$ are equivalent, i.e., there are constants $k, K > 0$, depending on d and t, such that*

$$k\|f\|_{L^2(\mathbb{C},\beta)} \leq \|f\|_{L^2(\mathbb{C},\lambda)} \leq K\|f\|_{L^2(\mathbb{C},\beta)}, \tag{9}$$

for all $f \in \mathcal{H}L^2(\mathbb{C},\lambda)$.

Proof. First, we will show that there is a constant $D > 0$, depending on d and t, such that

$$\|f\|^2_{L^2(\mathbb{C},\beta)} \leq D\,\|f\|^2_{L^2(\mathbb{C}-\mathbb{D},\lambda)}$$

for any $f \in \mathcal{H}L^2(\mathbb{C},\lambda)$, where $\mathbb{D} = \{w \in \mathbb{C} : |w| \leq 1\}$.

Let $w \in \mathbb{D}$. Denote by $A(w)$ the annulus $\{z \in \mathbb{C} : 2 \leq |z - w| \leq 3\}$. If f is in $\mathcal{H}L^2(\mathbb{C},\lambda)$ then a simple power series argument shows that

$$\int_{A(w)} f(v)\,dv = (9\pi - 4\pi)f(w) = 5\pi f(w).$$

This implies that

$$\begin{aligned} |f(w)| &= \frac{1}{5\pi}\Big|\int_{A(w)} f(v)\,dv\Big| \\ &= \frac{1}{5\pi}\Big|\Big\langle \chi_{A(w)}\frac{1}{\Lambda}, f\Big\rangle_{L^2(\mathbb{C}-\mathbb{D},\lambda)}\Big| \\ &\leq \frac{1}{5\pi}\Big\|\chi_{A^*}\frac{1}{\Lambda}\Big\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)}\|f\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)}, \end{aligned}$$

where $A^* = \{z \in \mathbb{C} : 1 < |z| < 4\}$, which contains each $A(w)$, $w \in \mathbb{D}$. It follows that there exists a constant c such that for any $w \in \mathbb{D}$

$$|f(w)| \leq c\|f\|_{L^2(\mathbb{C}-\mathbb{D},\lambda)}.$$

It now follows from Proposition 7 that

$$\begin{aligned} \int_{\mathbb{C}} |f(w)|^2\beta(w)\,dw &= \int_{\mathbb{D}} |f(w)|^2\beta(w)\,dw + \int_{\mathbb{C}-\mathbb{D}} |f(w)|^2\beta(w)\,dw \\ &\leq c^2\|f\|^2_{L^2(\mathbb{C}-\mathbb{D},\lambda)}\int_{\mathbb{D}}\beta(w)\,dw + \frac{1}{m}\|f\|^2_{L^2(\mathbb{C}-\mathbb{D},\lambda)} \\ &\leq D\,\|f\|^2_{L^2(\mathbb{C}-\mathbb{D},\lambda)} \end{aligned}$$

for some constant $D > 0$ depending on d and t. This gives the first inequality in (9). The second inequality in (9) can be proved in the same way. ■

Having established Corollary 8, it remains only to obtain pointwise bounds for elements in $\mathcal{H}L^2(\mathbb{C},\lambda)$. We do this by reducing to the standard Segal-Bargmann space (if d is odd) or to the space $\mathcal{H}L^2\big(\mathbb{C},(t\pi)^{-1}|w|e^{-|w|^2/t}\,dw\big)$ (if d is even). We now establish pointwise bound in the latter space.

Lemma 9. *The set* $\left\{\frac{w^n}{\left(t^{(2n+1)/2}\Gamma(n+\frac{3}{2})\right)^{1/2}}\right\}_{n=0}^{\infty}$ *is an orthonormal basis for the Hilbert space* $\mathcal{H}L^2\big(\mathbb{C},|w|\frac{e^{-|w|^2/t}}{t\pi}dw\big)$. *Hence for any* $g\in\mathcal{H}L^2\big(\mathbb{C},|w|\frac{e^{-|w|^2/t}}{t\pi}dw\big)$,

$$|g(w)|^2 \le \frac{e^{|w|^2/t}}{|w|}\operatorname{erf}\Big(\frac{|w|}{\sqrt{t}}\Big)\,\|g\|^2 \qquad (w\in\mathbb{C}),$$

where the error function erf *is defined by*

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}\,dy = e^{-x^2}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{\Gamma(n+\frac{3}{2})}.$$

Proof. The proof of the orthonormal basis part uses the same technique as in [1, 7] and [10], which we will omit. Then the pointwise bound for a function g in this space is

$$|g(w)|^2 \le \sum_{n=0}^{\infty}\frac{|w|^{2n}}{t^{(2n+1)/2}\Gamma(n+\frac{3}{2})}\|g\|^2 = \frac{e^{|w|^2/t}}{|w|}\operatorname{erf}\Big(\frac{|w|}{\sqrt{t}}\Big)\,\|g\|^2$$

for any $w\in\mathbb{C}$. ■

Theorem 10. *There is a constant* B*, depending on* d *and* t*, such that for any* $f\in\mathcal{H}L^2(\mathbb{C},\lambda)$ *and any* $w\in\mathbb{C}-\{0\}$,

$$|f(w)|^2 \le \frac{B}{|w|^{d-1}}\,e^{|w|^2/t}\,\|f\|^2_{L^2(\mathbb{C},\lambda)}. \tag{10}$$

