TAIWANESE JOURNAL OF MATHEMATICS
Vol. 11, No. 5, pp. 1369-1382, December 2007
This paper is available online at http://www.math.nthu.edu.tw/tjm/

AN IMPROVED STABILITY CRITERION WITH APPLICATION TO THE ARNEODO-COULLET-TRESSER MAP

B.-S. Du, S.-R. Hsiau, M.-C. Li and M. Malkin

Abstract. We give a criterion of discrete stability for polynomials of degree $n \leq 4$ in terms of polynomial inequalities which contain linear inequalities and only one inequality of degree $n-1$. The result then applies to study stability regions for fixed points of the Arneodo-Coullet-Tresser maps.

1. Introduction

Let F be a differentiable map on a manifold with a fixed point q and let $DF(q)$ denote the Jacobian matrix of F at q. It is well known that the local stability of F near q is determined by the spectral radius of $DF(q)$; more precisely, the following result is contained in many textbooks on dynamical system (see [9] for instance).

1. If the spectral radius of $DF(q)$ is less than one, then q is asymptotically stable, i.e., there is a neighborhood U of q such that if $y \in U$ then $F^n(y)$, the n^{th} iterate of y, tends to q as $n \to +\infty$; and
2. If the spectral radius of $DF(q)$ is bigger than one then q is unstable, i.e., there is a neighborhood U of q such that for any neighborhood V of q, there exists $y \in V$ such that $F^n(y) \notin U$ for some $n \geq 0$.

So the problem on local stability can be formulated in terms of characteristic polynomial of the Jacobian matrix as the following question: how can one easily determine whether a polynomial has all roots with moduli less than one?

Definition 1. A polynomial $P(x) \in \mathbb{C}[x]$ (or $\mathbb{R}[x]$) is said to be *stable* if all of its roots lie in $\{x \in \mathbb{C} : |x| < 1\}$.

Received September 27, 2005, accepted October 24, 2005.
Communicated by Wen-Wei Lin.
2000 *Mathematics Subject Classification*: 37C75, 37G10.
Key words and phrases: Stability, Bifurcations, Discrete dynamical systems.

Note that polynomials of this type of stability are called sometimes *discrete stable* or *Shur stable* to distinguish from the situation when all roots of polynomial have negative real parts (the latter applies usually to characteristic polynomials of linearization of vector fields at stable fixed points, in which case the term *Hurwitz stability* is sometimes used).

There are several criteria to determine stability of a polynomial in terms of relations on its coefficients (e.g., the Shur-Cohn criterion, see [7] and [8]). Though such criteria are equivalent to each other, their practical use depends on their complexity, i.e., on the number and degree of polynomial inequalities involved. For example, the usual Shur-Cohn criterion for a polynomial $P_n(z) = a_n z^n + \cdots + a_1 z + a_0 \in \mathbb{C}[z]$ contain polynomial inequalities in $2(n+1)$ variables $[a_0, \ldots, a_n, \bar{a}_0, \ldots, \ldots, \bar{a}_n]$ of maximal degree 2^n. For real polynomials $P_n(x)$ there are some methods using the so called inner determinants of square matrices (see [5] and [6]) which allow to reduce the maximal degree of inequalities to $n-1$. More precisely, the resulted system of inequalities obtained by those methods contain inequalities of maximal degree $n-1$ in $n+1$ variables $[a_0, \ldots, a_n]$ (or in n variables $[a_0, \ldots, a_{n-1}]$ for monic polynomials) with exactly two inequalities of maximal degree $n-1$. For example, for real polynomial $P(x) = x^3 + a_2 x^2 + a_1 x + a_0$, the system of inequalities in the stability criterion from [6] is the following

$$|a_0 + a_2| < 1 + a_1 \quad and \quad |a_1 - a_0 a_2| < 1 - a_0^2. \tag{1}$$

(To get a system of polynomial inequalities, one should replace any inequality of the form $|f| < g$ by two polynomial inequalities $f - g < 0$ and $f + g > 0$). So the above system contains two inequalities of degree 2 and two linear inequalities. Let us notice that the mentioned methods for producing stability criteria were algebraic ones. In this paper, in Section 2 we propose another method using dynamical (bifurcation) ideas which allow us to obtain for polynomial of degree $n \leq 4$, a stability criterion containing only one inequality of degree $n-1$ except for linear inequalities. For example, for $n = 3$ our criterion can be written as (see Corollary 2 below)

$$|a_0| < 1, \quad |a_0 + a_2| < 1 + a_1 \quad and \quad a_1 - a_0 a_2 < 1 - a_0^2. \tag{2}$$

So by comparing (1) and (2), one can see that the inequality $a_1 - a_0 a_2 > a_0^2 - 1$ involved in (1), is redundant provided that one takes into account the condition $-1 < a_0 < 1$ instead, which has simple "dissipative" meaning. This comparison can be regarded from the dynamical point of view by saying that this extra inequality in (1) has no dynamical meaning, while in (2) all the inequalities, being replaced by corresponding equalities, are responsible for important bifurcations (e.g., transcritical bifurcation, pitchfork bifurcation, period doubling bifurcation and Hopf bifurcation, as shown for characteristic polynomials of Arneodo-Coulle-Tresser maps in [4], see also Figure 2 below). To compare our criterion with stability criterion in [6] for polynomials of degree 4, see Remark 1 below.

