TAIWANESE JOURNAL OF MATHEMATICS
Vol. 11, No. 5, pp. 1287-1300, December 2007
This paper is available online at http://www.math.nthu.edu.tw/tjm/

WELL-POSEDNESS FOR VECTOR VARIATIONAL INEQUALITY AND CONSTRAINED VECTOR OPTIMIZATION

Ya-Ping Fang and Nan-Jing Huang

Abstract. In this paper we present some notions of parametric well-posedness for Stampacchia and Minty vector variational inequalities. We show that under suitable conditions, the parametric well-posedness of a Stampacchia vector variational inequality is equivalent to the parametric well-posedness of a vector optimization problem. Further, we introduce some concepts of well-posedness for a vector optimization problem with a Stampacchia vector variational inequality constraint. We prove that the well-posedness of this constrained vector optimization problem is closely related to the parametric well-posedness of its constrained vector variational inequality.

1. INTRODUCTION

Well-posedness of a given variational problem, generally speaking, deals with the behavior of the solution with respect to the problem's data perturbations, and it is closely related to stability analysis and sensitivity analysis issues. The issues of well-posedness have been studied intensively in various fields. An initial concept of well-posedness for a scalar global optimization problem was first introduced by Tykhonov [28], already known as Tykhonov well-posedness. Since then various notions of well-posedness for scalar global optimization problems have been introduced and studied (see, e.g., [1, 7, 31, 32]). Some concepts of well-posedness for vector optimization problems were considered in [2, 3, 10-12, 19, 26]. The well-posedness of Nash equilibria were studied in [4, 20, 23-25]. Recently, the notions of well-posedness was further generalized to fixed point problems [15],

Received September 8, 2005, accepted September 12, 2005.
Communicated by Wen-Wei Lin.
2000 *Mathematics Subject Classification*: 49J40, 49K40, 90C31.
Key words and phrases: Vector variational inequality, Vector optimization problem, Vector optimization problem with a vector variational inequality constraint, Well-posedness.
This work was supported by the Applied Research Project of Sichuan Province (05JY029-009-1), the National Natural Science Foundation of China (10671135) and the specialized Research Fund for the Doctoral Program of Higher Education (200606(0005)).

variational inequality problems [16, 17], inclusion problems [15]. Especially, Lignola and Morgan [16] introduced and studied the well-posedness for an optimization problem with a variational inequality constraint (for short OPVIC). The problem OPVIC has been investigated by many authors since it provides a unified mathematical model for some important problems arising in economics and engineering science. For details, we refer to [21, 22] and the references therein. On the other hand, vector generalizations of many important problems, such as variational inequality and optimization problems, have been studied intensively in recent years because they provide more general frameworks. The purpose of this paper is to generalize some corresponding results of Lignola and Morgan [16] to the vector case. We study the parametric well-posedness of Stampacchia and Minty vector variational inequality problems, and prove that under suitable conditions, the parametric well-posedness of a Stampacchia vector variational inequality is equivalent to the parametric well-posedness of a corresponding vector optimization problem. We introduce a notion of well-posedness for a vector optimization problem with a vector variational inequality constraint (for short, VOPVVIC) and show that the well-posedness of VOPVVIC is closely related to the parametric well-posedness of its constrained vector variational inequality.

2. Preliminaries and Notations

Throughout this paper, unless stated otherwise, we always suppose that P is a metric space, K is a nonempty, closed, and convex subset of a real Banach space X, and Y is a real Banach space endowed with a partial order induced by a pointed, closed and convex cone C with $\mathrm{int\,C} \neq \emptyset$ in the following ways:

$$x \leq_C y \Leftrightarrow y - x \in C,$$

$$x \leq_{\mathrm{int\,C}} y \Leftrightarrow y - x \in \mathrm{int\,C},$$

$$x \not\leq_C y \Leftrightarrow y - x \notin C,$$

$$x \not\leq_{\mathrm{int\,C}} y \Leftrightarrow y - x \notin \mathrm{int\,C},$$

where $\mathrm{int\,C}$ denotes the interior of C.

Let $A : P \times K \to L(X, Y)$ (the family of all continuous linear mappings from X into Y) be a mapping. The parametric (Stampacchia) vector variational inequality problem is to find $u \in K$ such that

$$\mathrm{VVI(p)} \quad \langle \mathrm{A(p,u), u - v} \rangle \not\geq_{\mathrm{int\,C}} 0, \quad \forall \mathrm{v} \in \mathrm{K}.$$

When $A(p, u)$ is independent on the parameter p, $\mathrm{VVI(p)}$ reduces to the classical vector variational inequality formulated by finding $u \in K$ such that

$$\mathrm{VVI} \quad \langle \mathrm{Au, u - v} \rangle \not\geq_{\mathrm{int\,C}} 0, \quad \forall \mathrm{v} \in \mathrm{K}.$$

