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ON RADIAL DISTRIBUTION OF JULIA SETS OF MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we mainly investigate the radial distribution of the Julia set of a transcendental meromorphic function with finitely many deficient values.

1. Introduction

Let $f(z)$ be a transcendental meromorphic function in the complex plane $\mathbf{C}$ and f^n be the nth iterate of f, i.e. $f^0 = 1, f^1 = f, f^2 = f(f), f^n = f(f^{n-1})$. For $n > 1$, $f^n(z)$ is well defined in $\mathbf{C}$ except for a possible countable set below:

$$\{z \in \mathbf{C} : f^k(z) = \infty, k = 1, 2, \cdots, n-1\}.$$

Fatou set $F(f)$ of $f(z)$ is defined by

$$F(f) = \{z \in \mathbf{C} : \{f^n\} \text{ is defined and normal in a neighborhood of } z\}.$$

Julia set $J(f)$ of $f(z)$ is the complement of $F(f)$ in $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. $F(f)$ is open and $J(f)$ is closed, non-empty.

For a $\theta \in [0, 2\pi)$, $\arg z = \theta$ is called the radial distribution of $J(f)$, if for any small $\epsilon > 0$, $\Omega(\theta - \epsilon, \theta + \epsilon) \cap J(f)$ is unbounded, where

$$\Omega(\theta - \epsilon, \theta + \epsilon) = \{z \in \mathbf{C} : \arg z \in (\theta - \epsilon, \theta + \epsilon)\}.$$

$RD(f)$ denotes the set of all radial distributions of $J(f)$. Obviously, $mesRD(f)$ is closed and measurable. $mesRD(f)$ denotes the linear measure of $RD(f)$.

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Some standard notations of Nevanlinna theory are used in this paper. $T(r, f)$, $N(r, f)$ and $N(r, \frac{1}{f})$ are defined in [2]. For $a \in \mathbf{C}$, if

$$\delta(a, f) = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)} > 0,$$

then a is called a Nevanlinna deficient value of $f(z)$, $\delta(a, f)$ is called the deficient number of $f(z)$ at a. $\delta(\infty, f)$ is the deficient number of $f(z)$ at ∞, which is defined by

$$\delta(\infty, f) = 1 - \limsup_{r\to\infty} \frac{N(r, f)}{T(r, f)}.$$

The growth order $\sigma(f)$ and lower order $\mu(f)$ of $f(z)$ are defined respectively by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}$$

and

$$\mu(f) = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

Let $W \subset \overline{\mathbf{C}}$ be a hyperbolic domain, that is, $\overline{\mathbf{C}} \backslash W$ contains at least three points. There exists the hyperbolic metric $\lambda_W(z)|dz|$ on W with Gaussian curvature -4. Let Δ be a unit disc and $h(z)$ be a holomorphic universal covering map of W from Δ, then the hyperbolic density λ_W on W is expressed as:

$$\lambda_W(h(z))|h'(z)| = \frac{1}{1-|z|^2}, z \in \Delta,$$

where the hyperbolic density λ_Δ on Δ is defined by:

$$\lambda_\Delta(z) = \frac{1}{1-|z|^2}.$$

For an $a \in \mathbf{C} \backslash W$, define

$$C_W(a) = \inf\{\lambda_W(z)\delta_W(a) : \forall z \in W\},$$

where $\delta_W(z)$ is a Euclidean distance between z and ∂W. For a finite number $a \in J(f)$, if there is a component U in $F(f)$ such that $C_U(a) > 0$, then we call $C_{F(f)}(a) > 0$, where $f(z)$ is a transcendental meromorphic function in $\mathbf{C}$. For example, $C_{\tan z}(0) > 0$, $0 \in J(\tan z)$.