As a consequence, we obtain explicit formulas for stability regions of some one-parameter families of polynomials (see Corollaries 2 and 4). In Section 3 we apply results from Section 2 to study stability regions of fixed points for one-parameter and two-parameter families of Arneode-Coullet-Tresser maps.

2. Stability Criterion

First we explain how to produce stability criteria by induction. Then we will give our bifurcation method for polynomials of degree not bigger than four and compare some results. The idea of the induction way for stability criterion can be found in [2].

Let

$$P_n(z) = a_n z^n + \cdots + a_1 z + a_0 \in \mathbb{C}[z].$$

Define

$$P_n^*(z) = \bar{a}_0 z^n + \bar{a_1} z^{n-1} + \cdots + \bar{a}_n.$$

Then on the unit circle $|z| = 1$, one has $|P_n(z)| = |P_n^*(z)|$. Furthemore, if $a_0 \neq 0$ and z_i is a zero of P_n then $\bar{z_i}^{-1}$ is a zero of P_n^*. Define a polynomial

$$P_{n-1}(z) = \bar{a}_0 P_n(z) - a_n P_n^*(z).$$

Then P_{n-1} has degree at most $n-1$, and the constant term of P_{n-1} is $|a_0|^2 - |a_n|^2$.

Lemma 1. *Let P_n have m zeros inside the unit circle and no zeros on the unit circle. If $|a_0| < |a_n|$ then P_{n-1} has $(n-m)$ zeros inside the unit circle and no zeros on the unit circle.*

Proof. Let Γ denote the unit circle $|z| = 1$. Then for $z \in \Gamma$ one has $|P_n(z)| = |P_n^*(z)| > 0$ and hence the hypotheses of the lemma imply $|\bar{a}_0 P_n(z)| < |a_n P_n^*(z)|$. It follows from this inequality that P_{n-1} has no zeros on Γ. Also this inequality along with the definition of P_{n-1} imply by using Rouché's theorem[1], that P_{n-1} has the same number of zeros inside Γ as P_n^*. Thus P_{n-1} has $(n-m)$ zeros inside Γ and no zeros on Γ. ■

Theorem 1. *P_n is stable if and only if $|a_0| < |a_n|$ and P_{n-1}^* is stable.*

[1]Here we give a general form of Rouché's theorem in [10]: Let Ω be the interior of a compact set K in the complex plane. Suppose that f and g are continuous on K and holomorphic (or analytic) in Ω, and $|f(z) - g(z)| < |f(z)|$ for all $z \in K \backslash \Omega$. Then f and g have the same number of zeros in Ω.

Proof. For "if" part: since all the zeros of P_{n-1}^* lie inside the unit circle Γ, the polynomial $P_{n-1} = (P_{n-1}^*)^*$ has all zeros outside Γ. Then the polynomial P_n cannot have zeros on Γ (otherwise P_{n-1} would have) and by using also the assumption $|a_0| < |a_n|$, we may apply Lemma 1. Thus, $n - m = 0$ and so, P_n has all the n zeros inside Γ.

For "only if" part: assume P_n is stable. Since a_0/a_n is the product of all zeros of P_n, it follows that $|a_0| < |a_n|$ and hence we may apply Lemma 1. Then P_{n-1} has no zeros inside Γ and no zeros on Γ. Thus, all zeros of P_{n-1}^* lie inside Γ and so, P_{n-1}^* is stable. ■

So, starting from the trivial criterion for degree 1 polynomials, one could produce by induction for polynomials $P \in \mathbb{C}[z]$ of degree n, the stability criterion as a system of polynomial inequalities in terms of coefficients of P (e.g., by using special determinants or Bézoutians, see [7] and [8]). However, the resulted system of inequalities would contain polynomials in $2(n+1)$ variables $[a_0, \ldots, a_n, \bar{a}_0, \ldots, \ldots, \bar{a}_n]$ of maximal degree 2^n. As for real polynomials, there are methods to reduce degrees of polynomial inequalities by using the so called innerwise determinants of square matrices (see [5] and [6]). Such methods produce inequalities of maximal degree $n-1$ in $n+1$ variables $[a_0, \ldots, a_n]$ (or in n variables $[a_0, \ldots, a_{n-1}]$ for monic polynomials) with exactly two inequalities of the maximal degree $n-1$.

By using bifurcation method we are able to give stability criteria for polynomials of degree $n \le 4$ which result in a system having just one inequality of degree $n-1$ and linear inequalities besides.

Theorem 2. *A polynomial* $P(x) = x^4 + Ax^3 + Bx^2 + Cx + D \in \mathbb{R}[x]$ *is stable if and only if*

$$|D| < 1, \quad |A - C| < 2(1 - D), \quad \textit{and} \quad \min\{P(1), P(-1), \tilde{\alpha}\} > 0,$$

where $\tilde{\alpha} = (A - C)(C - AD) + (1 + D - B)(1 - D)^2$.

Proof. Note that if $|D| \ge 1$ then (since the product of the four zeros of P is D), P is not stable. So we need to consider only the case when $|D| < 1$.

