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NONLOCAL CAUCHY PROBLEM FOR SECOND ORDER INTEGRODIFFERENTIAL EVOLUTION EQUATIONS IN BANACH SPACES

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Abstract. In this paper we derive a set of sufficient conditions for the existence of mild solutions of second order nonlinear integrodifferential evolution equations with nonlocal conditions in Banach spaces. The results are obtained by applying the Schaefer fixed point theorem. An application is provided to illustrate the technique.

1. INTRODUCTION

Byszewski [9] has studied the existence and uniqueness of mild, strong and classical solutions of the following nonlocal Cauchy problem:

$$\frac{du(t)}{dt} + Au(t) = f(t, u(t)), \qquad t \in (0, a] \tag{1}$$

$$u(t_0) + g(t_1, t_2, ..., t_p, u(.)) = u_0 \tag{2}$$

where $0 \leq t_0 < t_1 < ... < t_p \leq a, a > 0, -A$ is the infinitesimal generator of a C_0-semigroup in a Banach space $X, u_0 \in X$ and $f : [0, a] \times X \to X$, $g : [0, a]^p \times X \to X$ are given functions. Subsequently he has investigated the same type of problem for different kinds of evolution equations in Banach spaces [10-14]. Balachandran and Ilamaran [1, 5, 6], Balachandran and Chandrasekaran [2-4], Dauer and Balachandran [15] and Balachandran et al [7] have studied the nonlocal Cauchy problem for various classes of delay differential and integrodifferential equations. The nonlocal Cauchy problem for integrodifferential equation

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with resolvent operators has been discussed by Lin and Liu [20]. Ntouyas and Tsamatos [23]and Ntouyas [21] have established the global existence of solutions of semilinear evolution equations and functional integrodifferential equations with nonlocal conditions respectively. Physical motivation for this kind of problem is given in [16, 18].

In many cases it is advantageous to treat the second order abstract differential equations directly rather than to convert them to first order systems. For example Fitzgibbon [17] used the second order abstract differential equations for establishing the boundedness of solutions of the equation governing the transverse motion of an extensible beam. The occurrence of second order equations has been found in [8, 27]. Ntouyas [22] and Ntouyas and Tsamatos [24] have established the existence of solutions of second order delay integrodifferential equations with nonlocal conditions. In this paper we shall study the problem of existence of solutions for second order integrodifferential evolution equations with nonlocal conditions in Banach spaces by using the Schaefer fixed point theorem.

2. Preliminaries

Let X be a real reflexive Banach space and, for each $t \in [0,T]$, let $A(t) : X \to X$ be a closed densely defined operator. The fundamental solution for the second order evolution equation

$$x''(t) = A(t)x(t) \tag{3}$$

developed by Kozak [19] is as follows. Let us assume that the domain of $A(t)$ does not depend on $t \in [0,T]$ and denote it by $D(A)$ (for each $t \in [0,T]$, $D(A(t)) = D(A)$).

Definition 2.1. [19]. A family $\mathcal{S}$ of bounded linear operators $S(t,s) : X \to X$, $t,s \in [0,T]$, is called a fundamental solution of a second order equation if:

[Z_1] For each $x \in X$ the mapping $[0,T] \times [0,T] \ni (t,s) \to S(t,s)x \in X$ is of class C^1 and

(i) for each $t \in [0,T]$, $S(t,t) = 0$,

(ii) for all $t,s \in [0,T]$, and for each $x \in X$,

$$\frac{\partial}{\partial t}S(t,s)\Big|_{t=s} x = x, \qquad \frac{\partial}{\partial s}S(t,s)\Big|_{t=s} x = -x.$$

[Z_2] For all $t,s \in [0,T]$, if $x \in D(A)$, then $S(t,s)x \in D(A)$, the mapping $[0,T] \times [0,T] \ni (t,s) \to S(t,s)x \in X$ is of class C^2 and