The problem VVI has been studied intensively by many authors (see, e.g., [5, 6, 8, 14, 30]). In this paper we also consider the following Minty vector variational inequality problems [9, 29]:

$$\mathrm{MVVI(p)} \quad \text{find } \mathrm{u} \in \mathrm{K} \text{ such that } \quad \langle \mathrm{A(p,v), u-v} \rangle \not\geq_{\mathrm{int\,C}} 0, \quad \forall \mathrm{v} \in \mathrm{K}$$

and

$$\mathrm{VVI} \quad \text{find } \mathrm{u} \in \mathrm{K} \text{ such that } \quad \langle \mathrm{Av, u-v} \rangle \not\geq_{\mathrm{int\,C}} 0, \quad \forall \mathrm{v} \in \mathrm{K}.$$

The Stampacchia and Minty vector variational inequality problems are closely related to the following vector optimization problem:

$$\mathrm{VOP(p)} \qquad \min_{\mathrm{u} \in \mathrm{K}} \mathrm{F(p,u)},$$

where $F : P \times K \to Y$. When $F(p, u)$ is independent on the parameter p, $\mathrm{VOP(p)}$ reduces to

$$\mathrm{VOP} \qquad \min_{\mathrm{u} \in \mathrm{K}} \mathrm{F(u)}.$$

In the sequel we give some concepts.

Definition 2.1. Let $f : K \to Y$ be a mapping. A point $x_0 \in K$ is called a weakly efficient solution of f on K iff $f(u) \not\leq_{\mathrm{int\,C}} f(x_0)$, $\forall u \in K$. Denote by $\mathrm{argmin(K, f)}$ and $\mathrm{Min(K, f)}$ the set of weakly efficient solutions of f on K and the image of F on $\mathrm{argmin(K, f)}$ respectively.

Definition 2.2. The family $\{\mathrm{VOP(p) : p \in P}\}$ is said to be parametrically well-posed iff:

(i) there exists a unique weakly efficient solution x_p of $F(p, \cdot)$ on K for all $p \in P$;

(ii) for any sequences $\{p_n\} \subset P$ with $p_n \to p$ and $\{x_n\} \subset K$ such that $\exists \{\alpha_n\} \subset R_+$ with $\alpha_n \to 0$, and

$$F(p_n, x_n) \not\geq_{\mathrm{int\,C}} F(p_n, y) + \alpha_n e, \quad \forall y \in K,$$

it holds that $x_n \to x_p$.

The sequence $\{x_n\}$ in (ii) is called a minimizing sequence for $\mathrm{VOP(p)}$ corresponding to $\{p_n\}$.

Definition 2.3. The family $\{\mathrm{VOP(p) : p \in P}\}$ is said to be parametrically well-posed in the generalized sense iff:

(i) $\mathrm{argmin(K, F(p, \cdot))} \neq \emptyset, \forall \mathrm{p} \in \mathrm{P}$;

(ii) for any sequences $\{p_n\} \subset P$ with $p_n \to p$ and $\{x_n\} \subset K$ such that $\exists\{\alpha_n\} \subset R_+$ with $\alpha_n \to 0$, and

$$F(p_n, x_n) \not\geq_{\mathrm{int\,C}} F(p_n, y) + \alpha_n e, \quad \forall y \in K,$$

there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some point $x^* \in \mathrm{argmin}\,(K, F(p, \cdot))$ such that $x_{n_k} \to x^*$.

Definition 2.4. A mapping $f : K \to Y$ is said to be C-convex iff

$$f(tx + (1-t)y) \leq_C tf(x) + (1-t)f(y), \quad \forall x, y \in K, t \in [0, 1].$$

Definition 2.5. A mapping $T : K \to L(X, Y)$ is said to be C-monotone iff

$$\langle Tx - Ty, x - y\rangle \geq_C 0, \quad \forall x, y \in K.$$

Definition 2.6. A mapping $T : K \to L(X, Y)$ is said to be hemicontinuous iff for any $x, y \in K$ the mapping $t \mapsto \langle T(x + t(y - x)), y - x\rangle$ is continuous at 0_+.

Definition 2.7. Luc [18]. A mapping $f : K \to Y$ is said to be C-lower level closed iff for all $a \in Y$, the set $\{x \in K : f(x) \leq_C a\}$ is closed.

Definition 2.8. Let (E, d) be a metric space and let A, B be subsets of E. The Hausdorff metric $H(\cdot, \cdot)$ is defined by

$$H(A, B) := \max\{e(A, B), e(B, A)\},$$

where $e(A, B) := \sup_{a \in A} d(a, B)$ with $d(a, B) = \inf_{b \in B} d(a, b)$. Let $\{A_n\}$ be a sequence of subsets of E. We say A_n converges to $A \subset E$ in the sense of Hausdorff metric iff $H(A_n, A) \to 0$. It is easy to see that $e(A_n, A) \to 0$ if and only if $d(a_n, A) \to 0$ for all selection $a_n \in A_n$. For more details on this topic, see, e.g., [13].

3. Parametric Well-posedness for Vector Variational Inequalities

In this section we investigate the parametric well-posedness of Stampacchia and Minty vector variational inequalities. We first introduce the notions of parametric well-posedness for $\mathrm{VVP(p)}$ and $\mathrm{MVVP(p)}$. Fix an $e \in \mathrm{int\,C}$.