Let us define $P_\alpha(x) = P(x) + \alpha x^2$ as a one-parameter family of polynomials with parameter $\alpha \in \mathbb{R}$. Then for any fixed $x_0 \ne 0$, $P_\alpha(x_0)$ has the *increasing property* in α, i.e., $P_\alpha(x_0)$ increases as α increases. It is easily seen that there is a unique α, namely $\alpha = -P(1)$, such that $P_\alpha(1) = 0$. Similarly, $P_\alpha(-1) = 0$ for a unique α, namely for $\alpha = -P(-1)$. Next, we need to find the value of α, say $\hat{\alpha}$, for which P_α has two complex conjugate zeros on the unit circle. Let zeros of $P_{\hat{\alpha}}$ be x_1, $x_2 = \bar{x}_1$, and x_3, x_4 with $|x_1| = |x_2| = 1$, and denote $u = x_1 + x_2$,

$v = x_3 + x_4$. Obviously, $|u| \leq 2$. By Viète's theorem[2], we have that $u + v = -A$, $1 + D + uv = B + \hat{\alpha}$ and $v + Du = -C$. From these equalities we get a unique solution $u = \frac{C-A}{1-D}$, $v = \frac{AD-C}{1-D}$ and

(3) $$\hat{\alpha} = \frac{(A-C)(C-AD)}{(1-D)^2} - B + D + 1.$$

Therefore,

(4) $$P_{\hat{\alpha}}(x) = \left(x^2 + \frac{A-C}{1-D}x + 1\right)\left(x^2 + \frac{C-AD}{1-D}x + D\right).$$

Conversely, for $\alpha = \hat{\alpha}$ one has factorization (4) and if, in addition, $\left|\frac{A-C}{1-D}\right| \leq 2$ then $P_{\hat{\alpha}}$ has two complex conjugate zeros on the unit circle. Note that if $\left|\frac{A-C}{1-D}\right| = 2$ then both these zeros coincide with either 1 or -1.

So we have the following *bifurcation values* of the parameter: $\alpha_1 := -P(1)$, $\alpha_{-1} := -P(-1)$ associated to the zeros 1 and -1 respectively, and if $|A - C| \leq 2(1 - D)$ then there is also the third bifurcation value $\hat{\alpha}$, which is associated to two complex conjugate zeros on the unit circle. Note that these bifurcatinal values partition the real α-axis into open intervals such that for any α, α' from the same interval of this partition, either both polynomials P_α and $P_{\alpha'}$ are stable or both are not stable.

Let α_{max} be the maximum of all bifurcation values, so $\alpha_{max} = \max\{\alpha_1, \alpha_{-1}, \hat{\alpha}\}$ provided that $|A - C| \leq 2(1 - D)$; otherwise, $\alpha_{max} = \max\{\alpha_1, \alpha_{-1}\}$.

Claim (i). *P_α is not stable for all $\alpha \geq \alpha_{max}$.* For $\alpha = \alpha_{max}$ the claim is trivial. Suppose it is not true for some $\alpha' > \alpha_{max}$. Then for each $\alpha > \alpha_{max}$ the polynomial P_α is stable. Hence by Viète's theorem one would have that $|B+\alpha| \leq 6$, but this is impossible for α sufficiently large.

Claim (ii). *P_α is not stable for all $\alpha \leq \max\{\alpha_1, \alpha_{-1}\}$.* Indeed, the monotonicity property in α of $P_\alpha(1)$ implies that $P_\alpha(1) < P_{\alpha_1}(1) = 0$ for all $\alpha < \alpha_1$. On the other hand, the positiveness of the leading coefficient of P_α implies that $P_\alpha(x) > 0$ for some x large. Thus for all $\alpha < \alpha_1$ the polynomial P_α has a real root bigger than 1 and so P_α is not stable. The proof of the fact that P_α is not stable for all $\alpha < \alpha_{-1}$ is similar by considering $P_\alpha(-1)$.

---

[2]Viète's theorem states the relations between roots and coefficients of a polynomial as follows: Suppose that a polynomial $P(z) = \sum_{k=1}^{n} a_k z^k$ over $\mathbb{C}$ has roots $z_1, \ldots, z_n$. Then $a_0 = (-1)^n a_n \prod_{i=1}^{n} x_i$, $a_1 = (-1)^{n-1} a_n \sum_{i=1}^{n} (x_1 \cdots x_{i-1} x_{i+1} \cdots x_n)$, $\ldots$, $a_{n-2} = a_n \sum_{i \neq j} x_i x_j$, and $a_{n-1} = -a_n \sum_{i=1}^{n} x_i$. In other words, a_k is equal to $(-1)^{n-k} a_n$ times the sum of all products of k elements of $\{x_1, \ldots, x_n\}$.

Now we prove the "only if" part of the theorem. So we assume that the polynomial $P = P_0$ is stable. Then it follows from claims (i) and (ii) that there are precisely three bifurcation values $\{\alpha_1, \alpha_{-1}, \hat{\alpha}\}$ and

$$\max\{\alpha_1, \alpha_{-1}\} < 0 < \alpha_{max} = \hat{\alpha}. \tag{5}$$

Hence the desired inequalities of the theorem are fulfilled.

Next, we prove the "if" part of the theorem. By our assumptions we have precisely three bifurcation values of the parameter: α_1, α_{-1} and $\hat{\alpha}$, and for these values, (5) holds. Also, it follows from our assumptions that $P_{\hat{\alpha}}$ has two complex conjugate roots on the unit circle and (4) holds. Then by monotonicity property, $P_{\hat{\alpha}}(1) > 0$ and $P_{\hat{\alpha}}(-1) > 0$. Hence, either $P_{\hat{\alpha}}$ has two real zeros in the interval $(-1, 1)$ (which are the zeros x_3, x_4 of the second quadratic polynomial in (4) or the zeros x_3, x_4 are complex conjugate, in which case $|x_3| = |x_4| < 1$ because $x_3 x_4 = d$. So, in both cases the second quadratic polynomial in (4) is stable.