(i) $\dfrac{\partial^2}{\partial t^2}S(t,s)x = A(t)S(t,s)x,$

(ii) $\dfrac{\partial^2}{\partial s^2}S(t,s)x = S(t,s)A(s)x,$

(iii) $\dfrac{\partial}{\partial s}\dfrac{\partial}{\partial t}S(t,s)\Big|_{t=s} x = 0.$

[Z_3] For all $t, s \in [0,T]$, if $x \in D(A)$, then $\dfrac{\partial}{\partial s}S(t,s)x \in D(A)$, there exist $\dfrac{\partial^2}{\partial t^2}\dfrac{\partial}{\partial s}S(t,s)x$, $\dfrac{\partial^2}{\partial s^2}\dfrac{\partial}{\partial t}S(t,s)x$ and

(i) $\dfrac{\partial^2}{\partial t^2}\dfrac{\partial}{\partial s}S(t,s)x = A(t)\dfrac{\partial}{\partial s}S(t,s)x,$

(ii) $\dfrac{\partial^2}{\partial s^2}\dfrac{\partial}{\partial t}S(t,s)x = \dfrac{\partial}{\partial t}S(t,s)A(s)x,$

and the mapping $[0,T]\times[0,T] \ni (t,s) \to A(t)\dfrac{\partial}{\partial s}S(t,s)x$ is continuous.

The main aim of this paper is to establish the existence of mild solutions of the following nonlinear second order integrodifferential evolution equation with nonlocal condition

$$x''(t) = A(t)x(t)+f(t,x(t),x'(t))+\int_0^t q(t,s,x(s),x'(s))ds,\ t \in J = [0,T], \tag{4}$$

$$x(0) + g(x) = x_0, \qquad x'(0) = y_0, \tag{5}$$

where $x_0, y_0 \in X$, $A(t) : X \to X$ is a closed densely defined operator, $f : J \times X \times X \to X$, $q : J \times J \times X \times X \to X$ and $g : C(J;X) \to X$ are given functions. Similar to Pazy [25] we define the following solution.

Definition 2.2. Any continuous function $x : [0,T] \to X$ is called a mild solution of the problem (4)-(5) if $x(t) \in D(A(t))$, for each $t \in [0,T]$ and satisfies the following integral equation

$$\begin{aligned} x(t) =& -\frac{\partial}{\partial s}S(t,s)\Big|_{s=0}[x_0 - g(x)] + S(t,0)y_0 + \int_0^t S(t,s)f(s,x(s),x'(s))ds \\ &+ \int_0^t S(t,s)\int_0^s q(s,\tau,x(\tau),x'(\tau))d\tau ds. \end{aligned} \tag{6}$$

For simplicity let us introduce the notation $C(t,0) = -\frac{\partial}{\partial s}S(t,s)\big|_{s=0}$. Then the equation (6) becomes as

$$x(t) = C(t,0)[x_0 - g(x)] + S(t,0)y_0 + \int_0^t S(t,s)f(s,x(s),x'(s))ds$$
$$+ \int_0^t S(t,s)\int_0^s q(s,\tau,x(\tau),x'(\tau))d\tau ds, \quad t \in J.$$

To establish our main theorem we need the following assumptions.

(H_1) $x(t) \in D(A(t))$, for each $t \in [0,T]$.

(H_2) There exists a fundamental solution $S(t,s)$ of (3).

(H_3) $S(t,s)$ is compact for each $t,s \in [0,T]$ and there exist positive constants M, M^* and N, N^* such that

$$M = \sup\{\|S(t,s)\| : t,s \in J\}, \quad M^* = \sup\{\|C(t,0)\| : t \in J\},$$
$$\text{and} \quad N = \sup\{\|\tfrac{\partial}{\partial t}S(t,s)\| : t,s \in J\}, N^* = \sup\{\|\tfrac{\partial}{\partial t}C(t,0)\| : t \in J\}$$

respectively. Further, for $x \in X$ and $t_1, t_2, s \in J$,

$$[\frac{\partial}{\partial t_1}C(t_1,0) - \frac{\partial}{\partial t_2}C(t_2,0)]x \to 0, \ [\frac{\partial}{\partial t_1}S(t_1,s) - \frac{\partial}{\partial t_2}S(t_2,s)]x \to 0, \text{as } t_1 \to t_2.$$