Proof.
Let $f\in\mathcal{H}L^2(\mathbb{C},\lambda)$. Then $f\in\mathcal{H}L^2(\mathbb{C},\beta)$, and thus

$$\int_{\mathbb{C}}|w|^{d-1}|f(w)|^2\,\frac{e^{-|w|^2/t}}{\pi t}\,dw<\infty.$$

If $d-1$ is an even number, then

$$w^{(d-1)/2}f(w) \in \mathcal{H}L^2(\mathbb{C}, \frac{e^{-|w|^2/t}}{t\pi}\, dw).$$

This is the one-dimensional Segal-Bargmann space. Using Bargmann's pointwise bound (1) for this space, we obtain

$$|w|^{d-1}|f(w)|^2 \;\leq\; \|f\|^2_{L^2(\mathbb{C},\beta)}\, e^{|w|^2/t} \;\leq\; \tfrac{1}{k^2}\, \|f\|^2_{L^2(\mathbb{C},\lambda)}\, e^{|w|^2/t}$$

for all $w \in \mathbb{C}$, where k is the constant in Corollary 8.

On the other hand, if $d-1$ is an odd number, then

$$w^{(d-2)/2}f(w) \in \mathcal{H}L^2(\mathbb{C}, |w|\frac{e^{-|w|^2/t}}{t\pi} dw).$$

Following Lemma 9, we have

$$\begin{aligned}|w|^{d-2}|f(w)|^2 &\leq \|f\|^2_{L^2(\mathbb{C},\beta)} \frac{e^{|w|^2/t}}{|w|} \operatorname{erf}\Big(\frac{|w|}{\sqrt{t}}\Big) \\ &\leq \frac{1}{k^2} \|f\|^2_{L^2(\mathbb{C},\lambda)} \frac{e^{|w|^2/t}}{|w|}\end{aligned}$$

for all $w \in \mathbb{C} - \{0\}$.

In either case we obtain the pointwise (10) with $B = 1/k^2$. ∎

Proof of Theorem 1. We will transform the pointwise bound (10) to a function in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{\mathcal{O}}$. Let $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{\mathcal{O}}$. Then $F(w, 0, \ldots, 0) \in \mathcal{H}L^2(\mathbb{C}, \lambda)^e$, which implies

$$|F(z)|^2 = |F(w, 0, \ldots, 0)|^2 \leq B\, \frac{e^{|w|^2/t}}{|w|^{d-1}}\, \|F\|^2_{L^2(\mathbb{C}^d,\mu_t)}$$

where $w = \sqrt{(z,z)}$ for any $z \in \mathbb{C}^d$ with $(z,z) \neq 0$. In particular,

$$|F(z)|^2 \leq \frac{B\, e^{|(z,z)|/t}}{|(z,z)|^{(d-1)/2}} \|F\|^2_{L^2(\mathbb{C}^d,\mu_t)}.$$

On the other hand, from Proposition 4,

$$|F(z)|^2 \leq e^{|(z,z)|/t} \|F\|^2_{L^2(\mathbb{C}^d,\mu_t)} \quad \text{for any } z \in \mathbb{C}^d.$$

Applying the inequality

$$\min\left\{1, \frac{1}{x}\right\} \leq \frac{2}{x+1} \qquad \text{for each } x > 0,$$

we have

$$|F(z)|^2 \leq \frac{C\, e^{|(z,z)|/t}}{|(z,z)|^{(d-1)/2}+1} \|F\|^2_{L^2(\mathbb{C}^d,\mu_t)}$$

for each $z \in \mathbb{C}^d$, where C is a constant depending on d and t. This completes the proof of Theorem 1. ■

Remark on the sharpness. The bound in Theorem 1 is indeed sharp. We only outline the proof here since the argument relies heavily on properties of special functions. We can show that the reproducing kernel of the Hilbert space $\mathcal{H}L^2(\mathbb{C},\lambda)^e$ is given by

$$K(w,w) = \frac{\Gamma(d/2)}{a_0 2^{d/2+1}} \mathrm{BesselI}\Big(\frac{d-2}{2}, \frac{|w|^2}{t}\Big) \Big(\frac{t}{|w|^2}\Big)^{d/2-1}$$

where BesselI is the modified Bessel function of the first kind ([18, 19]). Asymptotically, BesselI$(\alpha, x) \sim \frac{e^x}{\sqrt{x}}$ if x is large enough when $\alpha > 0$ is fixed. Hence,

$$K(w,w) \sim C \frac{e^{|w|^2/t}}{|w|^{d-1}}$$

for any w such that $|w|$ is large enough, where C is a constant depending on d and t. The result follows by transforming this estimate back to the space $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{\mathcal{O}}$.

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