It is sufficient to show that P_α is stable for some $\alpha \in (\max\{\alpha_1, \alpha_{-1}\}, \hat{\alpha})$. We will prove that such an α can be taken slightly less than $\hat{\alpha}$. Let $\beta(\epsilon)$, $a(\epsilon)$ and $b(\epsilon)$ be real-valued functions on a small neighborhood of 0 such that

$$P_{\beta(\epsilon)}(x) = \Big(x^2 + a(\epsilon) \cdot x + 1 + \epsilon\Big) \left(x^2 + b(\epsilon) \cdot x + \frac{D}{1+\epsilon}\right). \tag{6}$$

By comparing the coefficients, one has that (6) is equivalent to the system

$$a(\epsilon) + b(\epsilon) = A, \tag{7}$$

$$\frac{D}{1+\epsilon} a(\epsilon) + (1+\epsilon) \cdot b(\epsilon) = C, \tag{8}$$

$$a(\epsilon) \cdot b(\epsilon) + 1 + \epsilon + \frac{D}{1+\epsilon} = B + \beta(\epsilon). \tag{9}$$

By solving the linear system of equations (7) and (8) we have unique functions $a(\epsilon)$ and $b(\epsilon)$ (for ϵ sufficiently small) because $D \neq 1$. For $\epsilon = 0$ their values are $a(0) = \frac{A-C}{1-D}$ and $b(0) = \frac{C-AD}{1-D}$. Then from (6) we get the function $\beta(\epsilon)$, and $\beta(0) = \hat{\alpha}$.

Differentiating both sides of (7) and (8) with respect to ϵ and solving the system of the obtained equations for $a'(\epsilon)$ and $b'(\epsilon)$, we get that for $\epsilon = 0$,

$$a'(0) = \frac{b(0) - Da(0)}{1-D} \quad \text{and} \quad b'(0) = -a'(0). \tag{10}$$

Then from (9) we obtain that

$$\beta'(0) = 1 - D + a'(0) \cdot b(0) + a(0) \cdot b'(0). \tag{11}$$

Plugging (10) into (11), we have

$$\begin{aligned}\beta'(0) &= \frac{(1-D)^2+\Big(b(0)-Da(0)\Big)\Big(b(0)-a(0)\Big)}{1-D}\\ &= \frac{\Big(2b(0)-a(0)-Da(0)\Big)^2+(1-D)^2\Big(4-\Big(a(0)\Big)^2\Big)}{4(1-D)}.\end{aligned}$$

Since $|D|<1$ and $|a(0)|=\frac{A-C}{1-D}<2$, we get $\beta'(0)>0$. Therefore, β is increasing in a small neighborhood of 0. By taking $\epsilon<0$ sufficiently close to 0, it follows from (6) that $P_{\beta(\epsilon)}$ is stable. The proof of the "if" part is completed and so is the proof of the theorem. ■

Remark 1. Note that the usual stability criterion (see [6]) for polynomials of degree 4 contains two degree 3 inequalities as follows:

$$\begin{aligned}|D|<1,\quad |A+C| &< 1+B+D, \quad and\\ |B(1-D)+D(1-D^2)+A(AD-C)| &< BD(1-D)+(1-D^2)+C(AD-C).\end{aligned}$$

From the proof of Theorem 2 (or directly from its statement), the following result easily follows.

Corollary 1. *Let $P_\alpha(x)=x^4+Ax^3+(B+\alpha)x^2+Cx+D\in\mathbb{R}[x]$ be a one-parameter family of polynomials with parameter $\alpha\in\mathbb{R}$. Then the set $J=\{\alpha: P_\alpha$ is stable$\}$ is either empty or is equal to a single open interval. More precisely, $J\neq\emptyset$ if and only if*

$$|D|<1,\ |A-C|<2(1-D)\quad \textit{and}\quad \max\{\alpha_1,\alpha_{-1}\}<\hat{\alpha},$$

where $\alpha_1=-P_0(1)$, $\alpha_{-1}=-P_0(-1)$ and $\hat{\alpha}$ is defined by (3); under these conditions $J=(\max\{\alpha_1,\alpha_{-1}\},\hat{\alpha})$.

The interval J from the above statement will be called later the *stability interval* of the family P_α.

The stability criterion for polynomials of degree three also can be derived from Theorem 2 as follows.

Corollary 2. *A polynomial $Q(x)=x^3+Ax^2+Bx+C\in\mathbb{R}[x]$ is stable if and only if*

$$|C|<1\quad \textit{and}\quad \min\{Q(1),-Q(-1),\hat{\alpha}\}>0,$$

where $\hat{\alpha}=-C^2+AC-B+1$.

Proof. Let $P(x)=xQ(x)$, then $P(1)=Q(1)$ and $P(-1)=-Q(-1)$. Applying Theorem 2 to $P(x)$ (with $D=0$) we have that Q is stable if and only if

$$|A-C|<2\quad \text{and}\quad \min\{Q(1),-Q(-1),\hat{\alpha}\}>0, \tag{12}$$

where $\hat{\alpha} = \hat{\alpha}(P) = \tilde{\alpha}(P) = -C^2 + AC - B + 1$. Since $\hat{\alpha} + Q(1) = (A - C + 2)(C + 1)$ and $\hat{\alpha} - Q(-1) = (A - C - 2)(C - 1)$, condition (10) is equivalent to

$$|C| < 1 \quad \text{and} \quad \min\{Q(1), -Q(-1), \hat{\alpha}\} > 0.$$

This completes the proof of the corollary. ∎

Now, for a family of polynomials of degree 3, a result similar to Corollary 1 is as follows.