(H_4) $g : C(J;X) \to X$ and there exists a constant $G > 0$ such that

$$\|g(x)\| \le G, \quad \text{for} \quad x \in C(J;X)$$

and the set $\{x(0) : x \in C(J;X), \|x\| \le k, x(0) = x_0 - g(x)\}$ is precompact in X

(H_5) $f(t,.,.) : X \times X \to X$ is continuous for each $t \in J$ and the function $f(.,x,y) : J \to X$ is strongly measurable for each $(x,y) \in X \times X$.

(H_6) For every positive constant k there exists $\alpha_k \in L^1(J)$ such that

$$\sup_{\|x\|,\|y\| \le k} \|f(t,x,y)\| \le \alpha_k(t) \quad \text{for a.a} \ \ t \in J.$$

(H_7) There exists an integrable function $m : J \to [0,\infty)$ such that

$$\|f(t,x,y)\| \le m(t)\Omega(\|x\| + \|y\|), \quad t \in J, \ \ x,y \in X,$$

where $\Omega : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(H_8) $q(t,s,\cdot,\cdot) : X \times X \to X$ is continuous for each $t,s \in J$ and the function $q(\cdot,\cdot,x,y) : J \times J \to X$ is strongly measurable for each $(x,y) \in X \times X$.

(H_9) For every positive constant k there exists $\beta_k \in L^1(J)$ such that

$$\sup_{\|x\|,\|y\|\leq k} \| \int_0^t q(t,s,x,y)ds\| \leq \beta_k(t), \quad t \in J \text{ a.e.}$$

(H_{10}) There exists an integrable function $n : J \to [0,\infty)$ such that

$$\| \int_0^t q(t,s,x,y)ds\| \leq n(t)\Omega_0(\|x\| + \|y\|), \quad t \in J, \quad x, y \in X,$$

where $\Omega_0 : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function and

$$(M+N)\int_0^T \tilde{m}(s)ds < \int_c^\infty \frac{ds}{\Omega(s)+\Omega_0(s)},$$

where $\tilde{m}(t) = \max\{m(t), n(t)\}$, $c = (M^*+N^*)(\|x_0\|+G)+(M+N)\|y_0\|$.

Lemma. (Schaefer's Theorem [26].) *Let E be a normed linear space. Let $F : E \to E$ be a completely continuous operator,that is, it is continuous and the image of any bounded set is contained in a compact set, and let*

$$\zeta(F) = \{x \in E : x = \lambda Fx \ \textit{ for some } 0 < \lambda < 1\}.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

3. Main Result

Theorem 3.1. *If the assumptions $(H_1)-(H_{10})$ hold, then the problem $(4)-(5)$ has a mild solution on J.*

Proof. Consider the space $Z = C^1(J,X)$ with norm $\|x\|^* = \max\{\|x\|_0, \|x\|_1\}$ where $\|x\|_0 = \sup\{|x(t)| : 0 \leq t \leq T\}$, $\|x\|_1 = \sup\{|x'(t)| : 0 \leq t \leq T\}$.

