TAIWANESE JOURNAL OF MATHEMATICS
Vol. 11, No. 5, pp. 1251-1285, December 2007
This paper is available online at http://www.math.nthu.edu.tw/tjm/

ON SOME INTEGRAL OPERATORS ON THE UNIT POLYDISK AND THE UNIT BALL

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Dedicated to the memory of Professor Huei-Shyong Lue

Abstract. Let $\mathbb{D}^n$ be the unit polydisk and B be the unit ball in $\mathbb{C}^n$ respectively. In this paper, we extend the Cesàro operator to the unit polydisk and the unit ball. We prove that the generalized Cesàro operator $\mathcal{C}^{\vec{b},\vec{c}}$ is bounded on the Hardy space $H^p(\mathbb{D}^n)$ and the mixed norm space $A^{p,q}_{\vec{\mu}}(\mathbb{D}^n)$, when $0 < q < \infty$, $p \in (0, 1]$ and $Re\,(b_j + 1) > Re\,c_j > 0$, $j = 1, \ldots, n$, or if $0 < q < \infty$, $p > 1$ and $Re\,(b_j+1) > Re\,c_j \geq 1$, $j = 1, \ldots, n$. Here $\vec{\mu} = (\mu_1, \ldots, \mu_n)$ and each μ_j, $j \in \{1, \ldots, n\}$ is a positive Borel measure on the interval $[0, 1)$. We also introduce a new class of averaging integral operators $\mathcal{C}^{b,c}_{\zeta_0}$ (the generalized Cesàro operators) on B and prove the boundedness of the operator on the Hardy space $H^p(B)$, $p \in (0,\infty)$, the mixed-norm space $\mathcal{A}^{p,q}_{\mu}(B)$, $0 < p, q < \infty$ and the α-Bloch space, when $\alpha > 1$. Finally, we study the boundedness and compactness of recently introduced Riemann-Stieltjes type operators T_g and L_g, from H^∞ and Bergman type spaces to α-Bloch spaces and little α-Bloch spaces on B.

1. Introduction

For an analytic function $f(z)$ in the unit disk $\mathbb{D}$ with Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the Cesàro operator acting on f is

$$\mathcal{C}f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n} a_k\right) z^n.$$

Received December 12, 2006, accepted December 19, 2006.
Communicated by Wen-Wei Lin.
2000 *Mathematics Subject Classification*: Primary 47B38, Secondary 46E15.
Key words and phrases: Cesàro operator, Riemann-Stieltjes operator, Hardy space, Bergman space, Bloch space, boundedness, compactness.
The first author is partially supported by a research grant from the United States Army Research Office and a competitive research grant at Georgetown University.
The second author is partially supported by the NNSF China (No.10671115), PHD Foundation (No. 20060560002) and NSF of Guangdong Province (No. 06105648).

After some simple calculations, we have

$$\mathcal{C}f(z) = \frac{1}{z}\int_0^z \frac{f(w)}{1-w}dw, \qquad z \in \mathbb{D}.$$

It is well known that Cesàro operator acts as a bounded linear operator on many analytic function spaces, however it is not bounded on the Bloch space (see, for example, [12, 16, 28, 31, 33-35, 40, 41, 48]).

For $b, c \in \mathbb{C}$ with $Re\,(b+1) > Re\, c > 0$, the generalized Cesàro operator $\mathcal{C}^{b,c}$ was recently defined in [1] in the following way

$$\mathcal{C}^{b,c}f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{A_n^{b+1,c}}\sum_{k=0}^{n} b_{n-k}a_k\right)z^n$$

where

$$A_k^{b,c} = \frac{(b,k)}{(c,k)} \quad \text{and} \quad b_k = \frac{(b+1-c)}{c}A_k^{b+1,c+1} = \frac{(b+1-c)}{b}A_k^{b,c},$$

and (a,n) is the shifted factorial defined by Appel's symbol

$$(a,n) = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad n \in \mathbb{N}$$

and $(a,0) = 1$ for $a \neq 0$.

It was shown in [1] that $\mathcal{C}^{b,c}$ can be written in the following form:

$$\mathcal{C}^{b,c}f(z) = \frac{\Gamma(b+1)}{\Gamma(c)\Gamma(b+1-c)}\int_0^1 f(tz)\frac{t^{c-1}(1-t)^{b-c}}{(1-tz)^{b+1-c}}F(c-1, c-b-1, c, tz)dt,$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic function on the unit disc $\mathbb{D}$ and $F(a,b,c,z)$ is the hypergeometric function. The hypergeometric function is defined by the power series expansion

$$F(a,b,c,z) = 1 + \sum_{n=1}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{z^n}{n!}, \quad (|z| < 1)$$

where a, b, c are complex numbers such that $c \neq -m, m \in \mathbb{N}_0$. It is assumed $c \neq -m,\ m \in \mathbb{N}_0$, to prevent the denominators vanishing. When $c = 1$ and $b = \alpha+1$, the operator $\mathcal{C}^{b,c}$ becomes the generalized Cesàro (or α-Cesàro) operator defined as in [37] (see, also [40, 41, 48]). When $c = 1$ and $b = 1$, operator $\mathcal{C}^{1,1}$ becomes the classical Cesàro operator $\mathcal{C}$.

In this paper, we generalize the Cesàro operator in three ways. We study the boundedness and compactness of these operators between certain spaces of analytic functions on the unit polydisk and unit ball.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preceq b$ means that there is a positive constant C such that $a \le Cb$. If both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. The Operator $\mathcal{C}^{\vec{b},\vec{c}}$ on the polydisk

Following the lines of paper [37], in [1] the authors have studied the generalized Cesàro operator $\mathcal{C}^{b,c}$ on the Hardy, Bloch, and BMOA space. However, for the case of the Hardy space $H^p(\mathbb{D})$ they have only proved that the operator $\mathcal{C}^{b,c}$ is bounded when $0 < p \le 1$. Here we prove that the generalized Cesàro operator $\mathcal{C}^{b,c}$ is bounded on the Hardy space $H^p(\mathbb{D})$ for every $p \in (0, \infty)$, moreover we extend naturally the generalized Cesàro operator on the polydisk and prove the boundedness on the corresponding Hardy and the mixed norm spaces.

Motivated by one-dimensional generalized Cesàro operator, we define an operator on the polydisk $\mathbb{D}^n$, as follows

$$\begin{aligned}\mathcal{C}^{\vec{b},\vec{c}}f(z) &= \prod_{j=1}^{n} \frac{\Gamma(b_j+1)}{\Gamma(c_j)\Gamma(b_j+1-c_j)} \int_{[0,1)^n} f(t_1 z_1, \ldots, t_n z_n) \\ &\quad \times \prod_{j=1}^{n} \frac{t_j^{c_j-1}(1-t_j)^{b_j-c_j}}{(1-t_j z_j)^{b_j+1-c_j}} F(c_j-1, c_j-b_j-1, c_j, t_j z_j)\, dt_j, \end{aligned} \tag{1}$$

where $Re\,(b_j+1-c_j) > 0$, $j = 1, \ldots, n$, where $f(z) = \sum_{|\mathbf{k}|=0}^{\infty} a_{\mathbf{k}} z^{\mathbf{k}}$, $\mathbf{k} \in (\mathbb{Z}_+)^n$ is an analytic function on the unit polydisk $\mathbb{D}^n$.

Now we introduce some notation. We write $z \cdot w = (z_1 w_1, \ldots, z_n w_n)$, $z, w \in \mathbb{C}^n$; $e^{i\theta} = (e^{i\theta_1}, \ldots, e^{i\theta_n})$, $d\theta = d\theta_1 \cdots d\theta_n$, $dt = dt_1 \cdots dt_n$ and u, v denote vectors in $\mathbb{C}$. When we write $0 \le r < 1$, where $r = (r_1, \ldots, r_n)$, this means that $0 \le r_i < 1$ $(i = 1, \ldots, n)$.

For the case of the unit polydisk we prove the following theorem in this section, which generalizes the main results in papers [6, 9, 31, 39].