Corollary 3. *Let $Q_\alpha(x) = x^3 + Ax^2 + (B+\alpha)x + C \in \mathbb{R}[x]$ be a one-parameter family of polynomials with parameter $\alpha \in \mathbb{R}$. Then the set $J = \{\alpha : Q_{\alpha}$ is stable$\}$ is either empty or is equal to a single open interval. More precisely, $J \neq \emptyset$ if and only if*

$$|C| < 1 \quad and \quad |A - C| < 2,$$

under these conditions $J = (\max\{\alpha_1, \alpha_{-1}\}, \hat{\alpha})$, where $\alpha_1 = -Q_0(1)$, $\alpha_{-1} = Q_0(-1)$ and $\hat{\alpha} = -C^2 + AC - B + 1$.

Proof. Let $P_\alpha(x) = xQ_\alpha(x)$. Then the stability intervals for the families Q_α and P_α are the same. From Corollary 1 we have that the stability interval of P_α exists if and only if

$$|C| < 1, \qquad \hat{\alpha} + P_0(1) > 0 \qquad \text{and} \qquad \hat{\alpha} + P_0(-1) > 0, \tag{13}$$

where $\hat{\alpha} = \hat{\alpha}(P_0) = -C^2 + AC - B + 1$. So we have inequalities

$$|C| < 1, \qquad \hat{\alpha} + Q_0(1) > 0 \qquad \text{and} \qquad \hat{\alpha} - Q_0(-1) > 0. \tag{14}$$

By simple computations one gets

$$\begin{aligned} \hat{\alpha} + Q_0(1) &= (C+1)(2 + A - C) \quad \text{and} \\ \hat{\alpha} - Q_0(-1) &= (1 - C)(2 - A + C). \end{aligned}$$

Therefore, (14) is equivalent to

$$|C| < 1 \qquad \text{and} \qquad |A - C| < 2.$$ ∎

Similarly to Corollary 2, one can easily get the criterion of stability for degree 2 polynomials $R(x) = x^2 + Ax + B$ from that for degree 3. The conditions of stability then are

$$\min\{R(1), R(-1), \hat{\alpha}\} > 0,$$

where $\hat{\alpha} = \hat{\alpha}(R) := -B + 1$.

3. Application to ACT Maps

As an example, we apply Corollary 1 to obtain stability regions of fixed points in the family of the Arneodo-Coullet-Tresser maps (ACT maps in abbreviation). We consider discrete dynamical systems induced by maps $F:\mathbb{R}^3 \to \mathbb{R}^3$ of the form

$$F(x,y,z) = (ax - b(y-z), bx + a(y-z), cx - dx^k + ez), \tag{15}$$

where a, b, c, d, e are real parameters with $bd \neq 0$ and $k > 1$ is an integer. These maps were introduced by Arneodo, Coullet and Tresser being motivated by the study of strange attractors in a family of differential equations on $\mathbb{R}^3$ with homoclinic points of Shilnikov type, refer to [3].

It is clear that the origin is a fixed point of F. By solving $F(x,y,z) = (x,y,z)$, we obtain that for even k, the map F has a unique nontrivial fixed point at

$$p_1 = (x_1, \frac{a^2+b^2-a}{b}x_1, \frac{(a-1)^2+b^2}{b}x_1)$$

where

$$x_1 = \sqrt[k-1]{\frac{bc-(1-e)[(a-1)^2+b^2]}{bd}} \tag{16}$$

and that for the case when k is odd and $\frac{bc-(1-e)[(a-1)^2+b^2]}{bd} > 0$, the map F has exactly two nontrivial fixed points at $\pm p_1$.

In this section, we will be concerned with stability region for the trivial fixed point i.e., regions in the parameter space for which these points are stable. As for stability regions for nontrivial fixed points and some periodic points, it can be done in similar way, see [4]. More precisely, we use the following definition.

Definition 2. Let $\mathbf{x} \mapsto G_{\mathbf{v}}(\mathbf{x}), \mathbf{x} \in X \subset \mathbb{R}^m, \mathbf{v} \in V \subset \mathbb{R}^l$ be a family of C^1 maps with parameter $\mathbf{v}$, and let $\mathbf{x}^*_{\mathbf{v}}$ be a fixed point of $G_{\mathbf{v}}$ for each $\mathbf{v} \in V$. A subset $J \subset V$ is called the *stability region* for the family of fixed points $\mathbf{x}^*_{\mathbf{v}}$ if J consists exactly of those parameters $\mathbf{v} \in V$ for which all eigenvalues of the Jacobian matrix $\frac{\partial}{\partial \mathbf{x}} G_{\mathbf{v}}(\mathbf{x}^*_{\mathbf{v}})$ lie inside the unit circle.