Definition 3.1. Let $p \in P$ and $\{p_n\} \subset P$ be a sequence converging to p. A sequence $\{x_n\} \subset K$ is said to be an approximating sequence for $\mathrm{VVI(p)}$ corresponding to $\{p_n\}$ iff there exists a sequence of positive numbers $\{\epsilon_n\}$ converging to zero such that

$$\langle A(p_n, x_n), x_n - y\rangle \not\geq_{\mathrm{int\,C}} \epsilon_n e, \quad \forall y \in K.$$

Definition 3.2. Let $p \in P$ and $\{p_n\} \subset P$ be a sequence converging to p. A sequence $\{x_n\} \subset K$ is said to be an approximating sequence for $\mathrm{MVVI(p)}$ corresponding to $\{p_n\}$ iff there exists a sequence of positive numbers $\{\epsilon_n\}$ converging to zero such that

$$\langle A(p_n, y), x_n - y\rangle \not\geq_{\mathrm{int\,C}} \epsilon_n e, \quad \forall y \in K.$$

Definition 3.3. The family $\{\mathrm{VVI(p) : p \in P}\}$ is said to be parametrically well-posed iff:

(i) there exists a unique solution x_p to $\mathrm{VVI(p)}$ for all $p \in P$;

(ii) for given $p \in P$ and $\{p_n\} \subset P$ with $p_n \to p$, every approximating sequence for $\mathrm{VVI(p)}$ corresponding to $\{p_n\}$ converges to x_p.

Definition 3.4. The family $\{\mathrm{VVI(p) : p \in P}\}$ is said to be parametrically well-posed in the generalized sense iff:

(i) the solution set $S(p)$ of $\mathrm{VVI(p)}$ is nonempty for all $p \in P$;

(ii) for given $p \in P$ and $\{p_n\} \subset P$ with $p_n \to p$, every approximating sequence for $\mathrm{VVI(p)}$ corresponding to $\{p_n\}$ has a subsequence converging to some point of $S(p)$.

Definition 3.5. The family $\{\mathrm{MVVI(p) : p \in P}\}$ is said to be parametrically well-posed iff:

(i) there exists a unique solution x_p to $\mathrm{MVVI(p)}$ for all $p \in P$;

(ii) for given $p \in P$ and $\{p_n\} \subset P$ with $p_n \to p$, every approximating sequence for $\mathrm{MVVI(p)}$ corresponding to $\{p_n\}$ converges to x_p.

Definition 3.6. The family $\{\mathrm{MVVI(p) : p \in P}\}$ is said to be parametrically well-posed in the generalized sense iff:

(i) the solution set $M(p)$ of $\mathrm{MVVI(p)}$ is nonempty for all $p \in P$;

(ii) for given $p \in P$ and $\{p_n\} \subset P$ with $p_n \to p$, every approximating sequence for $\mathrm{MVVI(p)}$ corresponding to $\{p_n\}$ has a subsequence converging to some point of $M(p)$.

To investigate the well-posedness of $\mathrm{VVI(p)}$ and $\mathrm{MVVI(p)}$, we consider the following sets:

$$T_p^S(\delta,\epsilon)=\cup_{p'\in B(p,\delta)}\{x\in K:\langle T(p',x),x-y\rangle\not\geq_{\mathrm{int\,C}}\epsilon e,\quad \forall y\in K\}$$

and

$$T_p^M(\delta,\epsilon)=\cup_{p'\in B(p,\delta)}\{x\in K:\langle T(p',y),x-y\rangle\not\geq_{\mathrm{int\,C}}\epsilon e,\quad \forall y\in K\}$$

for all $\delta,\epsilon\geq 0$, where $B(p,\delta)$ denotes the closed ball centered at p with radius δ.

Theorem 3.1. $\{\mathrm{VVI(p)}:\mathrm{p}\in\mathrm{P}\}$ *is parametrically well-posed if and only if for every $p\in P$, the solution set $S(p)$ of* $\mathrm{VVI(p)}$ *is nonempty and*

(1) $$\mathrm{diam\,T_p^S}(\delta,\epsilon)\to 0 \quad \textit{as} \quad (\delta,\epsilon)\to(0,0),$$

where diam means the diameter of a set.

Proof. Assume that $\{\mathrm{VVI(p)}:\mathrm{p}\in\mathrm{P}\}$ is parametrically well-posed. Then $S(p)$ is a singleton for all $p\in P$. If there exists some $p\in P$ such that $\mathrm{diamT_p^S}(\delta,\epsilon)\not\to 0$ as $(\delta,\epsilon)\to(0,0)$, then there exist positive number l and sequences $\{\delta_n\},\{\epsilon_n\},\{u_n\}$ and $\{v_n\}$ such that $\delta_n\to 0,\epsilon_n\to 0,u_n\in T_p^S(\delta_n,\epsilon_n),v_n\in T_p^S(\delta_n,\epsilon_n)$, and

$$\|u_n-v_n\|>l,\quad \forall n.$$

Obviously, $\{u_n\}$ and $\{v_n\}$ are approximating sequences for $\mathrm{VVI(p)}$. Then they have to converge to the unique solution of $\mathrm{VVI(p)}$. This arrives at a contradiction. Thus condition (1) holds.