2. Radial Distribution of Julia Sets

Let $f(z)$ be a transcendental entire function in $\mathbf{C}$. If $\sigma(f) < \infty$, Baker [1] proved that $J(f)$ cannot lie in finitely many lines beginning from the original point. But for an arbitrarily small $d > 0$, Baker[1] constructed an entire function $f(z)$, dependent on d, of infinite order satisfying

$$J(f) \subset \{z \in \mathbf{C} : |\arg z| < d, Rez > 0\}.$$

So $mesRD(f) < d$. We conclude $\mu(f) = \infty$ by the following Theorem A, see [3]:

Theorem A. *Let $f(z)$ be a transcendental entire function in $\mathbf{C}$ with $\mu(f) < \infty$. Then $mesRD(f) = 2\pi$ if $\mu(f) < \frac{1}{2}$; $mesRD(f) \geq \frac{\pi}{\mu(f)}$ if $\mu(f) \geq \frac{1}{2}$.*

For the proof of Theorem A, the Principle of Pragmén-Lindelöf was applied. But for the case of a meromorphic function with poles, the Principle of Pragmén-Lindelöf cannot be applied. The following theorem was proved in [7] by applying methods of Nevanlinna theory.

Theorem B. *Let $f(z)$ be a transcendental meromorphic function in $\mathbf{C}$ with $\mu(f) < \infty$ and $\delta(\infty, f) > 0$. If $\mu(f) = 0$, then $mesRD(f) = 2\pi$; if $\mu(f) > 0$ and $J(f)$ has an unbounded component, then*

$$mesRD(f) \geq \min\{2\pi, \frac{4}{\mu(f)} \arcsin \sqrt{\frac{\delta(\infty, f)}{2}}\}.$$

Now, we have a significant and interesting result in the following, which extends Theorem B to be a more general case. In this paper, p is a positive integer throughout.

Theorem 1. *Let $f(z)$ be a transcendental meromorphic function with lower order $\mu(f) \in (0, \infty)$. Suppose $f(z)$ has p mutually distinct deficient values $a_1, \cdots, a_p$ and the corresponding deficient numbers $\delta(a_1, f), \cdots, \delta(a_p, f)$. If there exists $a \in J(f)$ such that $C_{F(f)}(a) > 0$, then*

$$mesRD(f) \geq \min\{2\pi, \frac{4}{\mu} \Sigma_{j=1}^{p} \arcsin \sqrt{\frac{\delta(a_j, f)}{2}}\}.$$

If $C_{F(f)}(a) = 0$ for any $a \in J(f)$, does the conclusion of Theorem 1 still hold? This question seems be interesting, see [7] for a special case.

Next, considering the radial distribution of the common Julia sets of a transcendental meromorphic function and its derivatives, we have another interesting result as follows:

Theorem 2. *Let $f(z)$ be a transcendental meromorphic function of finite lower order $\mu > 0$ and $\delta(\infty, f) > 0$. If $J(f)$ has an unbounded component and for $k > 0$,*

$J(f^{(k)})$ *has an unbounded component, then*

$$mes(RD(f)\cap RD(f^{(k)}))\geq \min\{2\pi, \frac{4}{\mu}\arcsin\sqrt{\frac{\delta(\infty,f)}{2}}\}.$$

If $f(z)$ is an entire function with finite lower order $\mu(f)>0$, from [4], Theorem 2 and furthermore the following question holds.

Let $f(z)$ be a transcendental meromorphic function of finite lower order $\mu>0$ and $\delta(\infty,f)>0$. Do we always have the following, for some integer $k>0$,

$$mes(RD(f)\cap RD(f^{(k)}))\geq \min\{2\pi, \frac{4}{\mu}\arcsin\sqrt{\frac{\delta(\infty,f)}{2}}\}\ ?$$

3. Proofs of Theorems

Before the proof of the theorems, we need to quote two lemmas from [6] as follows:

Lemma A. *Suppose that* $f(z)$ *is a transcendental meromorphic function with lower order* $\mu<\infty$ *and order* $\sigma>0$. *Then for any* $\rho\in[\mu,\sigma]$, *there is a positive series* $\{r_k\}$, $\frac{r_k}{k}\to\infty$, *such that*

$$T(t,f)<(1+o(1))(\frac{t}{r_k})^{\rho}T(r_k,f),\ \forall t\in[\frac{r_k}{k},kr_k]$$