For the ACT family F, the Jacobian matrix of F at a point (x,y,z) is

$$\frac{\partial F(x,y,z)}{\partial(x,y,z)} = \begin{bmatrix} a & -b & b \\ b & a & -a \\ c - kdx^{k-1} & 0 & e \end{bmatrix}$$

and its characteristic polynomial is

$$P(\lambda) = \lambda^3 - (2a+e)\lambda^2 + [a^2 + b^2 + 2ae - bc + kbdx^{k-1})]\lambda - (a^2+b^2)e.$$

Note that the determinant of the Jacobian matrix of F is constant, that is, $|\frac{\partial F(x,y,z)}{\partial(x,y,z)}| = (a^2+b^2)e$, and if $e \neq 0$ then the map $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a diffeomorphism with the inverse

$$F^{-1}(x,y,z) = (\hat{x}, \frac{-bx+ay}{a^2+b^2} + \hat{z}, \hat{z}),$$

where $\hat{x} = \dfrac{ax+by}{a^2+b^2}$ and $\hat{z} = \dfrac{z - c\hat{x} + d\hat{x}^k}{e}$. So, F is a polynomial automorphism of $\mathbb{R}^3$ and thus, in the case of ACT maps, the Jacobian conjecture holds true (about the history and results concerning the Jacobian conjecture, see [1] and [11]). So to determine the local stability of fixed points, we need to know whether the characteristic polynomial of the Jacobian matrix of ACT map is stable, and for this we apply the results from Section 2.

In the following theorem, we deal with stability regions $J_{\text{tr}}(F_c)$ and $J_{\text{tr}}(F_{c,e})$ (recall that the subscripts here indicate the only parameters that vary). Note that if we find some functions $f_1(e)$, $f_2(e)$ of variables e, such that $J_{\text{tr}}(F_{c,e}) = \{(e,c) \in \mathbb{R}^2 : f_1(e) < c < f_2(e)\}$ then we will have for any c with $J_{\text{tr}}(F_c) \neq \emptyset$, that $J_{\text{tr}}(F_c) = \{c \in \mathbb{R} : f_1(e) < c < f_2(e)\}$, i.e., $J_{\text{tr}}(F_c)$ is described by the same inequalities. So in this case it is enough to give the corresponding formulas for $J_{\text{tr}}(F_{c,e})$ only.

Theorem 3. [stability regions for the trivial fixed point]. *Let F be the ACT family with $b \neq 0$. Let $J_{tr}(F_c)$ (resp. $J_{tr}(F_{c,e})$) denote the stability region of the origin for F_c (resp. for $F_{c,e}$). Then*

1. *$J_{tr}(F_c) \neq \emptyset$ if and only if*

$$-1 < (a^2+b^2)e < 1 \quad \textit{and} \quad 2a-2 < (a^2+b^2-1)e < 2a+2. \tag{17}$$

2. *For $F_{c,e}$, the following two statements hold:*
   (a) *Suppose $a^2+b^2-1 \leq 0$, then $J_{tr}(F_{c,e}) \neq \emptyset$.*
   (b) *Suppose $a^2+b^2-1 > 0$, then $J_{tr}(F_{c,e}) \neq \emptyset$ if and only if*

$$\max\left\{\frac{2a-2}{a^2+b^2-1}, \frac{-2a-2}{a^2+b^2-1}\right\} < \frac{1}{a^2+b^2}. \tag{18}$$

3. *If $J_{tr}(F_{c,e}) \neq \emptyset$, then*

$$J_{tr}(F_{c,e}) = \{(e,c) \in \mathbb{R}^2 : \max\{-c_1(e), c_{-1}(e)\} < -bc < \hat{c}(e)\},$$

where

$$\begin{aligned} c_1(e) &= (1-e)[(a-1)^2+b^2], \quad c_{-1}(e) = -(1+e)[(a+1)^2+b^2] \\ \textit{and} \quad \hat{c}(e) &= -(a^2+b^2-1)[(ae-1)^2+b^2e^2]. \end{aligned} \tag{19}$$

(*See Figures 1 and 2*).

Proof. Since $b \neq 0$, we may use $\alpha = -bc$ as parameter. Then the characteristic polynomial of the Jacobian matrix of F at the origin is

$$Q_\alpha(\lambda) = \lambda^3 - (2a+e)\lambda^2 + (a^2+b^2+2ae+\alpha)\lambda - (a^2+b^2)e.$$

By applying Corollary 3 to the above family Q_α with parameter α, we obtain that equation (17) is equivalent to the fact that the stability interval of Q_α is not empty, i.e., the stability region for F_c is not empty. Item 1 is completed.

The inequalities $a^2 + b^2 - 1 \leq 0$ and $b \neq 0$ imply $|a| < 1$, so the numbers $2a-2$ and $2a+2$ are of opposite signs, and therefore one gets item $2(a)$ from item 1 by taking $e = 0$.

For item 2(b), let us denote $e_\pm = \dfrac{\pm 1}{a^2+b^2}$, $e_l = \dfrac{2a-2}{a^2+b^2-1}$ and $e_r = \dfrac{2a+2}{a^2+b^2-1}$. The existence of e satisfying condition (18) is equivalent to the fact that the two intervals (e_-, e_+) and (e_l, e_r) overlap, i.e., $e_l < e_+$ and $e_- < e_r$. It is easy to see that the last two inequalities are the same as (18).