Theorem 2.1. *Assume that $p \in (0, 1]$ and $Re\,(b_j + 1) > Re\, c_j > 0$, $j = 1, \ldots, n$, or $p > 1$ and $Re\,(b_j + 1) > Re\, c_j \ge 1$, $j = 1, \ldots, n$. Then there is a constant C independent of f and r such that*

$$\int_{[0,2\pi]^n} |\mathcal{C}^{\vec{b},\vec{c}}(f)(r \cdot e^{i\theta})|^p d\theta \le C \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta,$$

for all $f \in H(\mathbb{D}^n)$ and $r \in (0, 1)$.

In order to prove Theorem 2.1, we need some auxiliary results which are incorporated in the following lemmas.

Lemma 2.1. ([7]). *Let $0 < p < \infty$ and $0 < r < 1$. Then there is a constant C independent of f and r such that*

$$\int_{[0,2\pi]^n} \sup_{0<\tau<1} |f(\tau \cdot r \cdot e^{i\theta})|^p d\theta \leq C \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta,$$

for all $f \in H(\mathbb{D}^n)$.

Lemma 2.2. ([13]). *For each $1 < s < \infty$ there is a positive constant $C = C(s)$ such that*

$$\int_{-\pi}^{\pi} \frac{1}{|1 - re^{i\theta}|^s} d\theta \leq \frac{C}{(1-r)^{s-1}}, \qquad 0 < r < 1.$$

Lemma 2.3. ([6]) *Let $0 < p < \infty$, $1 < s < \infty$ and $0 < r < 1$. Then there is a constant C independent of f and r such that*

$$\int_{[0,1)^n} \Big(\int_{[0,2\pi]^n} |f(\tau \cdot r \cdot e^{i\theta})|^{ps} d\theta \Big)^{1/s} \prod_{j=1}^{n} (1-\tau_j)^{-1/s} d\tau \leq C \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta,$$

for all $f \in H(\mathbb{D}^n)$.

For real y and $\sigma > -1$, set

$$H^\sigma(y) = \frac{1}{1+|y|} \begin{cases} 1 + |y|^\sigma, & \text{if } \sigma < 0; \\ \log(2 + 1/|y|), & \text{if } \sigma = 0; \\ 1, & \text{if } \sigma > 0. \end{cases}$$

Lemma 2.4. *For $Re\, \sigma > -1$ and $Re\, c \geq 1$, there is a constant C such that*

$$\int_0^1 \frac{|(1-x)^{c-1} x^{\sigma+1}| dx}{[x^2+\varphi^2][x^2+\theta^2]^{(\sigma+1)/2}} \leq C \frac{H^{Re\,\sigma}(\varphi/\theta)}{|\theta|}$$

for all real φ and $\theta \neq 0$.

Proof. Without loss of generality we may assume that σ and c are real numbers. Since $Re\, c \geq 1$ we have that

$$\int_0^1 \frac{(1-x)^{c-1} x^{\sigma+1} dx}{[x^2+\varphi^2][x^2+\theta^2]^{(\sigma+1)/2}} \leq \int_0^1 \frac{x^{\sigma+1} dx}{[x^2+\varphi^2][x^2+\theta^2]^{(\sigma+1)/2}}.$$

From this and by Lemma 2.1 in [5] the result follows.

For any measurable function $g(e^{i\theta})$, define $E_s g(e^{i\theta}) = E_{s_1,\cdots,s_n} g(e^{i\theta})$ by

$$E_s g(e^{i\theta}) = \begin{cases} g(e^{i(s+1)\theta}), & \text{if } |s_j\theta_j| \le \pi \text{ for all } j \in \{1,\dots,n\}; \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.5. ([9]) *Let $\sigma_j > -1$, $j = 1,\dots,n$, $1 < p < \infty$ and*

$$A_{\vec{\sigma},p} = 2^{n/p} \int_{\mathbb{R}^n} \prod_{j=1}^n \frac{H^{\sigma_j}(s_j)}{|s_j+1|^{1/p}} ds.$$

Then $A_{\vec{\sigma},p} < \infty$ and

$$\int_{[-\pi,\pi]^n} \Big(\int_{\mathbb{R}^n} \prod_{j=1}^n H^{\sigma_j}(s_j) E_s g(e^{i\theta}) ds \Big)^p d\theta \le A_{\vec{\sigma},p} \int_{[-\pi,\pi]^n} g^p(e^{i\theta}) d\theta$$

for all measurable $g \ge 0$.

Proof of Theorem 2.1. We follow the lines of the proof of Theorem 1 in [9]. We need to show that

$$M_p^p(\mathcal{C}^{\vec{b},\vec{c}}(f), r) \le C M_p^p(f, r)$$

for some constant $C > 0$. For the sake of simplicity, we assume that $b_j, c_j, j = 1,\dots,n$, are real numbers such that $b_j + 1 > c_j > 0$, $j = 1,\dots,n$.

Case $0 < p < 1$. Let $f \in H(\mathbb{D}^n)$ and let

$$\gamma := M_p^p(\mathcal{C}^{\vec{b},\vec{c}}(f), r) = \int_{[0,2\pi]^n} |\mathcal{C}^{\vec{b},\vec{c}}(f)(r \cdot e^{i\theta})|^p d\theta.$$

Let $t_k = 1 - 2^{-k}$, $k \in \mathbb{Z}_+$. Then $0 = t_0 < t_1 < t_2 < \cdots < 1$ forms a partition of the interval $[0,1)$. It is obvious that $t_k - t_{k-1} = 1 - t_k = 2(1 - t_{k+1})$. By Lemma 2.1, the boundedness of $F(c_j - 1, c_j - b_j - 1, c_j, tz)$, ($j = 1,\dots,n$) on the unit disk and some simple calculations, we obtain

$$\begin{aligned}\gamma &\le C \int_{[0,2\pi]^n} \Big(\int_{[0,1)^n} |f(t \cdot r \cdot e^{i\theta})| \\ &\quad \times \frac{\prod_{j=1}^n t_j^{c_j-1}(1-t_j)^{b_j-c_j}}{\prod_{j=1}^n |1 - t_j r_j e^{i\theta_j}|^{b_j+1-c_j}} |F(c_j-1, c_j-b_j-1, c_j, t_j r_j e^{i\theta_j})| dt \Big)^p d\theta \\ &\le C \sum_{k_1,\dots,k_n=1}^{\infty} \int_{[0,2\pi]^n} \Big(\int_{t_{k_1-1}}^{t_{k_1}} \cdots \int_{t_{k_n-1}}^{t_{k_n}} |f(t \cdot r \cdot e^{i\theta})|\end{aligned}$$

$$\times \frac{\prod_{j=1}^{n} t_j^{c_j-1}(1-t_j)^{b_j-c_j}}{\prod_{j=1}^{n}|1-t_jr_je^{i\theta_j}|^{b_j+1-c_j}}|F(c_j-1,c_j-b_j-1,c_j,t_jr_je^{i\theta_j})|dt\Big)^p d\theta$$
$$\leq C\sum_{k_1,\dots,k_n=1}^{\infty}\frac{1}{2^{p\sum_{j=1}^{n}k_j(b_j+1-c_j)}}\int_{[0,2\pi]^n}\sup_{t_{\mathbf{k}-1}<t<t_{\mathbf{k}}}\Big(\frac{|f(t\cdot r\cdot e^{i\theta})|}{\prod_{j=1}^{n}|1-t_jr_je^{i\theta_j}|^{b_j+1-c_j}}\Big)^p d\theta$$
$$\leq C\sum_{k_1,\dots,k_n=1}^{\infty}\frac{1}{2^{p\sum_{j=1}^{n}k_j(b_j+1-c_j)}}\int_{[0,2\pi]^n}\sup_{0<t<t_{\mathbf{k}}}\Big(\frac{|f(t\cdot r\cdot e^{i\theta})|}{\prod_{j=1}^{n}|1-t_jr_je^{i\theta_j}|^{b_j+1-c_j}}\Big)^p d\theta$$
$$\leq C\sum_{k_1,\dots,k_n=1}^{\infty}\frac{1}{2^{p\sum_{j=1}^{n}k_j(b_j+1-c_j)}}\int_{[0,2\pi]^n}\Big(\frac{|f(t_{\mathbf{k}}\cdot r\cdot e^{i\theta})|}{\prod_{j=1}^{n}|1-t_{k_j}r_je^{i\theta_j}|^{b_j+1-c_j}}\Big)^p d\theta.$$