and

$$\liminf_{k\to\infty}\frac{\log T(r_k,f)}{\log r_n}\geq\rho.$$

Lemma B. *Suppose that* $f(z)$ *is a transcendental meromorphic function with lower order* $\mu<\infty$ *and order* $\sigma>0$, $\rho\in[\mu,\sigma]$. *If* a *is a deficient value of* $f(z)$, $\delta(a,f)$ *is the deficient number, then we have*

$$\lim_{n\to\infty} mesE(r_n,\epsilon,a,\rho)\geq \min\{2\pi, \frac{4}{\rho}\arcsin\sqrt{\frac{\delta(a,f)}{2}}\},$$

where

$$E(r_n,\epsilon,a_j,\mu)=\{\theta\in[0,2\pi):\log\frac{1}{|f(r_ne^{i\theta})-a_j|}>r_n^{\mu-\epsilon}\}, a_j\neq\infty$$

or

$$E(r_n,\epsilon,a_j,\mu)=\{\theta\in[0,2\pi):\log|f(r_ne^{i\theta})|>r_n^{\mu-\epsilon}\}, a_j=\infty,$$

$j = 1, 2, \cdots, p$, *for* $\forall \epsilon \in (0, \mu)$.

Let $f(z)$ be a transcendental meromorphic function with finite order $\mu > 0$ and $f(z)$ has p mutually distinct deficient values a_j and the corresponding deficient numbers $\delta(a_j, f)$, $j = 1, 2, \cdots, p$. By Lemma A, there exists an unbounded positive series$\{r_n\}_{n=1}^{\infty}$ such that

$$\liminf_{n\to\infty} \frac{\log T(r_n, f)}{\log r_n} \geq \mu. \tag{1}$$

and for $\forall \epsilon \in (0, \mu)$, set

$$E(r_n, \epsilon, a_j, \mu) = \{\theta \in [0, 2\pi) : \log \frac{1}{|f(r_n e^{i\theta}) - a_j|} > r_n^{\mu-\epsilon}\}, a_j \neq \infty$$

or

$$E(r_n, \epsilon, a_j, \mu) = \{\theta \in [0, 2\pi) : \log |f(r_n e^{i\theta})| > r_n^{\mu-\epsilon}\}, a_j = \infty,$$

$j = 1, 2, \cdots, p$. By Lemma B, there exists N_j for all $n > N_j$, we have

$$mesE(r_n, \epsilon, a_j, \mu) > \min\{2\pi, \frac{4}{\mu} \arcsin \sqrt{\frac{\delta(a_j, f)}{2}}\} - \frac{\epsilon}{p},$$

$j = 1, 2, \cdots, p$. So, for all $n > \max\{N_1, \cdots, N_p\}$,

$$\sum_{j=1}^{p} mesE(r_n, \epsilon, a_j, \mu) > \min\{2\pi, \frac{4}{\mu} \arcsin \sqrt{\frac{\delta(a_j, f)}{2}}\} - \epsilon.$$

Therefore, we obtain

Lemma 1. *Let $f(z)$ be a transcendental meromorphic function in* $\mathbf{C}$ *with finite lower order $\mu > 0$. If $f(z)$ has p mutually distinct deficient values a_j and the corresponding deficient numbers $\delta(a_j, f)$, $j = 1, 2, \cdots, p$, then for any $\epsilon > 0$, there exist an unbounded positive number series satisfying (1) and integer $N > 0$, for all $n > N$, we have*

$$\sum_{j=1}^{p} mesE(r_n, \epsilon, a_j, \mu) > \min\{2\pi, \frac{4}{\mu} \arcsin \sqrt{\frac{\delta(a_j, f)}{2}}\} - \epsilon.$$

For $r > 0$ and $\theta_1, \theta_2 \in [0, 2\pi)$, $\theta_1 < \theta_2$, we define

$$\Omega(r; \theta_1, \theta_2) := \{z \in \mathbf{C} : \arg z \in (\theta_1, \theta_2), |z| > r\}.$$