In order to establish the existence of a mild solution to the problem (4)-(5), we have to apply the Lemma. First we obtain *a priori* bounds for the following nonlinear operator equation

(7) $$x(t) = \lambda Fx(t), \quad \lambda \in (0,1)$$

where $F : Z \to Z$ is defined by

(8) $$\begin{aligned} Fx(t) = \ & C(t,0)[x_0 - g(x)] + S(t,0)y_0 + \int_0^t S(t,s)f(s,x(s),x'(s))ds \\ & + \int_0^t S(t,s)\int_0^s q(s,\tau,x(\tau),x'(\tau))d\tau ds, \ t \in J. \end{aligned}$$

We have from (7)

$$\begin{aligned}\|x(t)\| \le\ & M^*[\|x_0\| + G] + M\|y_0\| \\ & + \int_0^t Mm(s)\Omega(\|x(s)\| + \|x'(s)\|)ds \\ & + \int_0^t Mn(s)\Omega_0(\|x(s)\| + \|x'(s)\|)ds.\end{aligned}$$

Denoting by $v(t)$ the right-hand side of the above inequality we have

$$\begin{aligned}\|x(t)\| &\le v(t),\ \ t \in J, \\ v(0) &= M^*[\|x_0\| + G] + M\|y_0\|, \\ v'(t) &= Mm(t)\Omega(\|x(t)\| + \|x'(t)\|) + +Mn(t)\Omega_0(\|x(t)\| + \|x'(t)\|),\ t \in J.\end{aligned}$$

From (7) and (8), we have

$$\begin{aligned}x'(t) =\ & \lambda \frac{\partial}{\partial t} C(t,0)[x_0 - g(x)] \\ & + \lambda \frac{\partial}{\partial t} S(t,0) y_0 + \lambda \int_0^t \frac{\partial}{\partial t} S(t,s) f(s, x(s), x'(s)) ds \\ & + \lambda \int_0^t \frac{\partial}{\partial t} S(t,s) \int_0^s q(s,\tau,x(\tau),x'(\tau)) d\tau ds,\ t \in J,\end{aligned} \tag{9}$$

and

$$\begin{aligned}\|x'(t)\| \le\ & N^*[\|x_0\| + G] + N\|y_0\| + N\int_0^t m(s)\Omega(\|x(s)\| + \|x'(s)\|)ds \\ & + N\int_0^t n(s)\Omega_0(\|x(s)\| + \|x'(s)\|)ds.\end{aligned}$$

Denoting by $r(t)$ the right-hand side of the above inequality we get

$$\begin{aligned}\|x'(t)\| &\le r(t),\ \ t \in J, \\ r(0) &= N^*[\|x_0\| + G] + N\|y_0\|, \\ r'(t) &= Nm(t)\Omega(\|x(t)\| + \|x'(t)\|) + Nn(t)\Omega_0(\|x(t)\| + \|x'(t)\|),\ t \in J.\end{aligned}$$

Let $\qquad w(t) = v(t) + r(t), \qquad t \in J.$

Then $\qquad w(0) = v(0) + r(0) = c,$ and

$$\begin{aligned} w'(t) &= v'(t) + r'(t) \\ &\leq (M+N)\tilde{m}(t)[\Omega_0(v(t)+r(t)) + \Omega(v(t)+r(t))] \\ &\leq (M+N)\tilde{m}(t)[\Omega_0(w(t)) + \Omega(w(t)). \end{aligned}$$

This gives

$$\begin{aligned} \int_{w(0)}^{w(t)} \frac{ds}{\Omega_0(s)+\Omega(s)} &\leq (M+N)\int_0^t \tilde{m}(s)ds \\ &\leq (M+N)\int_0^T \tilde{m}(s)ds \\ &< \int_c^\infty \frac{ds}{\Omega_0(s)+\Omega(s)}. \end{aligned}$$

This inequality implies that there is a constant K such that

$$w(t) = v(t) + r(t) \leq K, \qquad t \in J.$$

Thus

$$\|x(t)\| \leq v(t), \quad \|x'(t)\| \leq r(t), \quad t \in J,$$

and hence

$$\|x\|^* \leq K,$$

where K depends only on T and on the functions m, Ω_0 and Ω.