The last line of above inequality is bounded by

$$C\sum_{k_1,\dots,k_n=1}^{\infty}\int_{t_{k_1}}^{t_{k_1+1}}\cdots\int_{t_{k_n}}^{t_{k_n+1}}\int_{[0,2\pi]^n}\Big(\frac{|f(t\cdot r\cdot e^{i\theta})|}{\prod_{j=1}^{n}|1-t_jr_je^{i\theta_j}|^{b_j+1-c_j}}\Big)^p d\theta$$
$$\times\prod_{j=1}^{n}(1-t_j)^{p(b_j+1-c_j)-1}dt$$
$$\leq C\int_{[0,1)^n}\int_{[0,2\pi]^n}\Big(\frac{|f(t\cdot r\cdot e^{i\theta})|}{\prod_{j=1}^{n}|1-t_jr_je^{i\theta_j}|^{b_j+1-c_j}}\Big)^p d$$
$$\theta\prod_{j=1}^{n}(1-t_j)^{p(b_j+1-c_j)-1}dt \tag{2}$$

Here, $t_{\mathbf{k}}=(t_{k_1},\dots,t_{k_n})$. Now, we choose $a>1$ such that $\max_{j=1,\dots,n}\{1-p(b_j+1-c_j)\}<1/a<1$ and $1/a+1/b=1$. Then by Hölder's inequality and Lemma 2.2 we obtain

$$\int_{[0,2\pi]^n}\Big(\frac{|f(t\cdot r\cdot e^{i\theta})|}{\prod_{j=1}^{n}|1-t_jr_je^{i\theta_j}|^{b_j+1-c_j}}\Big)^p d\theta$$
$$\leq\Big(\int_{[0,2\pi]^n}|f(t\cdot r\cdot e^{i\theta})|^{pa}d\theta\Big)^{1/a}$$
$$\Big(\int_{[0,2\pi]^n}\frac{d\theta}{|\prod_{j=1}^{n}|1-t_jr_je^{i\theta_j}|^{b_j+1-c_j}|^{pb}}\Big)^{1/b} \tag{3}$$
$$\leq\Big(\int_{[0,2\pi]^n}|f(t\cdot r\cdot e^{i\theta})|^{pa}d\theta\Big)^{1/a}\prod_{j=1}^{n}(1-t_jr_j)^{-(b_j+1-c_j)p+1-1/a}$$
$$\leq\Big(\int_{[0,2\pi]^n}|f(t\cdot r\cdot e^{i\theta})|^{pa}d\theta\Big)^{1/a}\prod_{j=1}^{n}(1-t_j)^{-(b_j+1-c_j)p+1-1/a}.$$

Finally, from (2), (3) and Lemma 2.3 we can obtain

$$
\begin{aligned}
M_p^p(\mathcal{C}^{\vec{b},\vec{c}}(f),r) \le & C\int_{[0,1)^n}\Big(\int_{[0,2\pi]^n}|f(t\cdot r\cdot e^{i\theta})|^{pa}d\theta\Big)^{1/a}\prod_{j=1}^n(1-t_j)^{-1/a}dt \\
\le & C\int_{[0,2\pi]^n}|f(r\cdot e^{i\theta})|^p d\theta.
\end{aligned}
$$

Case $p>1$. Let $f\in H(\mathbb{D}^n)$ and $0<r<1$, set $f_r(e^{i\varphi})=f(r\cdot e^{i\varphi})$. Then for $0<t<1$, $f(t\cdot r\cdot e^{i\theta})$ is given by the following integral

$$
f(t\cdot r\cdot e^{i\theta})=\frac{1}{2\pi}\int_{[-\pi,\pi]^n}f_r(e^{i\varphi})\prod_{j=1}^n P(t_j,\varphi_j-\theta_j)d\varphi \tag{4}
$$

where $P(\xi,\eta)$ is the Poisson's kernel i.e.

$$
P(\xi,\eta)=\frac{1-\xi^2}{1-2\xi\cos\eta+\xi^2}.
$$

Combining (1) and (4) and using Fubini's theorem, we have

$$
\mathcal{C}^{\vec{b},\vec{c}}(f)(r\cdot e^{i\theta})=\prod_{j=1}^n\frac{\Gamma(b_j+1)}{2\pi\Gamma(c_j)\Gamma(b_j+1-c_j)}\int_{[-\pi,\pi]^n}f_r(e^{i(\theta+\varphi)})K_r^{\vec{b},\vec{c}}(\theta,\varphi)d\varphi
$$

here

$$
\begin{aligned}
K_r^{\vec{b},\vec{c}}(\theta,\varphi) = & \prod_{j=1}^n\int_0^1\frac{(1+t_j)t_j^{c_j-1}(1-t_j)^{b_j-c_j+1}}{(1-2t_j\cos\varphi_j+t_j^2)(1-t_jr_je^{i\theta_j})^{(b_j+1-c_j)}} \\
& \times F(c_j-1,c_j-b_j-1,c_j,t_jr_je^{\theta_j})dt_j.
\end{aligned}
$$

Using an estimate in [5] and the boundedness of $F(c_j-1,c_j-b_j-1,c_j,tz_j)$, $(j=1,\ldots,n)$ on the unit disk, we have that there is an constant C such that

$$
|K_r^{\vec{b},\vec{c}}(\theta,\varphi)|\le C\prod_{j=1}^n\int_0^1\frac{x^{b_j+1-c_j}dx}{[x^2+\varphi_j^2][x^2+\theta_j^2]^{\frac{b_j+1-c_j}{2}}}
$$

for $|\theta_j|\le\pi$, $|\varphi_j|\le\pi$, $j=1,\ldots,n$. Thus, by Lemma 2.4, we obtain

$$
|K_r^{\vec{b},\vec{c}}(\theta,\varphi)|\le C\prod_{j=1}^n\frac{H^{b_j-c_j}(\varphi_j/\theta_j)}{|\theta_j|}
$$

for $|\theta_j|<\pi$, $|\varphi_j|<\pi$, $j=1,\ldots,n$, $0<r<1$. Hence

$$
\begin{aligned}
|\mathcal{C}^{\vec{b},\vec{c}}(f)(r\cdot e^{i\theta})| \le & C\int_{[-\pi,\pi]^n}\prod_{j=1}^n\frac{H^{b_j-c_j}(\varphi_j/\theta_j)}{|\theta_j|}|f_r(e^{i(\theta+\varphi)})|d\varphi \\
\le & C\int_{\mathbb{R}^n}\prod_{j=1}^n H^{b_j-c_j}(s_j)E_s|f_r|(e^{i\theta})ds
\end{aligned}
$$

From this estimates, using Lemma 2.5 and 2π periodicity of the integrand, the result follows.

The Hardy space $H^p(\mathbb{D}^n)$ on $\mathbb{D}^n$ can be defined as follows:

$$H^p = H^p(\mathbb{D}^n) = \{f \mid f \in H(\mathbb{D}^n), \quad \|f\|_{H^p(\mathbb{D}^n)} < \infty\},$$

where

$$\|f\|^p_{H^p(\mathbb{D}^n)} = \frac{1}{(2\pi)^n} \sup_{0\le r<1} \int_{[0,2\pi]^n} |f(r\cdot e^{i\theta})|^p d\theta.$$

Given $0 < p, q < \infty$, and positive Borel measure $\mu_j, j = 1, \ldots, n$ on the interval $(0,1)$, the weighted space $A^{p,q}_{\vec{\mu}}(\mathbb{D}^n)$ consists of those functions f analytic on $\mathbb{D}^n$ for which

$$\|f\|_{A^{p,q}_{\vec{\mu}}(\mathbb{D}^n)} = \left(\int_{(0,1)} \left(\int_{[0,2\pi]} |f(r\cdot e^{i\theta})|^p d\theta\right)^{q/p} \prod_{j=1}^n d\mu_j(r_j)\right)^{1/q} < \infty.$$

Corollary 2.1. *Assume that $p \in (0,1]$ and $Re\,(b+1) > Re\,c > 0$, or $p > 1$ and $Re\,(b+1) > Re\,c \ge 1$, and $0 < r < 1$. Then there is a constant C independent of f and r such that*

$$\int_0^{2\pi} |\mathcal{C}^{b,c}(f)(re^{i\theta})|^p d\theta \le C \int_0^{2\pi} |f(re^{i\theta})|^p d\theta,$$

for all $f \in H(\mathbb{D})$.