Conversely, assume that $S(p)$ is nonempty and condition (1) holds. Obviously $S(p)$ is a singleton otherwise condition (1) does not hold. Let $p_n\to p$ and $\{x_n\}$ be an approximating sequence for $VVI(p)$ corresponding to $\{p_n\}$, i.e., there exists $\{\epsilon_n\}\subset R_+$ converging to zero such that

$$\langle A(p_n,x_n),x_n-y\rangle\not\geq_{\mathrm{int\,C}}\epsilon_n e,\quad \forall y\in K.$$

For given $\delta,\epsilon>0$, we have $x_n\in T_p^S(\delta,\epsilon)$ for all sufficiently large n. Let x_p be the unique solution of $\mathrm{VVI(p)}$. It follows that

$$\|x_n-x_p\|\leq \mathrm{diamT_p^S}(\delta,\epsilon)\to 0.$$

Thus $\{\mathrm{VVI(p)}:\mathrm{p}\in\mathrm{P}\}$ is parametrically well-posed. ■

Theorem 3.2. $\{\mathrm{VVI(p)}:\mathrm{p}\in\mathrm{P}\}$ *is parametrically well-posed in the generalized sense if and only if for every $p\in P$, the solution set $S(p)$ of* $\mathrm{VVI(p)}$ *is nonempty compact and*

(2) $$e(T_p^S(\delta,\epsilon),S(p))\to 0 \quad \text{as} \quad (\delta,\epsilon)\to(0,0).$$

Proof. Assume that $\{\mathrm{VVI(p)} : \mathrm{p} \in \mathrm{P}\}$ is parametrically well-posed in the generalized sense. Then $S(p) \neq \emptyset$ for all $p \in P$. Let $\{x_n\}$ be a sequence in $S(p)$. Obviously $\{x_n\}$ is an approximating sequence for $\mathrm{VVI(p)}$. Since $\{\mathrm{VVI(p)} : \mathrm{p} \in \mathrm{P}\}$ is parametrically well-posed in the generalized sense, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some point of $S(p)$. This proves that $S(p)$ is compact. To prove (2), suppose by contradiction that $e(T_p^S(\delta,\epsilon), S(p)) \not\to 0$ as $(\delta,\epsilon) \to (0,0)$. Then there exists a sequence $\{x_n\}$ with $x_n \in T_p^S(\frac{1}{n},\frac{1}{n})$ such that $d(x_n, S(p)) \not\to 0$ as $n \to \infty$, i.e., there exists some $\tau > 0$ such that

(3) $$x_n \notin S(p) + B(0,\tau), \quad \forall n.$$

Since $\{\mathrm{VVI(p)} : \mathrm{p} \in \mathrm{P}\}$ is parametrically well-posed in the generalized sense and $\{x_n\}$ is an approximating sequence for $\mathrm{VVI(p)}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some point of $S(p)$. This contradicts (3).

Conversely, assume that $S(p)$ is nonempty compact for all $p \in P$ and condition (2) holds. Let $p_n \to p$ and $\{x_n\}$ be an approximating sequence for $\mathrm{VVI(p)}$ corresponding to $\{p_n\}$. For given $\delta, \epsilon > 0$, we have $x_n \in T_p^S(\delta,\epsilon)$ for all sufficient large n. From (2), there exists a sequence $\{\bar{x}_n\}$ in $S(p)$ such that

$$\|x_n - \bar{x}_n\| \to 0.$$

Since $S(p)$ is compact, there exists a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ converging to $\bar{x} \in S(p)$. Hence the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $\bar{x}$. Thus $\{\mathrm{VVI(p)} : \mathrm{p} \in \mathrm{P}\}$ is parametrically well-posed in the generalized sense. ■

Theorem 3.3. $\{\mathrm{MVVI(p)} : \mathrm{p} \in \mathrm{P}\}$ *is parametrically well-posed if and only if for every* $p \in P$*, the solution set* $M(p)$ *of* $\mathrm{MVVI(p)}$ *is nonempty and*

$$\mathrm{diam}\, \mathrm{T_p^M}(\delta,\epsilon) \to 0 \quad \text{as} \quad (\delta,\epsilon) \to (0,0).$$

Proof. The conclusion follows from the similar arguments as in Theorem 3.1. ■

Theorem 3.4. $\{\mathrm{MVVI(p)} : \mathrm{p} \in \mathrm{P}\}$ *is parametrically well-posed in the generalized sense if and only if for every* $p \in P$*, the solution set* $M(p)$ *of* $\mathrm{MVVI(p)}$ *is nonempty compact and*

$$e(T_p^M(\delta,\epsilon), M(p)) \to 0 \quad \text{as} \quad (\delta,\epsilon) \to (0,0).$$

Proof. The conclusion follows from the similar arguments as in Theorem 3.2. ■

Now we recall the Minty lemma for the vector variational inequality:

Lemma 3.1. See [5, 8, 9, 29]. *Let K be nonempty, closed and convex and $A : K \to L(X, Y)$ be hemicontinuous and C-monotone. Then u is a solution of* VVI *if and only if it is a solution of* MVVI.