Now we shall prove that the operator $F : Z \to Z$ is a completely continuous operator. Let $B_k = \{x \in Z : \|x\|^* \leq k\}$ for some $k \geq 1$. We first show that F maps B_k into an equicontinuous family. Let $x \in B_k$ and $t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \leq T$,

$$\begin{aligned} &\|(Fx)(t_1) - (Fx)(t_2)\| \\ &\leq \|[C(t_1,0) - C(t_2,0)][x_0 - g(x)]\| + \|[S(t_1,0) - S(t_2,0)]y_0\| \\ &\quad + \int_0^{t_1} \|S(t_1,s) - S(t_2,s)\| [\alpha_k(s) + \beta_k(s)]ds \\ &\quad + \int_{t_1}^{t_2} \|S(t_2,s)\| [\alpha_k(s) + \beta_k(s)]ds \\ &\quad \to 0 \quad \text{as} \quad t_1 \to t_2 \end{aligned}$$

and similarly

$$
\begin{aligned}
&\|(Fx)'(t_1) - (Fx)'(t_2)\| \\
&\le \; [\frac{\partial}{\partial t_1}C(t_1,0) - \frac{\partial}{\partial t_2}C(t_2,0)][x_0 - g(x)]\| + \|[\frac{\partial}{\partial t_1}S(t_1,0) - \frac{\partial}{\partial t_2}S(t_2,0)]y_0\| \\
&\quad + \int_0^{t_1} \|[\frac{\partial}{\partial t_1}S(t_1,s) - \frac{\partial}{\partial t_2}S(t_2,s)]\|[\alpha_k(s) + \beta_k(s)]ds \\
&\quad + \int_{t_1}^{t_2} \|\frac{\partial}{\partial t_2}S(t_2,s)\|[\alpha_k(s) + \beta_k(s)]ds \quad \to 0,
\end{aligned}
$$

as $t_1 \to t_2$. Thus F maps B_k into an equicontinuous family of functions. It is easy to see that the family FB_k is uniformly bounded.

Next we show that $\overline{FB_k}$ is compact. Since we have shown FB_k is an equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that F maps B_k into a precompact set in X.

Let $0 < t \le T$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $x \in B_k$ we define

$$
\begin{aligned}
(F_\epsilon x)(t) \,=\, & C(t,0)[x_0 - g(x)] + S(t,0)y_0 + \int_0^{t-\epsilon} S(t,s)f(s,x(s),x'(s))ds \\
& + \int_0^{t-\epsilon} S(t,s)\int_0^s q(s,\tau,x(\tau),x'(\tau)d\tau ds, \quad t \in J.
\end{aligned}
$$

Since $S(t,s)$ is a compact operator, the set $Y_\epsilon(t) = \{(F_\epsilon x)(t) : x \in B_k\}$ is precompact in X for every ϵ, $0 < \epsilon < t$. Moreover for every $x \in B_k$ we have

$$
\begin{aligned}
\|(Fx)(t) - (F_\epsilon x)(t)\| \,\le\, & \int_{t-\epsilon}^t \|S(t,s)f(s,x(s),x'(s))\|ds \\
& + \int_{t-\epsilon}^t \|S(t,s)\int_0^s q(s,\tau,x(\tau),x'(\tau)d\tau\|ds \\
\le\, & \int_{t-\epsilon}^t \|S(t,s)\|[\alpha_k(s) + \beta_k(s)]ds \quad \to 0 \quad \text{as} \quad \epsilon \to 0
\end{aligned}
$$

and

$$
\begin{aligned}
\|(Fx)'(t) - (F_\epsilon x)'(t)\| \,\le\, & \int_{t-\epsilon}^t \|\frac{\partial}{\partial t}S(t,s)f(s,x(s),x'(s))\|ds \\
& + \int_{t-\epsilon}^t \|\frac{\partial}{\partial t}S(t,s)\int_0^s q(s,\tau,x(\tau),x'(\tau)d\tau\|ds \\
\le\, & \int_{t-\epsilon}^t \|\frac{\partial}{\partial t}S(t,s)\|[\alpha_k(s) + \beta_k(s)]ds \quad \to 0 \quad \text{as} \quad \epsilon \to 0.
\end{aligned}
$$

Therefore there are precompact sets arbitrarily close to the set $\{(Fx)(t) : x \in B_k\}$. Hence the set $\{(Fx)(t) : x \in B_k\}$ is precompact in X.