Lemma 2. ([7, Lemma 2.2]). *Let $f(z)$ be analytic in $\Omega(r; \theta_1, \theta_2), r > 0$, U a hyperbolic domain and*

$$f:\Omega(r;\theta_1,\theta_2)\to U.$$

If there exists a point $a\in\partial U\backslash\{\infty\}$ *such that* $C_U(a)>0$*, then there exists a constant* $d>0$ *such that for arbitrary* $\epsilon>0$, $\theta_2-\theta_1-2\epsilon>0$*, it has*

$$|f(z)|=O(|z|^d), z\to\infty, z\in\Omega(r;\theta_1+\epsilon,\theta_2-\epsilon).$$

Proof of Theorem 1. Assume that by contradiction,

$$mesRD(f)<l=\min\{2\pi,\frac{4}{\mu}\Sigma_{j=1}^p\arcsin\sqrt{\frac{\delta(a_j,f)}{2}}\}.$$

Since $RD(f)$ is closed, $RD(f)^c=[0,2\pi)\backslash RD(f)$ is an union set of at most countable open intervals I. From I, we chosen $m\geq 1$ intervals I_j, $j=1,\cdots,m$, such that

$$mes(RD(f)^c\backslash\cup_{j=1}^m I_j)<\frac{t}{2},$$

where

$$t=l-mesRD(f)-q, 0<q<l-mesRD(f).$$

By the hypotheses of Theorem 1 and Lemma 1, for any $\epsilon>0$, there exists an unbounded positive number series $\{r_n\}_{n=1}^\infty$ and integer $N\geq 1$, if $n>N$, then

$$mes\cup_{j=1}^p E(r_n,\epsilon,a_j,\mu)>l-q>0.$$

And then for $n>N$, we have

$$\begin{aligned}&mes((\cup_{j=1}^p E(r_n,\epsilon,a_j,\mu))\cap RD(f)^c)\\&=mes(\cup_{j=1}^p E(r_n,\epsilon,a_j,\mu)\backslash((\cup_{j=1}^p E(r_n,\epsilon,a_j,\mu))\cap RD(f)))\\&=mes(\cup_{j=1}^p E(r_n,\epsilon,a_j,\mu))-mes((\cup_{j=1}^p E(r_n,\epsilon,a_j,\mu))\cap RD(f))\\&>l-q-mesRD(f)=t>0\end{aligned}$$

and

$$\begin{aligned}&mes((\cup_{j=1}^m I_j)\cap(\cup_{j=1}^p E(r_n,\epsilon,a_j,\mu)))\\&\geq mes(RD(f)^c\cap(\cup_{j=1}^p E(r_n,\epsilon,a_j,\mu)))\\&-mes((RD(f)^c)\backslash\cup_{j=1}^m I_j)\\&>t-\tfrac{t}{2}=\tfrac{t}{2}.\end{aligned}$$

There exists a $j_0,\ 1\leq j_0\leq m$ such that $I_{j_0}\subset RD(f)^c$, and for infinitely many n it has

$$mes[(\cup_{j=1}^m I_j)\cap(\cup_{j=1}^p E(r_n,\epsilon,a_j,\mu))]\leq m\cdot mes(I_{j_0}\cap\cup_{j=1}^p E(r_n,\epsilon,a_j,\mu)).$$

So, for infinitely many n, it has

$$mes(I_{j_0} \cap (\cup_{j=1}^{p} E(r_n, \epsilon, a_j, \mu))) > \frac{t}{2m}. \tag{2}$$

Without loss of generality, assume (2) is valid for all n. Set

$$I_{j_0} = (\theta_1, \theta_2), 0 < \theta_2 - \theta_1 < 2\pi.$$

Take a positive number $s > 1$ such that $\theta_2 - \theta_1 - \frac{2\epsilon}{s} > 0$ and

$$mes[(\theta_1 + \frac{\epsilon}{s}, \theta_2 - \frac{\epsilon}{s}) \cap (\cup_{j=1}^{p} E(r_n, \epsilon, a_j, \mu))] > \frac{t}{3m}.$$

Hence, according to the fact (see [6] or the Notes following the end of the proof.),

$$E(r_n, \epsilon, a_j, \mu) \cap E(r_n, \epsilon, a_k, \mu) = \emptyset, j \neq k, j, k = 1, \cdots, p,$$

there is some a_j, say a_1, for infinitely many n it has

$$mes[(\theta_1+\frac{\epsilon}{s}, \theta_2-\frac{\epsilon}{s})\cap(\cup_{j=1}^{p} E(r_n, \epsilon, a_j, \mu))] \leq p \cdot mes((\theta_1+\frac{\epsilon}{s}, \theta_2-\frac{\epsilon}{s})\cap E(r_n, \epsilon, a_1, \mu)).$$