In terms of Lemma 3.1, we have the following result:

Theorem 3.5. *Let K be nonempty, closed and convex and $A(p, \cdot)$ be hemicontinuous and C-monotone for all $p \in P$. Then $\{\mathrm{VVI(p) : p \in P}\}$ is parametrically well-posed whenever $\{\mathrm{MVVI(p) : p \in P}\}$ is parametrically well-posed.*

Proof. Assume that $\{\mathrm{MVVI(p) : p \in P}\}$ is parametrically well-posed. By Lemma 3.1, there exists a unique solution u_p to $\mathrm{VVI(p)}$ for all $p \in P$. Let $p_n \to p$ and $\{u_n\}$ be an approximating sequence for $\mathrm{VVI(p)}$ corresponding to $\{p_n\}$, i.e., there exists $\{\epsilon_n\} \subset R_+$ converging to zero such that

$$\langle A(p_n, u_n), u_n - v\rangle \not\geq_{\mathrm{int\,C}} \epsilon_n e, \quad \forall v \in K.$$

By the C-monotonicity of $A(p, \cdot)$, we have

$$\langle A(p_n, v), u_n - v\rangle \not\geq_{\mathrm{int\,C}} \epsilon_n e, \quad \forall v \in K.$$

This means that $\{u_n\}$ is an approximating sequence for $\mathrm{MVVI(p)}$ corresponding to $\{p_n\}$. Thus $u_n \to u_p$ and so $\{\mathrm{VVI(p) : p \in P}\}$ is parametrically well-posed. ■

Theorem 3.6. *Let K be nonempty, closed and convex and $A(p, \cdot)$ be hemicontinuous and C-monotone for all $p \in P$. Then $\{\mathrm{VVI(p) : p \in P}\}$ is parametrically well-posed in the generalized sense whenever $\{\mathrm{MVVI(p) : p \in P}\}$ is parametrically well-posed in the generalized sense.*

Proof. The conclusion follows from the similar arguments as in Theorem 3.5. ■

Now we suppose that $\mathrm{VVI(p)}$ arises from $\mathrm{VOP(p)}$. Under suitable conditions, we derive the equivalence of well-posedness for $\mathrm{VVI(p)}$ and $\mathrm{VOP(p)}$. First we recall the following result:

Lemma 3.2. See [6, 9]. *Let K be nonempty, closed and convex, $F : K \to Y$ be C-convex and differentiable on an open set containing K such that $A(u) = F'(u)$. Then x is a solution of* VVI *if and only if it is a weakly efficient solution of* VOP.

Theorem 3.7. *Let $F(p, \cdot)$ be C-convex and differentiable on an open set containing K for all $p \in P$ and $F'_u(p, u) = A(p, u)$. Then* $\{\mathrm{VVI(p) : p \in P}\}$

is parametrically well-posed whenever $\{\mathrm{VOP(p) : p \in P}\}$ *is parametrically well-posed.*

Proof. Assume that $\{\mathrm{VOP(p) : p \in P}\}$ is parametrically well-posed. By Lemma 3.2, $\mathrm{VVI(p)}$ has a unique solution for all $p \in P$. Let $p_n \to p$ and $\{u_n\}$ be an approximating sequence for $\mathrm{VVI(p)}$ corresponding to $\{p_n\}$, i.e., there exists $\{\epsilon_n\} \subset R_+$ converging to zero such that

$$\langle A(p_n, u_n), u_n - v\rangle \not\geq_{\mathrm{int\,C}} \epsilon_n e, \quad \forall v \in K.$$

Since $F(p, \cdot)$ is C-convex, we have

$$F(p_n, u_n) - F(p_n, v) \leq_C \langle A(p_n, u_n), u_n - v\rangle \not\geq_{\mathrm{int\,C}} \epsilon_n e.$$

This yields

$$F(p_n, u_n) \not\geq_{\mathrm{int\,C}} F(p_n, v) + \epsilon_n e, \quad \forall v \in K$$

and so $\{u_n\}$ is a minimizing sequence for $\mathrm{VOP(p)}$ corresponding to $\{p_n\}$. By the parametrical well-posedness of $\{\mathrm{VOP(p) : p \in P}\}$, u_n converges to the unique weakly efficient solution u_p of $\mathrm{VOP(p)}$. Again from Lemma 3.2, u_p is the unique weakly efficient solution of $\mathrm{VVI(p)}$. This proves that $\{\mathrm{VVI(p) : p \in P}\}$ is parametrically well-posed. ■

To prove the converse, we need the following concept and result.

Let $\xi : Y \to R$ be defined by

$$\xi(y) = \min\{t \in R : y \in te - C\}, \quad \forall y \in Y.$$

For the properties of ξ, we refer to [18, 27] and the references therein.