It remains to show that $F : Z \to Z$ is continuous. Let $\{x_n\}_0^\infty \subseteq Z$ with $x_n \to x$ in Z. Then there is an integer ν such that $\|x_n(t)\| \le \nu, \|x'_n(t)\| \le \nu$ for all n and $t \in J$, so $\|x(t)\| \le \nu, \|x'(t)\| \le \nu$ and $x, x' \in B_\nu$. By (H_5) and (H_8)

$$f(s, x_n(s), {x_n}'(s)) \to f(s, x(s), x'(s)),$$
$$\int_0^t q(t, s, x_n(s), x'_n(s))ds \to \int_0^t q(t, s, x(s), x'(s))ds$$

for each $t \in J$ and since

$$\|f(t, x_n(t), {x_n}'(t)) - f(t, x(t), x'(t))\| \le 2\alpha_\nu(t),$$
$$\|\int_0^t q(t, s, x_n(s), x'_n(s))ds - \int_0^t q(t, s, x(s), x'(s))ds\| \le 2\beta_\nu(t)$$

we have by dominated convergence theorem

$$\begin{aligned}
&\|Fx_n - Fx\| \\
&= \sup_{t\in J}\Big\| \int_0^t S(t,s)f(s, x_n(s), x'_n(s))ds - \int_0^t S(t,s)f(s, x(s), x'(s))ds \\
&\quad + \int_0^t S(t,s)[\int_0^s q(s, \tau, x_n(\tau), x'_n(\tau))d\tau - \int_0^s q(s, \tau, x(\tau), x'(\tau))d\tau]ds\Big\| \\
&\le \int_0^t \|S(t,s)[f(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s))]\|ds \\
&\quad + \int_0^t \|S(t,s)\int_0^s [q(s, \tau, x_n(\tau), x'_n(\tau)) - q(s, \tau, x(\tau), x'(\tau))]d\tau\|ds \\
&\quad \to 0 \ \text{ as } \ n \to \infty
\end{aligned}$$

and

$$\begin{aligned}
&\|(Fx_n)' - (Fx)'\| \\
&= \sup_{t\in J}\Big\| \int_0^t \frac{\partial}{\partial t}S(t,s)f(s, x_n(s), x'_n(s))ds - \int_0^t \frac{\partial}{\partial t}S(t,s)f(s, x(s), x'(s))ds \\
&\quad + \int_0^t \frac{\partial}{\partial t}S(t,s)[\int_0^s q(s, \tau, x_n(\tau), x'_n(\tau))d\tau - \int_0^s q(s, \tau, x(\tau), x'(\tau))d\tau]ds\Big\| \\
&\le \int_0^t \|\frac{\partial}{\partial t}S(t,s)[f(s, x_n(s), x'_n(s)) - f(s, x(s), x'(s))]\|ds \\
&\quad + \int_0^t \|\frac{\partial}{\partial t}S(t,s)\int_0^s [q(s, \tau, x_n(\tau), x'_n(\tau) - q(s, \tau, x(\tau), x'(\tau))]d\tau\|ds \\
&\quad \to 0 \ \text{ as } \ n \to \infty.
\end{aligned}$$

Thus F is continuous. This completes the proof that F is completely continuous.

We have already proved that the set $\zeta(F) = \{x \in Z : x = \lambda Fx, \lambda \in (0,1)\}$ is bounded. Hence by the Schaefer fixed point theorem the operator F has a fixed point in Z. This means that any fixed point of F is a mild solution of (4)-(5) on J satisfying $(Fx)(t) = x(t)$. Thus the initial value problem (4)-(5) has at least one mild solution on J.