So that, for infinitely many n, it gets

$$mes((\theta_1 + \frac{\epsilon}{s}, \theta_2 - \frac{\epsilon}{s}) \cap E(r_n, \epsilon, a_1, \mu)) > \frac{t}{3pm} > 0. \tag{3}$$

Obviously, we may assume (3) is valid for all n. Set

$$\phi(z) = \frac{1}{z - a_1}.$$

Write

$$\alpha = \theta_1 + \frac{\epsilon}{s}, \beta = \theta_2 - \frac{\epsilon}{s}.$$

There exists a sufficiently large $R > 0$,

$$\phi \circ f : \Omega(R; \alpha, \beta) \to \phi(F(f))$$

is an analytic map. Note that

$$C_{\phi(F(f))}(\phi(a)) = C_{F(f)}(a) > 0,$$

By Lemma 2, for an arbitrarily small $\zeta > 0$, we have

$$\beta - \alpha - 2\zeta > 0$$

and

$$\log^+ |\phi(f(z))| = O(\log(|z|)), z \in \Omega(R; \alpha+\zeta, \beta-\zeta), |z| \to \infty.$$

So

$$\log^+ |\frac{1}{f(z)-a_1}| = O(\log(|z|)), z \in \Omega(R; \alpha+\zeta, \beta-\zeta), |z| \to \infty. \tag{4}$$

On the another hand, noting that ζ may be chosen as small as we like, from (3), for all n, it follows

$$mes[(\alpha+\zeta, \beta-\zeta) \cap E(r_n, \epsilon, a_1, f)] > 0.$$

And then, there is an unbounded series

$$\{r_n e^{i\theta_n}\}_{n=1}^{\infty}, \theta_n \in (\alpha+\zeta, \beta-\zeta) \cap E(r_n, \epsilon, a_1, f),$$

such that for all sufficiently large n, it has

$$\log^+ |\frac{1}{f(r_n e^{i\theta_n}) - a_1}| > {r_n}^{\mu(f)-\epsilon}. \tag{5}$$

Since the unbounded series $\{r_n e^{i\theta_n}\}_{n=N}^{\infty}$ satisfying (4) for some $N \geq 1$, namely

$$\log^+ |\frac{1}{f(r_n e^{i\theta_n}) - a_1}| = O(\log(r_n)), n \to \infty. \tag{6}$$

When $n \to \infty$, it derives a contradiction from (5) and (6). The proof is complete.

Notes. Let's prove

$$E(r_n, \epsilon, a_j, \mu) \cap E(r_n, \epsilon, a_k, \mu) = \emptyset, j \neq k, j, k = 1, \cdots, p,$$

for sufficiently large n.

If assume that

$$E(r_n, \epsilon, a_j, \mu) \cap E(r_n, \epsilon, a_k, \mu) \neq \emptyset, j \neq k$$

for sufficiently large n, without loss of generality, assume $a_j \neq \infty, a_k \neq \infty$, and there is a θ such that

$$\theta \in E(r_n, \epsilon, a_j, \mu) \cap E(r_n, \epsilon, a_k, \mu),$$

then by the definition of $E(r_n, \epsilon, a, \mu)$, we may have

$$|f(r_n e^{i\theta}) - a_j| < e^{-r_n^{\mu-\epsilon}},$$

and

$$|f(r_n e^{i\theta}) - a_k| < e^{-r_n^{\mu-\epsilon}}.$$

But from the following

$$\begin{aligned}&|f(r_n e^{i\theta}) - a_k| = |(f(r_n e^{i\theta}) - a_j) + (a_j) - a_k|\\ &\geq |a_j - a_k| - |f(r_n e^{i\theta}) - a_k|\\ &> |a_j - a_k| - e^{-r_n^{\mu-\epsilon}}\\ &> \frac{1}{2}|a_j - a_k|,\end{aligned}$$

we have the following contradiction:

$$\log \frac{1}{|a_j - a_k|} > \log \frac{2}{|f(r_n e^{i\theta}) - a_k|} > r_n^{\mu-\epsilon}.$$

The contradiction shows that the fact cited is right.