Lemma 3.3. Huang [11]. *Let $f : K \to Y$ be C-lower level closed and $\xi(f)$ be bounded below. Given $\epsilon > 0$ and an x^* satisfying $f(x^*) \not\geq_C f(x) + \epsilon e, \forall x \in K$, then for any real number $\delta > 0$, there exists $x' \in K$ such that:*

(i) $f(x') \leq_C f(x^*)$;

(ii) $\|x' - x^*\| \leq \delta$;

(iii) $f(x') - f(x) \not\geq_{\mathrm{int\,C}} \frac{\epsilon}{\delta}\|x - x'\|e, \quad \forall x \in K.$

Theorem 3.8. *Let K be bounded, $F(p, \cdot)$ be C-lower level closed, differentiable on an open set containing K, and $\xi(F(p, \cdot)$ be bounded below for all $p \in P$. Let $A(p, u) = F'_u(p, u)$. Then $\{\mathrm{VOP(p) : p \in P}\}$ is parametrically well-posed whenever $\{\mathrm{VVI(p) : p \in P}\}$ is parametrically well-posed.*

Proof. Assume that $\{\mathrm{VVI(p) : p \in P}\}$ is parametrically well-posed. By Lemma 3.2, $\mathrm{VOP(p)}$ has a unique weakly efficient solution u_p for all p. Let $p_n \to p$ and

$\{u_n\}$ be a minimizing sequence for $\mathrm{VOP(p)}$ corresponding to $\{p_n\}$,i.e., there exists $\{\epsilon_n\} \subset R_+$ converging to zero such that for all n,

$$F(p_n, u_n) \not\geq_{\mathrm{int\,C}} F(p_n, v) + \epsilon_n e, \quad \forall v \in K.$$

By Lemma 3.3, there exists $\bar{u}_n \in K$ such that

$$\|\bar{u}_n - u_n\| \leq \sqrt{\epsilon_n} \tag{4}$$

and

$$F(p_n, x) - F(p_n, \bar{u}_n) \not\leq_{\mathrm{int\,C}} -\sqrt{\epsilon_n}\|x - \bar{u}_n\|e, \quad \forall x \in K. \tag{5}$$

For given $v \in K$, substituting $x = \bar{u}_n + t(v - \bar{u}_n), t \in (0, 1)$, in (5), we obtain

$$\frac{F(p_n, \bar{u}_n + t(v - \bar{u}_n)) - F(p_n, \bar{u}_n)}{t} \not\leq_{\mathrm{int\,C}} -\sqrt{\epsilon_n}\|v - \bar{u}_n\|e.$$

It follows that

$$\langle F'_u(p_n, \bar{u}_n), v - \bar{u}_n\rangle \not\leq_{\mathrm{int\,C}} -\sqrt{\epsilon_n}\|v - \bar{u}_n\|e \geq_C -\sqrt{\epsilon_n}\mathrm{diamK\,e}, \quad \forall \mathrm{v} \in \mathrm{K},$$

i.e.,

$$\langle A(p_n, \bar{u}_n), \bar{u}_n - v\rangle \not\geq_{\mathrm{int\,C}} \sqrt{\epsilon_n}\mathrm{diamK\,e}, \quad \forall \mathrm{v} \in \mathrm{K}.$$

This means that $\{\bar{u}_n\}$ is an approximating sequence for $\mathrm{VVI(p)}$ corresponding to $\{p_n\}$. By the parametrical well-posedness of $\{\mathrm{VVI(p) : p \in P}\}$, we obtain $\bar{u}_n \to u_p$. It follows from (4) that $u_n \to u_p$, and so $\{\mathrm{VOP(p) : p \in P}\}$ is parametrically well-posed. ■

Theorem 3.9. *Let $F(p, \cdot)$ be C-convex and differentiable on an open set containing K for all $p \in P$ and $F'_u(p, u) = A(p, u)$. Then $\{\mathrm{VVI(p) : p \in P}\}$ is parametrically well-posed in the generalized sense whenever $\{\mathrm{VOP(p) : p \in P}\}$ is parametrically well-posed in the generalized sense.*

Proof. The conclusion follows from the similar arguments as in Theorem 3.7. ■

Theorem 3.10. *Let K be bounded, $F(p, \cdot)$ be C-lower level closed, differentiable on an open set containing K, and $\xi(F(p, \cdot)$ be bounded below for all $p \in P$. Let $A(p, u) = F'_u(p, u)$. Then $\{\mathrm{VOP(p) : p \in P}\}$ is parametrically well-posed in the generalized sense whenever $\{\mathrm{VVI(p) : p \in P}\}$ is parametrically well-posed in the generalized sense.*

Proof. The conclusion follows from similar arguments as in Theorem 3.8. ■

4. Well-posedness for Vector Optimization with Vector Variational Inequality Constraint

In this section we investigate the well-posedness of the following vector optimization problem with a vector variational inequality constraint:

$$\text{VOPVVIC} \qquad \inf_{(\mathrm{p},\mathrm{u})\in\Delta} \mathrm{F}(\mathrm{p},\mathrm{u})$$

where

$$\Delta = \{(p,u) \in P \times K : u \text{ is a solution of } \mathrm{VVI(p)}\}.$$

Definition 4.1. A sequence $\{(p_n, u_n)\}$ is said to be a minimizing sequence for $\mathrm{VOPVVIC}$ iff:

(i) $d(F(p_n, u_n), \mathrm{Min}(\Delta, \mathrm{F})) \to 0$ as $n \to \infty$;

(ii) there exists $\{\epsilon_n\} \subset R_+$ with $\epsilon_n \to 0$ such that $F(p_n, x_n) \not\geq_{\mathrm{int\,C}} F(p_n, y) + \epsilon_n e, \quad \forall y \in K$.