Corollary 2.2. *Assume that $p \in (0,1]$ and $Re\,(b_j+1) > Re\,c_j > 0$, $j = 1, \ldots, n$, or $p > 1$ and $Re\,(b_j+1) > Re\,c_j \ge 1$, $j = 1, \ldots, n$. Then the generalized Cesàro operator $\mathcal{C}^{\vec{b},\vec{c}}$ is bounded on $H^p(\mathbb{D}^n)$.*

Corollary 2.3. *Assume that $p \in (0,1]$ and $Re\,(b_j+1) > Re\,c_j > 0$, $j = 1, \ldots, n$, or $p > 1$ and $Re\,(b_j+1) > Re\,c_j \ge 1$, $j = 1, \ldots, n$. The generalized Cesàro operator $\mathcal{C}^{\vec{b},\vec{c}}$ is bounded on $A^{p,q}_{\vec{\mu}}(\mathbb{D}^n)$ for every $q \in (0,\infty)$. Moreover, there is a constant C independent of f, such that*

$$\|\mathcal{C}^{\vec{b},\vec{c}} f\|_{A^{p,q}_{\vec{\mu}}(\mathbb{D}^n)} \le C\|f\|_{A^{p,q}_{\vec{\mu}}(\mathbb{D}^n)}.$$

Remark 1. If $p = \infty$ and $Re\,(b_j+1) > Re\,c_j > 0$, $j = 1, \ldots, n$, then the operator $\mathcal{C}^{\vec{b},\vec{c}}$ is not bounded. Choose $g(z) \equiv 1 \in H^\infty(\mathbb{D}^n)$. Then we have,

$$\mathcal{C}^{\vec{b},\vec{c}}(g) = \prod_{j=1}^n \frac{b_j - c_j + 1}{b_j} \prod_{j=1}^n \sum_{k=0}^\infty \frac{A_k^{b_j,c_j}}{A_k^{b_j+1,c_j}} z_j^k = C \prod_{j=1}^n \sum_{k=0}^\infty \frac{1}{b_j+k} z_j^k.$$

It is easy to see that each sum in the last product is unbounded on $\mathbb{D}$, therefore $\mathcal{C}^{\vec{b},\vec{c}}(g) \notin H^\infty(\mathbb{D}^n)$.

3. The Operator $\mathcal{C}^{b,c}_{\zeta_0}$ on the Unit Ball

Let $B = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$ and let $S = \partial B = \{z \in \mathbb{C}^n : |z| = 1\}$ be its boundary. Let dv denote the normalized Lebesgue volume measure on the unit ball B such that $v(B) = 1$, $d\sigma$ be the normalized rotation invariant measure on the boundary S of B such that $\sigma(S) = 1$, $H(B)$ the class of all holomorphic functions on the unit ball and $H^\infty = H^\infty(B)$ the space of all bounded holomorphic functions on the unit ball.

Assume that $f \in H(B)$ with Taylor series expansion $f(z) = \sum_{|\beta|\geq 0} a_\beta z^\beta$ where $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ is a multi-index and $z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}$. Denote

$$\mathcal{R}f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z)$$

the radial derivative of f. It is well known that

$$\mathcal{R}f(z) = \sum_{|\beta|\geq 0} |\beta| a_\beta z^\beta = \sum_{(\beta_1,\dots,\beta_n)\in(\mathbb{Z}_+)^n} (\beta_1 + \cdots + \beta_n) a_\beta z^\beta,$$

see, for example [51].

Let $\alpha > 0$. The α-Bloch space $\mathcal{B}^\alpha = \mathcal{B}^\alpha(B)$ is the space of all holomorphic functions f on B such that

$$b_\alpha(f) = \sup_{z\in B} (1 - |z|^2)^\alpha \, |\mathcal{R}f(z)| < \infty.$$

It is clear that $\mathcal{B}^\alpha$ is a normed space under the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + b_\alpha(f)$, and $\mathcal{B}^{\alpha_1} \subset \mathcal{B}^{\alpha_2}$ for $\alpha_1 < \alpha_2$. Let $\mathcal{B}^\alpha_0$ denote the subspace of $\mathcal{B}^\alpha$ consisting of those $f \in \mathcal{B}^\alpha$ for which

$$(1 - |z|^2)^\alpha |\mathcal{R}f(z)| \to 0 \quad \text{as} \quad |z| \to 1.$$

This space is called the little α-Bloch space.

The Hardy space $H^p(B)$ $(0 < p < \infty)$ is defined on B by

$$H^p(B) = \{\, f \mid f \in H(B) \quad \text{and} \quad \|f\|_{H^p(B)} = \sup_{0\leq r<1} M_p(f, r) < \infty \,\},$$

where

$$M_p(f, r) = \left(\int_S |f(r\zeta)|^p d\sigma(\zeta) \right)^{1/p}.$$

The mixed-norm space $\mathcal{A}_\mu^{p,q}(B)$, $p, q \in (0, \infty)$, μ is Borel measure on $(0, 1)$, is the space of all analytic functions f on B for which

$$\|f\|^q_{\mathcal{A}_\mu^{p,q}(B)} = \int_0^1 M_p^q(f, r) d\mu(r) < \infty.$$

When $p = q$ and $d\mu(r) = (1 - r^2)^\alpha r^{2n-1} dr$ the mixed norm space becomes the weighted Bergman space.

Now we generalize the Cesàro operator $\mathcal{C}^{b,c}$ on B in the following way:

$$\mathcal{C}^{b,c}_{\zeta_0} f(z) = \frac{\Gamma(b+1)}{\Gamma(c)\Gamma(b+1-c)} \int_0^1 f(tz) \frac{t^{c-1}(1-t)^{b-c}}{(1 - \langle tz, \zeta_0\rangle)^{b+1-c}} F(c-1, c-b-1, c, \langle tz, \zeta_0\rangle) dt,$$

where $Re\,(b+1-c) > 0$, $f(z) = \sum_{|\beta|=0}^\infty a_\beta z^\beta$ is an analytic function on the unit ball B, $F(a, b, c, z)$ is the hypergeometric function, ζ_0 is a fixed point lying on S and $\langle z, w\rangle = z_1 w_1 + \cdots + z_n w_n$. This operator is also a natural generalization of the operator introduced in [45].

For the case of the unit ball we prove the following results in this section.

Theorem 3.1. *Assume that $p \in (0, 1]$ and $b, c \in \mathbb{C}$ such that $Re\,(b+1) > Re\, c > 0$, or $p > 1$ and $Re\,(b+1) > Re\, c \geq 1$. Then the operator $\mathcal{C}^{b,c}_{\zeta_0}$ is bounded on $H^p(B)$. Moreover, there is a positive constant C such that*

$$M_p(\mathcal{C}^{b,c}_{\zeta_0} f, r) \leq C M_p(f, r), \qquad 0 < r < 1$$

for every $f \in H^p(B)$.