Proof of Theorem 2. By contradiction, assume that

$$mes(RD(f) \cap RD(f^{(k)})) < \nu = \min\{2\pi, \frac{4}{\mu} \arcsin \sqrt{\frac{\delta(\infty, f)}{2}}\}. \tag{7}$$

There must exist an open interval

$$I = (\alpha, \beta) \subset RD(f^{(k)})^c, 0 < \beta - \alpha < \nu,$$

such that $\forall \epsilon > 0$ and

$$\lim_{n\to\infty} mes(I \cap RD(f) \cap E(r_n, \epsilon, \infty, \mu)) > 0. \tag{8}$$

In fact, we have

$$\lim_{n\to\infty} mes(E(r_n, \epsilon, \infty, \mu) \backslash RD(f)) = 0. \tag{9}$$

Otherwise, suppose there is a subseries $\{n_k\}$ such that

$$\lim_{k\to\infty} mes(E(r_{n_k}, \epsilon, \infty, \mu) \backslash RD(f)) > 0,$$

for some $\epsilon > 0$, then there exist $\theta_0 \in RD(f)^c$ and $\eta > 0$ satisfying

$$\lim_{k\to\infty} mes((\theta_0 - \eta, \theta_0 + \eta) \cap (E(r_{n_k}, \epsilon, \infty, \mu) \backslash RD(f))) > 0. \tag{10}$$

As $\arg z = \theta_0$ is not a radial distribution of $J(f)$, there is $R > 0$, $f(z)$ is analytic in

$$\Omega(R; \theta_0 - \eta, \theta_0 + \eta)$$

and

$$f(\Omega(R; \theta_0 - \eta, \theta_0 + \eta)) \subset F(f).$$

Note that $J(f)$ has an unbounded component, by Lemma 2, for any $\zeta > 0$, $\zeta < \eta$,

$$\log |f(z)| = O(\log |z|), z \in \Omega(R; \theta_0 - \eta + \zeta, \theta_0 + \eta - \zeta), |z| \to \infty. \tag{11}$$

Since ζ may be chosen sufficiently small, from (10)

$$\lim_{k\to\infty} mes((\theta_0 - \eta + \zeta, \theta_0 + \eta - \zeta) \cap E(r_{n_k}, \epsilon, \infty, \mu)) > 0,$$

we can find an infinite series $\{r_{n_k} e^{i\theta_{n_k}}\}$ such that for all sufficiently large k,

$$\log |f(r_{n_k} e^{i\theta_{n_k}})| > r_{n_k}^{\mu-\epsilon}, \tag{12}$$

where

$$\theta_{n_k} \in ((\theta_0 - \eta + \zeta, \theta_0 + \eta - \zeta) \cap E(r_{n_k}, \epsilon, \infty, \mu)).$$

But, from (11), it has

$$\log |f(r_{n_k} e^{i\theta_{n_k}})| = O(\log r_{n_k}), k \to \infty. \tag{13}$$

When $k \to \infty$, (12) contradicts to (13). This contradiction implies (9) is valid.

Because

$$mesRD(f) \geq \nu,$$

and for all sufficiently large n,

$$mesE(r_n, \epsilon, \infty, \mu) > \nu - \epsilon,$$

it follows

$$\lim_{n\to\infty} mesRD(f) \cap E(r_n, \epsilon, \infty, \mu) \geq \nu.$$

By (7), there exists an open interval

$$I_j \subset RD(f^{(k)})^c (j = 1, 2, \cdots, m; m \geq 1)$$

such that for sufficiently large n

$$mes((\cup_{j=1}^m I_j) \cap RD(f) \cap E(r_n, \epsilon, \infty, \mu)) > \frac{1}{2}(\nu - mes(RD(f) \cap RD(f^{(k)}))) > 0.$$

For infinitely many n, there are some I_{j_0} satisfying

$$mes(I_{j_0} \cap RD(f) \cap E(r_n, \epsilon, \infty, \mu)) > \frac{1}{2m}(\nu - mes(RD(f) \cap RD(f^{(k)}))) > 0.$$

We may assume the above is true for all n. For this case, we prove that (8) is valid.