Definition 4.2. $\mathrm{VOPVVIC}$ is said to be well-posed iff:

(i) $\mathrm{VOPVVIC}$ has a unique weakly efficient solution $(\bar{p}, u_{\bar{p}})$;

(ii) every minimizing sequence $\{(p_n, u_n)\}$ for $\mathrm{VOPVVIC}$ converges to $(\bar{p}, u_{\bar{p}})$.

Definition 4.3. $\mathrm{VOPVVIC}$ is said to be well-posed in the generalized sense iff:

(i) $\mathrm{VVI(p)}$ has at least a solution for all $p \in P$;

(ii) $\mathrm{argmin}(\Delta, \mathrm{F})$ is nonempty;

(iii) every minimizing sequence $\{(p_n, u_n)\}$ for $\mathrm{VOPVVIC}$ has a subsequence convergent to some point of $\mathrm{argmin}(\Delta, \mathrm{F})$.

Theorem 4.1. *Let P be sequentially compact, $F : P \times K \to Y$ be continuous, $\{\mathrm{VVI(p)} : \mathrm{p} \in \mathrm{P}\}$ be parametrically well-posed in the generalized sense and $\mathrm{Min}(\Delta, \mathrm{F})$ be nonempty closed. Then* $\mathrm{VOPVVIC}$ *is well-posed in the generalized sense.*

Proof. Let $\{(p_n, u_n)\}$ be a minimingzing sequence for $\mathrm{VOPVVIC}$. Then there exists $\{\epsilon_n\} \subset R_+$ such that $\epsilon_n \to 0$ and

$$F(p_n, u_n) \not\geq_{\mathrm{int\,C}} F(p_n, v) + \epsilon_n e, \quad \forall v \in K.$$

Since P is sequentially compact, there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $p_{n_k} \to p$. Then $\{u_{n_k}\}$ is an approximating sequence for $\mathrm{VVI(p)}$ corresponding to $\{p_{n_k}\}$. By the parametric well-posedness of $\{\mathrm{VVI(p) : p \in P}\}$, $\{u_{n_k}\}$ has a subsequence $\{u_{n_{k_l}}\}$ convergent to a solution u_p of $\mathrm{VVI(p)}$. By using the continuity of F, we have

$$\begin{aligned} & d(F(p,u_p), \mathrm{Min}(\Delta, \mathrm{F})) \\ \leq\ & d(F(p,u_p), F(p_{n_{k_l}}, u_{n_{k_l}})) + d(F(p_{n_{k_l}}, u_{n_{k_l}}), \mathrm{Min}(\Delta, \mathrm{F})) \to 0. \end{aligned}$$

This implies that (p, u_p) is a weakly efficient solution of VOPVVIC since $\mathrm{Min}(\Delta, \mathrm{F})$ is closed. Thus VOPVVIC is well-posed in the generalized sense. ∎

Theorem 4.2. *Let P be sequentially compact, $F : P \times K \to Y$ be continuous, $\{\mathrm{VVI(p) : p \in P}\}$ be parametrically well-posed and* VOPVVIC *has a unique solution $(\bar{p}, u_{\bar{p}})$. Then* VOPVVIC *is well-posed.*

Proof. Let $\{(p_n, u_n)\}$ be a minimizing sequence for VOPVVIC. By the same arguments as in Theorem 4.1, we know that $\{(p_n, u_n)\}$ has a subsequence convergent to $(\bar{p}, u_{\bar{p}})$. Further, we can prove that any converging subsequence of $\{(p_n, u_n)\}$ converges to $(\bar{p}, u_{\bar{p}})$. Thus $(p_n, u_n) \to (\bar{p}, u_{\bar{p}})$. Thus VOPVVIC is well-posed. ∎

References

1. H. Attouch and R. J. B. Wets, Quantitative stability of variational systems, II. A framework for nonlinear conditioning, *SIAM J. Optim.*, **3** (1993), 359-381.

2. E. Bednarczuk, Well posedness of vector optimization problem, In: J. Jahn and W. Krabs (eds) Recent Advances and Historical Development of Vector Optimization Problems, Lecture Notes in Economics and Mathematical systems, Vol. 294, Berlin: Springer-verlag, pp. 51-61, 1987.

3. E. Bednarczuk, An approach to well-posedness in vector optimization: consequences to stability, Parametric optimization, *Control Cybernet.*, **23** (1994), 107-122.

4. E. Cavazzuti, and J. Morgon, Well-posed saddle point problems, in: Optimization Thoery and Algorithms, Edited by J.B. Hiriart-Urruty, W.Oettli, and J. Stoer, Marcel Dekker, New York, NY, pp. 61-76, 1983.

5. G. Y. Chen, Existence of solutions for a vector variational inequality: an extension of the Hartman-Stampacchia theorem, *J. Optim. Theory and Appl.*, **74** (1992), 445-456.