The stability interval of Q_α is given by $\max\{-Q_0(1), Q_0(-1)\} < \alpha < \hat{\alpha}$ (for the definition of $\hat{\alpha}$ see item 2 of Corollary 3). By evaluating the values $Q_0(1)$, $Q_0(-1)$ and $\hat{\alpha}$ and using the fact that $\alpha = -bc$, item 3 is proved. ■

Let us give some remarks on Figure 1. The dashed lines there are $e = \pm\frac{1}{a^2+b^2}$, which corresponds to the cases when $C = \pm 1$ in Corollaries 2 and 3. So the stability region $J_{\mathrm{tr}}(F_{c,e})$ must belong to the strip between the dashed lines. It is easy to see that the intersection of the bifurcation curves $c = \frac{\hat{c}(e)}{-b}$ and $c = \frac{c_1(e)}{b}$ consists of either one or two points depending on whether $a^2 + b^2 - 1$ is zero or not. In the former case, the point of intersection has coordinates $(e, c) = (\frac{1}{a^2+b^2}, 0)$, which corresponds to the eigenvalues $\lambda_1 = 1$ and $\lambda_{2,3} = a \pm i\sqrt{1-a^2}$. In the latter case, the two intersection points are $M'(e', c')$ and $M''(e'', c'')$, where $c' = \frac{(1-\frac{1}{a^2+b^2})[(a-1)^2+b^2]}{b}$, $e' = \frac{1}{a^2+b^2}$, $c'' = \frac{(1-\frac{2a-2}{a^2+b^2-1})[(a-1)^2+b^2]}{b}$, and $e'' = \frac{2a-2}{a^2+b^2-1}$, which corresponds to the eigenvalues $\lambda_1' = 1$, $|\lambda_2'| = |\lambda_3'| = 1$ and $\lambda_1'' = \lambda_2'' = 1$, $\lambda_3'' \in \mathbb{R}$. Note that the coordinates of M' and M'' satisfy the equalities $C = 1$ and $C = A+2$ respectively in Corollary 3. Furthermore, the condition (17) on existence of stability region $J_{\mathrm{tr}}(F_{c,e})$ implies that $e'' < e'$ if $a^2+b^2-1 > 0$. Similar geometric interpretation of the conditions of Corollary 3 and Theorem 3 can be done in terms of intersections, say N' and N'', of the lines $c = \hat{c}$ and $c = c_{-1}$. Note that $J_{\mathrm{tr}}(F_{c,e})$ has either two "sides" or three "sides" depending on whether both points M'' and N'' lie outside the strip $|e| < \frac{1}{a^2+b^2}$ or one of M'' and N'' lies inside (one can easily see that the points M'' and N'' cannot lie simultaneously inside the strip).

Remark 2. Note that formula (16) for appearance of nontrivial fixed points of ACT map F coincides with the bifurcation equation $c = \frac{c_1(e)}{b}$. So F has transcritical or pitchfork) bifurcation at the bifurcation curve c_1 in Figure 1 depending on whether k is even or odd. Moreover, by solving the system of equations $F(x, y, z) = (-x, -y, -z)$ and $F(-x, -y, -z) = (x, y, z)$, we get that F has periodic points of period 2 symmetric to the origin, say $\pm p_2$, if and only if k is odd and $\frac{bc-(1+e)[(a+1)^2+b^2]}{bd} > 0$; in this case, $p_2 = (x_2, \frac{-a^2-b^2-a}{b}x_2, \frac{-(a+1)^2-b^2}{b}x_2)$, where