4. Application

As an application of the above theorem we consider the following integrodifferential equation with nonlocal condition of the form

$$x''(t) = A(t)x(t) + f(t, x(t), \textstyle\int_0^t a(t,s)h(s, x(s), x'(s)ds, x'(t)), \quad t \in J \tag{10}$$

$$x(0) + g(x) = x_0, \tag{11}$$

where $A(t), g$ are as in the previous section and $f : J \times X \times X \times X \to X, a : J \times J \to R, h : J \times X \times X \to X$ are given functions. If $x(t)$ is a solution of the problem (10)-(11) then

$$\begin{aligned} x(t) &= C(t,0)[x_0 - g(x)] + S(t,0)y_0 \\ &\quad + \int_0^t S(t,s)f(s, x(s), \\ &\qquad \int_0^s a(s,\tau)h(\tau, x(\tau), x'(\tau))d\tau, x'(s))ds, \ t \in J. \end{aligned} \tag{12}$$

The above equation is more general than equation (10) and every solution of this is called mild solution of (10)-(11).

Assume the following conditions:

(C_1) $f(t,.,.,.) : X \times X \times X \to X$ is continuous for each $t \in J$ and the function $f(.,x,y,z) : J \to X$ is strongly measurable for each $(x,y,z) \in X \times X \times X$.

(C_2) For every positive constant k there exists $\gamma_k \in L^1(J)$ such that

$$\sup_{\|x\|,\|y\|,\|z\| \le k} \|f(t,x,y,z)\| \le \gamma_k(t) \quad \text{for a.a } t \in J.$$

(C_3) $h : J \times X \times X \to X$ is continuous and there exists an integrable function $m : J \to [0,\infty)$ such that

$$\|h(t,x,y)\| \le m(t)\Omega_1(\|x\| + \|y\|), \quad t \in J, \ \ x, y \in X,$$

where $\Omega_1 : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(C_4) $f : J \times X \times X \times X \to X$ is continuous and there exists an integrable function $p : J \to [0, \infty)$ such that

$$\|f(t, x, y, z)\| \leq p(t)\Omega_2(\|x\| + \|y\| + \|z\|), \quad t \in J, \ \ x, y, z \in X,$$

where $\Omega_2 : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

(C_5) $a : J \times J \to R$ is measurable and there exists a constant L such that

$$|a(t, s)| \leq L, \ \text{for} \ \ t \geq s \geq 0.$$

Theorem 4.1. *If the assumptions $(H_1) - (H_4)$ and $(C_1) - (C_5)$ hold and if*

$$\int_0^T \tilde{m}(s) ds < \int_c^\infty \frac{ds}{\Omega_1(s) + \Omega_2(s)},$$

where $\tilde{m}(t) = \max\{(M + N)p(t), Lm(t)\}$, then the problem (10)-(11) has at least one mild solution on J.

Proof. Consider the space $Z = C^1(J, X)$ with norm $\|x\|^*$. In order to study the problem (10)-(11), we have to apply the Lemma to the following equation

$$x(t) = \lambda P x(t), \qquad \lambda \in (0, 1), \tag{13}$$

where $P : Z \to Z$ is defined by

$$\begin{aligned} Px(t) \ &= \ C(t, 0)[x_0 - g(x)] + S(t, 0)y_0 \\ &\quad + \int_0^t S(t, s) f(s, x(s), \\ &\qquad \int_0^s a(s, \tau) h(\tau, x(\tau), x'(\tau)) d\tau, x'(s)) ds, \quad t \in J, \end{aligned} \tag{14}$$

we have

$$\begin{aligned} \|x(t)\| \ &\leq \ M^*[\|x_0\| + G] + M\|y_0\| \\ &\quad + \int_0^t M p(s) \Omega_2(\|x(s)\| + L \int_0^s m(\tau) \Omega_1(\|x(\tau)\| \\ &\quad + \|x'(\tau)\|) d\tau + \|x'(s)\|) ds. \end{aligned}$$