From (8), we know there are θ and $\tilde{\eta} > 0$ such that

$$(\theta - 2\tilde{\eta}, \theta + 2\tilde{\eta}) \subset I$$

and

$$\lim_{n\to\infty} mes((\theta - 2\tilde{\eta}, \theta + 2\tilde{\eta}) \cap RD(f) \cap E(r_n, \epsilon, \infty, \mu)) > 0. \tag{14}$$

There exists $R > 0$ such that $f^{(k)}(z)$ is analytic in

$$\Omega(R; \theta - 2\tilde{\eta}, \theta + 2\tilde{\eta})$$

and

$$f^{(k)}(\Omega(R; \theta - 2\tilde{\eta}, \theta + 2\tilde{\eta})) \subset F(f^{(k)}).$$

Noting that $J(f^{(k)})$ has an unbounded component, By Lemma 2,

$$\log |f^{(k)}(z)| = O(\log |z|), z \in \Omega(R; \theta - 2\tilde{\eta}, \theta + 2\tilde{\eta}), |z| \to \infty. \tag{15}$$

From (14), we can select an unbounded series $\{r_n e^{i\theta_n}\}$, for all sufficiently large n, it has

$$\log |f(r_n e^{i\theta_n})| > r_n^{\mu-\epsilon}, \tag{16}$$

where

$$\theta_n \in (((\theta - 2\tilde{\eta}, \theta + 2\tilde{\eta}) \cap RD(f) \cap E(r_n, \epsilon, \infty, \mu)).$$

Fix $r_N e^{i\theta_N} \in \{r_n e^{i\theta_n}\}$, and take a $r_n e^{i\theta_n} \in \{r_n e^{i\theta_n}\}$, $n > N$. Take a simple Jordan arc γ in

$$\Omega(R; \theta - 2\tilde{\eta}, \theta + 2\tilde{\eta}),$$

which connecting $r_N e^{i\theta_N}$ to $r_N e^{i\theta_n}$ along $\{|z| = r_N\}$, and connecting $r_N e^{i\theta_n}$ to $r_n e^{i\theta_n}$ along $\arg z = \theta_n$. For any $z \in \gamma$, γ_z denotes a part of γ, which connecting $r_N e^{i\theta_N}$ to z. Let $L(\gamma)$ be the length of γ. Obviously,

$$L(\gamma) = O(r_n), n \to \infty.$$

For some $M > 0$, from (15), it follows

$$\begin{aligned} |f^{(k-1)}(z)| &\leq \int_{\gamma_z} |f^{(k)}(z)||dz| + c_k \\ &\leq O(|z|^M L(\gamma)) + c_k \\ &\leq O(r_n^{M+1}), n \to \infty. \end{aligned}$$

Similarly, it follows

$$\begin{aligned}
|f^{(k-2)}(z)| &\leq \int_{\gamma_z} |f^{(k-1)}(z)||dz| + c_{k-1} \\
&\leq O(r_n^{M+1} L(\gamma)) + c_{k-1} \\
&\leq O(r_n^{M+2}), n \to \infty; \\
&\vdots \\
|f'(z)| &\leq \int_{\gamma_z} |f''(z)||dz| + c_2 \\
&\leq O(r_n^{M+k-2} L(\gamma)) + c_2 \\
&\leq O(r_n^{M+k-1}), n \to \infty; \\
|f(z)| &\leq \int_{\gamma_z} |f'(z)||dz| + c_1 \\
&\leq O(r_n^{M+k-1} L(\gamma)) + c_1 \\
&\leq O(r_n^{M+k}), n \to \infty,
\end{aligned}$$

where $c_1, \cdots, c_k$ are constants, which are independent of n. Therefore,

$$\log |f(r_n e^{i\theta_n})| \leq O(\log r_n), n \to \infty. \tag{17}$$

When $n \to \infty$, (16) contradicts to (17).

All in all, (7) is false. The proof is complete.

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