Theorem 3.2. *Assume that $p \in (0, 1]$ and $b, c \in \mathbb{C}$ such that $Re\,(b+1) > Re\, c > 0$, or $p > 1$ and $Re\,(b+1) > Re\, c \geq 1$. Then the operator $\mathcal{C}^{b,c}_{\zeta_0}$ is bounded on $\mathcal{A}_\mu^{p,q}(B)$ for every and $q \in (0, \infty)$.*

Theorem 3.3. *Let $b, c \in \mathbb{C}$ such that $Re\,(b+1) > Re\, c > 0$. Then the operator $\mathcal{C}^{b,c}_{\zeta_0}$ is bounded on $\mathcal{B}^\alpha$ if $\alpha > 1$.*

In order to prove the above Theorems, we need the following lemmas.

Lemma 3.1. ([30]). *The following identity holds.*

$$\int_{\partial B} f d\sigma = \int_{\partial B} d\sigma(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}\zeta) d\theta.$$

Lemma 3.2. ([1]). *Let $a, b \in \mathbb{C}$ are such that $Re(b+1) > Re\, c > 0$ and $t_k = 1 - 2^{-k}$, $k \in \mathbb{N} \cup \{0\}$. Then there is a positive constant C such that*

$$\int_{t_{k-1}}^{t_k} |t^{Re(c)-1}(1-t)^{Re(b-c)}|dt \leq \frac{C}{2^{kRe(b+1-c)}}.$$

Lemma 3.3. ([32]). *For $\beta > -1$ and $m > 1+\beta$ we have*

$$\int_0^1 \frac{(1-r)^\beta}{(1-\rho r)^m} dr \leq C(1-\rho)^{1+\beta-m}, \qquad 0 < \rho < 1.$$

Proof of Theorem 3.1. Without loss of generality we may assume that b and c are real numbers. For a function f holomorphic on the unit ball B and $\xi \in S$, the slice function f_ξ is well defined for $|w| < 1$, by $f_\xi(w) = f(w\xi)$. For fixed $\zeta_0, \xi \in S$, write $\langle \xi, \zeta_0 \rangle = re^{i\theta}$. First, assume that $r \neq 0$. Then we have

$$\begin{aligned} &\mathcal{C}^{b,c}_{\zeta_0}(f)_\xi(w) \\ &= \mathcal{C}^{b,c}_{\zeta_0}(f)(w\xi) \\ &= C\int_0^1 f(tw\xi)\frac{t^{c-1}(1-t)^{b-c}}{(1-\langle tw\xi, \zeta_0\rangle)^{b+1-c}} F(c-1, c-b-1, c, \langle tw\xi, \zeta_0\rangle)dt \qquad (5) \\ &= C\int_0^1 f_{e^{-i\theta}\xi/r}(twre^{i\theta})\frac{t^{c-1}(1-t)^{b-c}}{(1-trwe^{i\theta})^{b+1-c}} F(c-1, c-b-1, c, trwe^{i\theta})dt \\ &= C\, \mathcal{C}^{b,c}(f_{e^{-i\theta}\xi/r})(rwe^{i\theta}), \end{aligned}$$

where $w \in \mathbb{D}$.

Let $w = |w|e^{i\varphi}$. From (5), by the boundedness of the operator $\mathcal{C}^{b,c}$ in one variable (Corollary 2.1) it follows that

$$\begin{aligned} M_p^p\big(\mathcal{C}^{b,c}_{\zeta_0}(f)_\xi, |w|\big) &= \int_0^{2\pi} \big|\mathcal{C}^{b,c}_{\zeta_0}(f)_\xi(|w|e^{i\varphi})\big|^p d\varphi \\ &= C\int_0^{2\pi} \big|\mathcal{C}^{b,c}(f_{e^{-i\theta}\xi/r})(r|w|e^{i(\varphi+\theta)})\big|^p d\varphi \qquad (6) \\ &\leq C\int_0^{2\pi} \big|f_{e^{-i\theta}\xi/r}(r|w|e^{i(\varphi+\theta)})\big|^p d\varphi \\ &= C\int_0^{2\pi} \big|f_\xi(|w|e^{i\varphi})\big|^p d\varphi = CM_p^p(f_\xi, |w|), \end{aligned}$$

for all $\xi, \zeta_0 \in S$ with $\langle \xi, \zeta_0 \rangle \neq 0$.

Now assume that $\langle \xi, \zeta_0 \rangle = 0$. If $p \geq 1$, then by Minkowski's inequality and the monotonicity of the integral means we have

$$
\begin{aligned}
&M_p\big(\mathcal{C}_{\zeta_0}^{b,c}(f)_\xi, |w|\big)\\
&= L_{b,c}\left(\int_0^{2\pi}\left|\int_0^1 f(t|w|\xi e^{i\theta})t^{c-1}(1-t)^{b-c}dt\right|^p d\theta\right)^{1/p}\\
&\leq L_{b,c}\int_0^1\left(\int_0^{2\pi}|f(t|w|\xi e^{i\theta})|^p d\theta\right)^{1/p} t^{c-1}(1-t)^{b-c}dt\\
&\leq L_{b,c}\left(\int_0^1 t^{c-1}(1-t)^{b-c}dt\right)\left(\int_0^{2\pi}|f(|w|\xi e^{i\theta})|^p d\theta\right)^{1/p}\\
&= \left(\int_0^{2\pi}|f(|w|\xi e^{i\theta})|^p d\theta\right)^{1/p},
\end{aligned}
\tag{7}
$$

where $L_{b,c} = \frac{\Gamma(b+1)}{\Gamma(c)\Gamma(b+1-c)}$.

Now assume that $p \in (0,1)$ and let $t_k = 1-2^{-k}$, $k \in \mathbb{N}\cup\{0\}$, then by Lemma 2.1, Lemma 3.2 and the monotonicity of the integral means we have

$$
\begin{aligned}
&M_p^p\big(\mathcal{C}_{\zeta_0}^{b,c}(f)_\xi, |w|\big)\\
&= \int_0^{2\pi}\left|\int_0^1 f(t|w|\xi e^{i\theta})t^{c-1}(1-t)^{b-c}dt\right|^p d\theta\\
&= \int_0^{2\pi}\sum_{k=1}^{\infty}\left|\int_{t_{k-1}}^{t_k} f_\xi(t|w|e^{i\theta})t^{c-1}(1-t)^{b-c}dt\right|^p d\theta\\
&\leq C\sum_{k=1}^{\infty}\frac{1}{2^{pk(b+1-c)}}\int_0^{2\pi}\sup_{t_{k-1}<t<t_k}|f_\xi(t|w|e^{i\theta})|^p d\theta\\
&\leq C\sum_{k=1}^{\infty}\frac{1}{2^{pk(b+1-c)}}\int_0^{2\pi}\sup_{0<t<t_k}|f_\xi(t|w|e^{i\theta})|^p d\theta\\
&\leq C\sum_{k=1}^{\infty}\frac{1}{2^{pk(b+1-c)}}\int_0^{2\pi}|f_\xi(t_k|w|e^{i\theta})|^p d\theta\\
&\leq C\sum_{k=1}^{\infty}\int_{t_k}^{t_{k+1}}\int_0^{2\pi}|f_\xi(t|w|e^{i\theta})|^p d\theta\,(1-t)^{p(b+1-c)-1}dt\\
&\leq C\int_0^1\int_0^{2\pi}|f_\xi(t|w|e^{i\theta})|^p d\theta\,(1-t)^{p(b+1-c)-1}dt
\end{aligned}
\tag{8}
$$

$$\leq C\int_0^1 (1-t)^{p(b+1-c)-1}dt\int_0^{2\pi}|f_\xi(|w|e^{i\theta})|^p d\theta$$
$$\leq C$$

From (5)-(8) we see that there is a positive constant C independent of ξ, $r\in(0,1)$ and f, such that

$$M_p^p\big(\mathcal{C}_{\zeta_0}^{b,c}(f)_\xi, r\big)\leq CM_p^p\big(f_\xi, r\big). \tag{9}$$

Integrating (9) over S and applying Lemma 3.1, we obtain

$$M_p^p(\mathcal{C}_{\zeta_0}^{b,c}f, r)\leq CM_p^p(f,r), \qquad 0<r<1, \tag{10}$$

from which the result follows.