Denoting by $v(t)$ the right-hand side of the above inequality we have

$$\|x(t)\| \ \leq \ v(t), \ \ t \in J, \text{and} \ \ v(0) = M^*[\|x_0\| + G] + M\|y_0\|,$$

$$v'(t) \ = \ M p(t) \Omega_2(\|x(t)\| + L \int_0^t m(\tau) \Omega_1(\|x(\tau)\| + \|x'(\tau)\|) d\tau + \|x'(t)\|), \ t \in J.$$

From (13) and (14), we have

$$
\begin{aligned}
x'(t) &= \lambda\frac{\partial}{\partial t}C(t,0)[x_0 - g(x)] + \lambda\frac{\partial}{\partial t}S(t,0)y_0 \\
&\quad +\lambda\int_0^t \frac{\partial}{\partial t}S(t,s)f(s,x(s), \qquad (15)\\
&= \int_0^s a(s,\tau)h(\tau,x(\tau),x'(\tau))d\tau, x'(s))ds,\ t\in J,
\end{aligned}
$$

and

$$
\begin{aligned}
\|x'(t)\| &\le N^*[\|x_0\| + G] + N\|y_0\| \\
&\quad + \int_0^t Np(s)\Omega_2(\|x(s)\| \\
&\quad +L\int_0^s m(\tau)\Omega_1(\|x(\tau)\| + \|x'(\tau)\|)d\tau + \|x'(s)\|)ds.
\end{aligned}
$$

Denoting by $r(t)$ the right-hand side of the above inequality we get

$$
\begin{aligned}
\|x'(t)\| &\le r(t),\quad t\in J, \text{and}\quad r(0) = N^*[\|x_0\| + G] + N\|y_0\|, \\
r'(t) &= Np(t)\Omega_2(\|x(t)\| + L\int_0^t m(\tau)\Omega_1(\|x(\tau)\| \\
&\quad +\|x'(\tau)\|)d\tau + \|x'(t)\|),\quad t\in J.
\end{aligned}
$$

Hence

$$
\begin{aligned}
v'(t) &\le Mp(t)\Omega_2(v(t) + r(t) + L\int_0^s m(\tau)\Omega_1(v(\tau) + r(\tau))d\tau) \\
r'(t) &\le Np(t)\Omega_2(v(t) + r(t) + L\int_0^s m(\tau)\Omega_1(v(\tau) + r(\tau))d\tau)\quad t\in J.
\end{aligned}
$$

Let

$$
w(t) = v(t) + r(t) + L\int_0^t m(s)\Omega_1(v(s) + r(s))ds,\quad t\in J.
$$

Then $w(0) = v(0) + r(0) = c$, and

$$
\begin{aligned}
w'(t) &= v'(t) + r'(t) + Lm(t)\Omega_1(v(t) + r(t)) \\
&\le (M+N)p(t)\Omega_2(w(t)) + Lm(t)\Omega_1(w(t)) \\
&\le \tilde{m}(t)[\Omega_2(w(t)) + \Omega_1(w(t))].
\end{aligned}
$$

This gives

$$
\int_{w(0)}^{w(t)} \frac{ds}{\Omega_2(s) + \Omega_1(s)} \le \int_0^t \tilde{m}(s)ds \le \int_0^T \tilde{m}(s)ds < \int_c^\infty \frac{ds}{\Omega_2(s) + \Omega_1(s)}.
$$

This inequality implies that there is a constant K such that

$$v(t) + r(t) \leq w(t) \leq K, \qquad t \in J.$$

Thus

$$\|x(t)\| \leq v(t), \quad \|x'(t)\| \leq r(t), \quad t \in J,$$

and hence

$$\|x\|^* \leq K,$$

where K depends only on T and on the functions m, Ω_2 and Ω_1.
Moreover using the same method as in Theorem 3.1 we can prove that the corresponding operator is completely continuous. Thus the problem (10)-(11) has at least one mild solution on J.

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