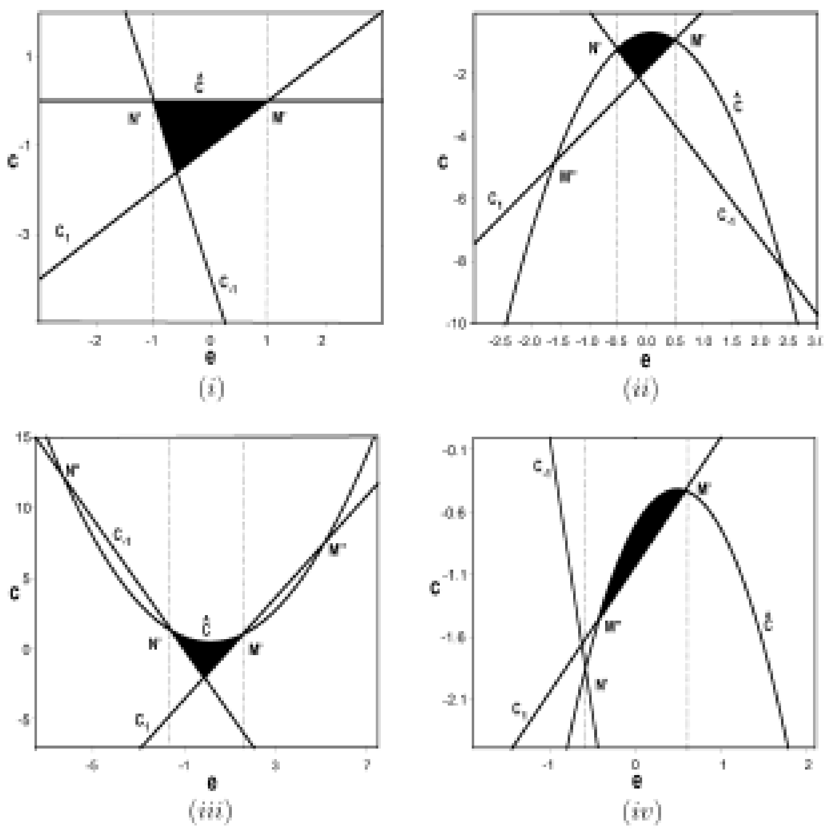

Fig. 1. The graphs of the bifurcation curves $c = \frac{c_1(e)}{b}$, $c = \frac{c_{-1}(e)}{-b}$ and $c = \frac{\hat{c}(e)}{-b}$, indicated simply as c_1, c_{-1} and $\hat{c}$, are shown in the (e, c)-plane. The dashed lines in the figures are $e = \frac{\pm 1}{a^2+b^2}$. In figures $1(i)$-(iii), the stability regions $J_{\text{tr}}(F_{c,e})$ (shaded in black) are shown for three cases: when $a^2+b^2-1 = 0, > 0$ and < 0, all together with $b < 0$, namely (i) $a = 0.6$ and $b = -0.8$, (ii) $a = 0.2$ and $b = -1.4$, and (iii) $a = 0.1$ and $b = -0.8$. Figure $1(iv)$ corresponds to the subcase of (ii) when the stability region has two sides (M'' lies inside the strip between the dashed lines); here $a = 0.85$ and $b = -1$.

$$x_2 = \sqrt[k-1]{\frac{bc-(1+e)[(a+1)^2+b^2]}{bd}}. \tag{20}$$

Formula (20) coincides with the bifurcation equation $c = \frac{c_{-1}(e)}{-b}$. So when k is odd, F has period-doubling bifurcation at the bifurcation curve c_{-1}. Also it can be shown that at the bifurcation curve $\hat{c}$ the map F has Hopf bifurcation generically.

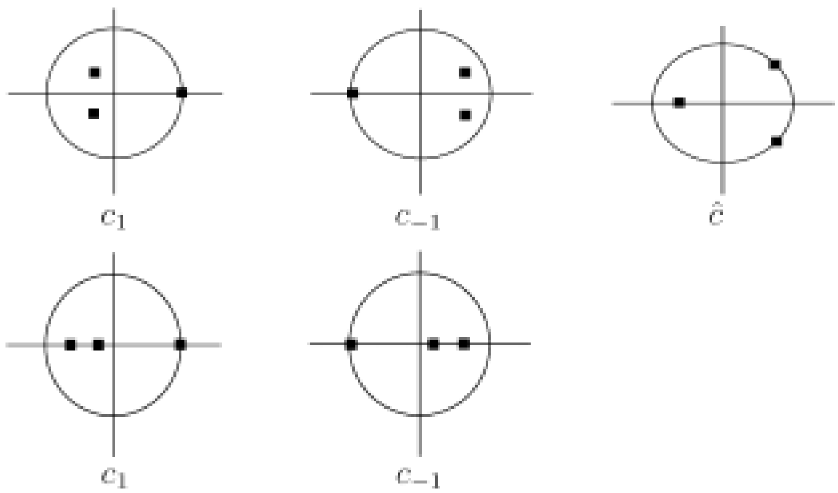

Fig. 2. Eigenvalues of ACT map at the bifurcation curves c_1, c_{-1}, and $\hat{c}$ on the border of stability region, which correspond to transcritical or pitchfork bifurcation (depending on whether k is even or odd) at c_1, to period-doubling biburcation when k is odd at c_{-1}, and to Hopf bifurcation at $\hat{c}$.

Acknowledgment

Li and Malkin are grateful to the Institute of Mathematics at Academia Sinica of Taiwan for hospitality during their visit. Li was partially supported by NSC grant 93-2115-M-018-001 and Malkin was partially supported by RFBR grants 05-01-00501 and 05-01-00558.

References

1. H. Bass, E. H. Connell, and D. Wright, The Jacobian conjecture: Reduction of degree and formal expansion of the inverse, *Bull. Amer. Math. Soc.*, **7** (1982), 287-330.
2. A. Cohn, Über die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise, *Math. Z.*, **14** (1922), 110-148.
3. B.-S. Du, Bifurcation of periodic points of some diffeomorphisms on $\mathbb{R}^3$, *Nonlinear Analysis, Theory, Methods & Applications*, **9** (1985), 309-319.

4. B.-S. Du, M.-C. Li, and M. Malkin, Topological horseshoes for Arneodo-Coullet-Tresser maps, *Rugular & Chaotic Dynamics*, **11** (2006), 181-190.
5. E. I. Jury, *Inners and stability of dynamic systems*, Wiley-Interscience, New York, 1974.
6. J. P. LaSalle, *The stability and control of discrete processes*, Springer-Verlag, New York, 1986.
7. M. Marden, *Geometry of polynomials*, 2nd ed., Mathematical Surveys and Monographs, vol. 3, American Mathematical Society, Providence, R.I., 1989.
8. M. Mignotte and D. Stefanescu, *Polynomials: An Algorithmic Approach*, Springer, Singapore, New York, 1999.
9. C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, second ed., Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1999.
10. W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill Book Co., New York, 1987.
11. W. Rudin, *Injective polynomial maps are automorphisms*, Amer. Math. Monthly **102** (1995), 540-543.

B.-S. Du
Institute of Mathematics,
Academia Sinica,
Taipei 115,
Taiwan
E-mail: dubs@math.sinica.edu.tw

S.-R. Hsiau
Department of Mathematics,
National Changhua University of Education,
Changhua 500,
Taiwan
E-mail: srhsiau@math.ncue.edu.tw

M.-C. Li
Department of Applied Mathematics,
National Chiao Tung University,
Hsinchu 300,
Taiwan
E-mail: mcli@math.nctu.edu.tw

M. Malkin
Department of Mathematics,
Nizhny Novgorod State University,
Nizhny Novgorod,
Russia
E-mail: malkin@uic.nnov.ru