Proof of Theorem 3.2. Taking inequality (10) to the $\frac{q}{p}$th power, multiplying by $d\mu(r)$ and then integrating from 0 to 1 we obtain

$$\|\mathcal{C}_{\zeta_0}^{b,c}f\|^q_{\mathcal{A}^{p,q}_\mu(B)}=\int_0^1 M_p^q(\mathcal{C}_{\zeta_0}^{b,c}f,r)d\mu(r)\leq C\int_0^1 M_p^q(f,r)d\mu(r)=\|f\|^q_{\mathcal{A}^{p,q}_\mu(B)},$$

finishing the proof of the result.

Proof of Theorem 3.3. Let $\alpha>1$. Then it is well known that

$$\|f\|_{\mathcal{B}^\alpha}\asymp\|f\|'_{\mathcal{B}^\alpha}=\sup_{z\in B}(1-|z|)^{\alpha-1}|f(z)|. \tag{11}$$

Then we have

$$\begin{aligned}
|\mathcal{C}_{\zeta_0}^{b,c}f(z)| &\leq L_{b,c}\max_{z\in\overline{U}}|F|\int_0^1|f(tz)|\frac{t^{c-1}(1-t)^{b-c}}{|1-\langle tz,\zeta_0\rangle|^{b+1-c}}dt\\
&\leq C\int_0^1\frac{|f(tz)|(1-t\,|z|)^{\alpha-1}}{(1-t\,|z|)^{\alpha+b-c}}t^{c-1}(1-t)^{b-c}dt\\
&\leq C\,\|f\|'_{\mathcal{B}^a}\int_0^1\frac{t^{c-1}(1-t)^{b-c}}{(1-t\,|z|)^{\alpha+b-c}}dt\\
&\leq C\,\|f\|'_{\mathcal{B}^a}\left(1+\int_0^1\frac{(1-t)^{b-c}}{(1-t\,|z|)^{\alpha+b-c}}dt\right)\\
&\leq C\,\|f\|'_{\mathcal{B}^a}\frac{1}{(1-|z|)^{\alpha-1}} \qquad \text{(by Lemma 3.3)},
\end{aligned}$$

for each $z \in B$. Hence

$$(1-|z|)^{\alpha-1}|\mathcal{C}_{\zeta_0}^{b,c} f(z)| \leq C\|f\|_{\mathcal{B}^\alpha}, \tag{12}$$

for every $z \in B$. Taking the supremum in (12) over $z \in B$ and using the relationship (11) for the function $\mathcal{C}_{\zeta_0}^{b,c} f$ we obtain the result.

4. The Operators T_g and L_g on the Unit Ball

A positive continuous function ϕ on $[0, 1)$ is normal, if there exist positive numbers s and t ($s < t$), such that

$$\frac{\phi(r)}{(1-r)^s} \downarrow 0, \qquad \frac{\phi(r)}{(1-r)^t} \uparrow \infty$$

as $r \to 1^-$. For $0 < p < \infty$, and a normal function ϕ, let $H(p, p, \phi)$ denote the space of all holomorphic functions f on the unit ball such that

$$\|f\|_{H(p,p,\phi)} = \int_B |f(z)|^p \frac{\phi^p(|z|)}{1-|z|} dv(z) < \infty.$$

If $1 \leq p < \infty$, the space $H(p, p, \phi)$ is a Banach space. When $0 < p < 1$, $H(p, p, \phi)$ is a Fréchet space but not a Banach space. $H(p, p, \phi)$ is called the Bergman type space. In particular, if $\phi(r) = (1-r)^{1/p}$, then $H(p, p, \phi)$ is the Bergman space A^p. For some basic properties of Bergman spaces, see for example, [6, 8, 11, 17, 38, 42, 43, 51].

Note that the integral form of the Cesàro operator $\mathcal{C}$ is

$$\mathcal{C}(f)(z) = \frac{1}{z}\int_0^z f(\zeta)\frac{1}{1-\zeta} d\zeta = \frac{1}{z}\int_0^z f(\zeta)\left(\ln\frac{1}{1-\zeta}\right)' d\zeta,$$

taking simply as a path the segment joining 0 and z, we have that

$$\mathcal{C}(f)(z) = \int_0^1 f(tz)\left(\ln\frac{1}{1-\zeta}\right)'\Bigg|_{\zeta=tz} dt.$$

The following operator

$$z\mathcal{C}(f)(z) = \int_0^z \frac{f(\zeta)}{1-\zeta} d\zeta,$$

is closely related to the previous operator and on many spaces the boundedness of these two operators is equivalent.

From this point of view, it is natural to extend the Cesàro operator in the following way. Suppose that $g : \mathbb{D} \to \mathbb{C}^1$ is an analytic map and $f \in H(\mathbb{D})$. A class of integral operator introduced by Pommerenke is defined by (see [29])

$$J_g f(z) = \int_0^z f dg = \int_0^1 f(tz) z g'(tz) dt = \int_0^z f(\xi) g'(\xi) d\xi, \qquad z \in \mathbb{D}.$$

The operator J_g can be viewed as a generalization of the Cesàro operator. In [29] Pommerenke showed that J_g is a bounded operator on the Hardy space H^2 if and only if $g \in BMOA$. Alemann and Siskakis showed that J_g is bounded (compact) on the Hardy space H^p, $1 \le p < \infty$, if and only if $g \in BMOA\,(g \in VMOA)$, and that J_g is bounded (compact) on the Bergman space A^p if and only if $g \in \mathcal{B}$ ($g \in \mathcal{B}_0$), see [3, 4]. Some other results on the operator J_g can be found in [2-4, 22, 25, 36, 47] (see, also the related references therein). Closely related operators on the unit polydisk were investigated in [10] and [46].

It is natural to generalize the operator J_g for the case of the unit ball. Suppose that $g : B \to \mathbb{C}^1$ is a holomorphic map of the unit ball, for a holomorphic function $f : B \to \mathbb{C}^1$, define

$$T_g f(z) = \int_0^1 f(tz) \frac{dg(tz)}{dt} = \int_0^1 f(tz) \mathcal{R} g(tz) \frac{dt}{t}, \qquad z \in B.$$

This operator is called extended-Cesàro operator (or Riemann-Stieltjes operator), it was introduced in [17], and studied in [17-20, 23, 24, 26, 44, 49, 50].

Similarly, another integral operator was defined as follows (see [23]):

$$L_g f(z) = \int_0^1 \mathcal{R} f(tz) g(tz) \frac{dt}{t}, \qquad z \in B.$$

In [24], we proved that $T_g : H^2 \to H^2$ is bounded if and only if $g \in BMOA$ and $T_g : H^2 \to H^2$ is compact if and only if $g \in VMOA$. $L_g : H^2 \to H^2$ is bounded if and only if

$$\sup_{a \in B} \int_B \left(\frac{1 - |a|^2}{|1 - \langle a, z \rangle|^2} \right)^{n+2} |g(z)|^2 (1 - |z|^2) dv(z) < \infty \tag{12}$$

and $L_g : H^2 \to H^2$ is compact if and only if

$$\lim_{|a| \to 1} \int_B \left(\frac{1 - |a|^2}{|1 - \langle a, z \rangle|^2} \right)^{n+2} |g(z)|^2 (1 - |z|^2) dv(z) = 0.$$

In this section, we study the boundedness and compactness of operators T_g and L_g from Bergman type spaces and $H^\infty(B)$ to α-Bloch spaces and little α-Bloch spaces on the unit ball. Our discussion will be divided into four parts.

4.1. The boundedness and compactness of $T_g, L_g : H^\infty \to \mathcal{B}^\alpha$

In this subsection, we discuss the boundedness and the compactness of operators T_g and L_g from H^∞ to the α-Bloch space $\mathcal{B}^\alpha$. In order to prove our main results, we need some auxiliary results which are incorporated in the following lemmas.

Lemma 4.1. *For every $f, g \in H(B)$ it holds*

$$\mathcal{R}[T_g(f)](z) = f(z)\mathcal{R}g(z) \quad \text{and} \quad \mathcal{R}[L_g(f)](z) = \mathcal{R}f(z)g(z).$$

Proof. The first identity was proved in [17], while the proof of the second identity is similar and is omitted.