6. G. Y. Chen and X. Q. Yang, The vector complementary problem and its equivalences with the weak minimal elements in ordered spaces, *J. Math. Anal. Appl.*, **153(1)** (1990), 136-158.

7. A. L. Dontchev and T. Zolezzi, Well-Posed Optimization Problems, Lecture Notes in Math., 1543, Berlin: Springer-Verlag, 1993.

8. F. Giannessi, Theorems of Alterative, Quadratic Progams and Complementarity Problems, In: R. W. Cottle, F. Giannessi, J. L. Lions (ed.) Variational Inequalities and Complementarity Problems, Wiley, New York, 1980.

9. F. Giannessi, On Minty variational principle, in: New trends in mathematical programming, 93-99, Appl. Optim., 13, Kluwer Acad. Publ., Boston, MA, 1998.

10. X. X. Huang, Pointwise well-posedness of perturbed vector optimization problems in a vector-valued variational principle, *J. Optim. Theory Appl.*, **108** (2001), 671-684.

11. X. X. Huang, Extended well-posedness properties of vector optimization problems, *J. Optim. Theory Appl.*, **106** (2000), 165-182.

12. X. X. Huang, Extended and strongly extended well-posedness of set-valued optimization problems, *Math. Methods Oper. Res.*, **53** (2001), 101-116.

13. E. Klein and A. C. Thompson, Theory of Correspondences. John Wiley & Sons, Inc., New York, 1984.

14. I. V. Konnov and J. C. Yao, On the generalized vector variational inequality problem, *J. Math. Anal. Appl.*, **206(1)** (1997), 42-58.

15. B. Lemaire, C. Ould Ahmed Salem and J. P. Revalski, Well-posedness by perturbations of variational problems, *J. Optim. Theory Appl.*, **115(2)** (2002), 345-368.

16. M. B. Lignola and J. Morgan, Well-posedness for optimization problems with constraints defined by variational inequalities having a unique solution, *J. Global Optim.*, **16(1)** (2000), 57-67.

17. M. B. Lignola and J. Morgan, Approximating solutions and α-well-posedness for variational inequalities and Nash equilibria, in: *Decision and Control in Management Science, Kluwer Academic Publishers*, pp. 367-378, 2002.

18. D. T. Luc, Theory of Vector Optimization, Lecture Notes in Economics and Mathematical Systems, No. 319, Berlin: Springer-Verlag, 1989.

19. R. Lucchetti, Well posedness, towards vector optimization, In: J. Jahn and W. Krabs (eds) Recent Advances and Historical Development of Vector Optimization Problems, Lecture Notes in Economics and Mathematical systems, Vol 294, Berlin: Springer-Verlag, pp. 194-207, 1987.

20. R. Lucchetti and J. Revalski (eds), Recent Developments in Well-Posed Variational Problems, Kluwer Academic Publishers, Dordrecht, Holland, 1995.

21. Z. Q. Luo, J. S. Pang, D. Ralph and S. Q. Wu, Exact penalization and stationarity conditions of mathematical programs with equilibrium constraints, *Math. Programming Ser. A*, **75 (1)** (1996), 19-76.

22. P. Marcotte and D. L. Zhu, Exact and inexact penalty methods for the generalized bilevel programming problem, *Math. Programming Ser. A*, **74 (2)** (1996), 141-157.

23. M. Margiocco, F. Patrone and L. Pusillo, A new approach to Tikhonov well-posedness for Nash equilibria, *Optimization*, **40(4)** (1997), 385-400.

24. M. Margiocco, F. Patrone and L. Pusillo, Metric characterizations of Tikhonov well-posedness in value, *J. Optim. Theory Appl.*, **100(2)** (1999), 377-387.

25. M. Margiocco, F. Patrone and L. Pusillo, On the Tikhonov well-posedness of concave games and Cournot oligopoly games, *J. Optim. Theory Appl.*, **112(2)** (2002), 361-379.

26. E. Miglierina and E. Molho, Well-posedness and convexity in vector optimization, *Math. Methods Oper. Res.*, **58** (2003), 375-385.

27. Chr. Tammer, A generalization of Ekeland's variational principle, *Optimization*, **25** (1992), 129-141.

28. A. N. Tykhonov, On the stability of the functional optimization problem, *USSR J. Comput. Math. Math. Physics*, **6** (1966), 631-634.

29. X. M. Yang, X. Q. Yang and K. L. Teo, Some remarks on the Minty vector variational inequality, *J. Optim. Theory Appl.*, **121 (1)** (2004), 193-201.

30. X. Q. Yang, Vector variational inequality and its duality, *Nonlinear Anal. TMA*, **95** (1993), 729-734.

31. T. Zolezzi, Well-posedness criteria in optimization with application to the calculus of variations, *Nonlinear Anal. TMA*, **25** (1995), 437-453.

32. T. Zolezzi, Extended well-posedness of optimization problems, *J. Optim. Theory Appl.*, **91** (1996), 257-266.

Ya-Ping Fang
Department of Mathematics,
Sichuan University,
Chengdu Sichuan 610064,
P. R. China

Nan-Jing Huang
Department of Mathematics,
Sichuan University,
Chengdu Sichuan 610064,
P. R. China