The next lemma can be proved in a standard way, and its proof will be omitted.

Lemma 4.2. *The operator T_g (or L_g) : H^∞ (or $H(p,p,\phi)$) $\to \mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$) is compact if and only if the operator T_g (or L_g) : H^∞ (or $H(p,p,\phi)$) $\to \mathcal{B}^\alpha$ (or $\mathcal{B}_0^\alpha$) is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in H^∞ (or $H(p,p,\phi)$) which converges to zero uniformly on compact subsets of B, we have $\|T_g f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$ (or $\|L_g f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$).*

Theorem 4.1. *Suppose that g is a holomorphic function on B and $\alpha > 0$, then $T_g : H^\infty \to \mathcal{B}^\alpha$ is bounded if and only if $g \in \mathcal{B}^\alpha$. Moreover, the following relationship*

$$\|T_g\|_{H^\infty \to \mathcal{B}^\alpha} \asymp \sup_{z\in B} (1-|z|^2)^\alpha |\mathcal{R}g(z)| \tag{13}$$

holds.

Proof. Suppose that $g \in \mathcal{B}^\alpha$. Let $f \in H^\infty$, it is easy to see that $T_g f(0) = 0$, by Lemma 4.1 we have

$$\begin{aligned}(1-|z|^2)^\alpha |\mathcal{R}(T_g f)(z)| &= (1-|z|^2)^\alpha |f(z)||\mathcal{R}g(z)| \\ &\le \|f\|_\infty (1-|z|^2)^\alpha |\mathcal{R}g(z)|.\end{aligned} \tag{14}$$

Taking the supremum in (14) over $z \in B$ it follows that $T_g : H^\infty \to \mathcal{B}^\alpha$ is bounded.

On the other hand, suppose $T_g : H^\infty \to \mathcal{B}^\alpha$ is bounded. Set

$$f_w(z) = \frac{1-|w|^2}{1-\langle z, w\rangle}, \qquad w \in B, \tag{15}$$

then $f_w \in H^\infty$ and $\|f_w\|_\infty \le 2$ for every $w \in B$. Therefore, we have that

$$(16)\qquad \begin{aligned}(1-|w|^2)^\alpha|\mathcal{R}g(w)| &= (1-|w|^2)^\alpha|f_w(w)||\mathcal{R}g(w)|\\ &= (1-|w|^2)^\alpha|\mathcal{R}(T_g f_w)(w)|\\ &\le \sup_{z\in B}(1-|z|^2)^\alpha|\mathcal{R}(T_g f_w)(z)|\\ &= \|T_g f_w\|_{\mathcal{B}^\alpha} \le 2\|T_g\|_{H^\infty\to\mathcal{B}^\alpha}\end{aligned}$$

which implies that $g \in \mathcal{B}^\alpha$. From (14) and (16) it follows (13).

Theorem 4.2. *Suppose that g is a holomorphic function on B, then*

$$L_g : H^\infty \to \mathcal{B}^\alpha \ \text{ is bounded } \Leftrightarrow \begin{cases} g \in \mathcal{B}^\alpha, & \alpha > 1;\\ g \in H^\infty, & \alpha = 1;\\ g \equiv 0, & \alpha \in (0,1).\end{cases}$$

Moreover, if $\alpha \ge 1$, then the following relationship

$$(17)\qquad \|L_g\|_{H^\infty\to\mathcal{B}^\alpha} \asymp \sup_{z\in B}(1-|z|^2)^{\alpha-1}|g(z)|$$

holds.

Proof. Let $f \in H^\infty$. Then it is known that $H^\infty \subset \mathcal{B}$ and moreover

$$\|f\|_{\mathcal{B}} \le C\|f\|_\infty,$$

see, for example, [51]. It follows that

$$|\mathcal{R}f(z)|(1-|z|^2) \le C\|f\|_\infty,$$

for every $z \in B$. From this, Lemma 4.1 and using the fact that $L_g f(0) = 0$, we have

$$(18)\qquad \begin{aligned}(1-|z|^2)^\alpha|\mathcal{R}(L_g f)(z)| &= (1-|z|^2)^\alpha|\mathcal{R}f(z)||g(z)|\\ &\le \|f\|_{\mathcal{B}}(1-|z|^2)^{\alpha-1}|g(z)|\\ &\le C\|f\|_\infty(1-|z|^2)^{\alpha-1}|g(z)|\\ &\le C\|f\|_\infty\|g\|_{\mathcal{B}^\alpha},\end{aligned}$$

where in the last inequality we have used the well known fact that

$$\sup_{z\in B}(1-|z|^2)^{\alpha-1}|g(z)| \asymp \|g\|_{\mathcal{B}^\alpha}$$

for $\alpha > 1$. Hence, if $g \in \mathcal{B}^\alpha$ for some $\alpha > 1$, it follows that L_g is bounded.

If $\alpha = 1$ we have

$$(1-|z|^2)|\mathcal{R}(L_g f)(z)| \le C\|f\|_\infty |g(z)| \le C\|f\|_\infty \|g\|_\infty$$

from which it follows that $g \in H^\infty$ implies the boundedness of $L_g : H^\infty \to \mathcal{B}^1$.

If $\alpha \in (0,1)$ and $g \equiv 0$ it is easy to see that $L_g : H^\infty \to \mathcal{B}^\alpha$ is bounded.

Conversely, suppose $L_g : H^\infty \to \mathcal{B}$ is bounded. Let $\beta(z,w)$ denote the Bergman metric between two points z and w in B. For $a \in B$ and $r > 0$, the set $D(a,r) = \{z \in B : \beta(a,z) < r\}, a \in B$ is a Bergman metric ball centered at a with radius r. It is well know that

$$\frac{(1-|a|^2)^{n+1}}{|1-\langle a,z\rangle|^{2(n+1)}} \asymp \frac{1}{(1-|z|^2)^{n+1}} \asymp \frac{1}{(1-|a|^2)^{n+1}} \asymp \frac{1}{|D(a,r)|} \tag{19}$$

when $z \in D(a,r)$ and where $|D(a,r)|$ is the volume of the Bergman ball $D(a,r)$ (see [51]). For $w \in B$, let f_w be defined by (15), then

$$\begin{aligned} |g(w)|^2|w|^2 &\le \frac{C|w|^2}{(1-|w|^2)^{n+1}} \int_{D(w,r)} |g(z)|^2 dv(z) \\ &\le C \int_{D(w,r)} \frac{1}{(1-|z|^2)^{n-1}} |\mathcal{R} f_w(z)|^2 |g(z)|^2 dv(z) \\ &\asymp \|L_g f_w\|^2_{\mathcal{B}^\alpha} \int_{D(w,r)} \frac{dv(z)}{(1-|z|^2)^{2\alpha+n-1}} \\ &\le \frac{C\|L_g f_w\|^2_{\mathcal{B}^\alpha}}{(1-|w|^2)^{2\alpha-2}}. \end{aligned} \tag{20}$$

If $\alpha > 1$, from (20) we obtain $\sup_{1/2 \le |z| < 1} |g(z)|(1-|z|^2)^{\alpha-1} < \infty$. From this and since

$$\begin{aligned} \sup_{|z|\le 1/2} |g(z)|(1-|z|^2)^{\alpha-1} &\le \sup_{|z|\le 1/2} |g(z)| = \sup_{|z|=1/2} |g(z)| \\ &= \left(\frac{4}{3}\right)^{\alpha-1} \sup_{|z|=1/2} |g(z)|(1-|z|^2)^{\alpha-1} \\ &\le \left(\frac{4}{3}\right)^{\alpha-1} \sup_{1/2\le|z|<1} |g(z)|(1-|z|^2)^{\alpha-1}, \end{aligned}$$

we have that

$$\sup_{z\in B} |g(z)|(1-|z|^2)^{\alpha-1} < \infty,$$

which is equivalent to $g \in \mathcal{B}^\alpha$, in the case.

When $\alpha = 1$, from (20) and the maximum modulus principle we obtain $g \in H^\infty$. Moreover

$$\sup_{z\in B}|g(z)| \le 2C\|L_g\|_{H^\infty\to\mathcal{B}^\alpha},$$

which along with (18) implies (17) in this case.

If $\alpha \in (0,1)$, then (20) can be written in the form

$$|g(w)||w| \le C\|L_g f_w\|_{\mathcal{B}^\alpha}(1-|w|^2)^{1-\alpha}. \tag{21}$$

Letting $|w| \to 1$ in (21) and applying the maximum modulus principle we obtain $g \equiv 0$, as desired.

Theorem 4.3. *Suppose that g is a holomorphic function on B and $\alpha > 0$. Then $T_g : H^\infty \to \mathcal{B}^\alpha$ is compact if and only if $g \in \mathcal{B}^\alpha_0$.*

Proof. First assume that $g \in \mathcal{B}^\alpha_0$. In order to prove that T_g is compact it suffices to show that if $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in H^∞ which converges to 0 uniformly on compact subsets of B, then $\|T_g f_k\|_{\mathcal{B}^\alpha} \to 0$, as $k \to \infty$. Hence, assume that $(f_k)_{k\in\mathbb{N}}$ is a sequence in H^∞ with $\sup_{k\in\mathbb{N}}\|f_k\|_\infty \le K$ and $f_k \to 0$ uniformly on compact subsets of B as $k \to \infty$. By the assumption, for every $\varepsilon > 0$, there is a constant $\delta \in (0,1)$, such that

$$(1-|z|^2)^\alpha|\mathcal{R}g(z)| < \varepsilon/K,$$

whenever $\delta < |z| < 1$.

Let $E = \{z \in B : |z| \le \delta\}$, then we have

$$\begin{aligned}\|T_g f_k\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^\alpha|\mathcal{R}(T_g f_k)(z)|\\ &= \sup_{z\in E}(1-|z|^2)^\alpha|\mathcal{R}g(z)f_k(z)| + \sup_{z\in B\setminus E}(1-|z|^2)^\alpha|\mathcal{R}g(z)f_k(z)|\\ &\le b_\alpha(g)\sup_{z\in E}|f_k(z)| + \varepsilon.\end{aligned}$$

By the condition $f_k \to 0$ on compacts as $k \to \infty$, and since E is a compact subset of B, we obtain $\limsup_{k\to\infty}\|T_g f_k\|_{\mathcal{B}^\alpha} \le \varepsilon$. Since ε is an arbitrary positive number we have that $\lim_{k\to\infty}\|T_g f_k\|_{\mathcal{B}^\alpha} = 0$, and therefore, $T_g : H^\infty \to \mathcal{B}^\alpha$ is compact.

Conversely, suppose $T_g : H^\infty \to \mathcal{B}^\alpha$ is compact. Assume that $(z_k)_{k\in\mathbb{N}}$ is a sequence in B such that $|z_k| \to 1$ as $k \to \infty$, and set

$$f_k(z) = \frac{1-|z_k|^2}{1-\langle z, z_k\rangle}, \quad k \in \mathbb{N}. \tag{22}$$

Then $\sup_{k\in\mathbb{N}} \|f_k\|_\infty \le 2$ and f_k converges to 0 uniformly on compact subsets of B as $k\to\infty$. Since T_g is compact, we have $\|T_g f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k\to\infty$. Therefore

$$\begin{aligned}(1-|z_k|^2)^\alpha|\mathcal{R}g(z_k)| &= (1-|z_k|^2)^\alpha|f_k(z_k)||\mathcal{R}g(z_k)|\\ &\le \sup_{z\in B}(1-|z|^2)^\alpha|f_k(z)||\mathcal{R}g(z)|\\ &= \sup_{z\in B}(1-|z|^2)^\alpha|\mathcal{R}(T_g f_k)(z)| = \|T_g f_k\|_{\mathcal{B}^\alpha}\to 0,\end{aligned}$$

as $k\to\infty$, which implies that $\lim_{|z|\to 1}(1-|z|^2)^\alpha|\mathcal{R}g(z)|=0$.

Theorem 4.4. *Suppose that g is a holomorphic function on B and $\alpha>0$. Then*

$$L_g : H^\infty \to \mathcal{B}^\alpha \text{ is compact } \Leftrightarrow \begin{cases} \lim_{|z|\to 1}(1-|z|^2)^{\alpha-1}|g(z)|=0, & \alpha>1;\\ g\equiv 0, & \alpha\in(0,1].\end{cases}$$

Proof. When $\alpha\in(0,1]$, then it is obvious that $g\equiv 0$ implies that $L_g : H^\infty\to \mathcal{B}^\alpha$ is compact. Now we consider the case $\alpha>1$, we assume

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha-1}|g(z)|=0$$

holds. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in H^∞ such that $\sup_{k\in\mathbb{N}}\|f_k\|_\infty\le K$ and $f_k\to 0$ uniformly on compact subsets of B as $k\to\infty$. By the assumption, for every $\varepsilon>0$, there is a constant $\delta\in(0,1)$, such that

$$(1-|z|^2)^{\alpha-1}|g(z)|<\varepsilon/K$$

whenever $\delta<|z|<1$. Let $E=\{z\in B : |z|\le\delta\}$. We have

$$\begin{aligned}\|L_g f_k\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^\alpha|\mathcal{R}(L_g f_k)(z)|\\ &= \sup_{z\in E}(1-|z|^2)^\alpha|g(z)\mathcal{R}f_k(z)| + \sup_{z\in B\setminus E}(1-|z|^2)^\alpha|g(z)\mathcal{R}f_k(z)|\\ &\le N\sup_{z\in E}(1-|z|^2)|\mathcal{R}f_k(z)| + C\varepsilon.\end{aligned}$$

where $N=\sup_{z\in B}(1-|z|^2)^{\alpha-1}|g(z)|$. By Cauchy's estimate the condition $f_k\to 0$ as $k\to\infty$ uniformly on compact subsets of B, implies that $\mathcal{R}f_k\to 0$ as $k\to\infty$ uniformly on compact subsets of B. Hence, we have $\|L_g f_k\|_{\mathcal{B}^\alpha}\to 0$ as $k\to\infty$. Therefore, $L_g : H^\infty\to\mathcal{B}^\alpha$ is compact.

Conversely, suppose $L_g : H^\infty\to\mathcal{B}^\alpha$ is compact. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|z_k|\to 1$ as $k\to\infty$, and $(f_k)_{k\in\mathbb{N}}$ be the sequence defined by (22). We know that $\sup_{k\in\mathbb{N}} \|f_k\|_\infty \le 2$ and f_k converges to 0 uniformly on compact subsets of B as $k \to \infty$. Since L_g is compact, we have $\|L_g f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$. Thus

$$\begin{aligned}(1-|z_k|^2)^{\alpha-1}|g(z_k)||z_k|^2 &= (1-|z_k|^2)^\alpha|g(z_k)|\,|\mathcal{R}f(z_k)|\\ &\le \sup_{z\in B}(1-|z|^2)^\alpha|g(z)|\,|\mathcal{R}f_k(z)|\\ &= \sup_{z\in B}(1-|z|^2)^\alpha|\mathcal{R}(L_g f_k)(z)|\\ &= \|L_g f_k\|_{\mathcal{B}^\alpha} \to 0,\end{aligned}$$

as $k \to \infty$, which implies that $\lim_{|z|\to 1}(1-|z|^2)^{\alpha-1}|g(z)| = 0$ when $\alpha > 1$.

When $\alpha \in (0, 1]$, the last inequality can be written as follows

$$|g(z_k)||z_k|^2 \le (1-|z_k|^2)^{1-\alpha}\|L_g f_k\|_{\mathcal{B}^\alpha}.$$

Letting $k \to \infty$ in the last inequality and using the maximum modulus principle we obtain that $g(z) \equiv 0,\ z \in B$.

4.2. The boundedness and compactness of $T_g, L_g : H^\infty \to \mathcal{B}_0^\alpha$

In this section, we characterize the boundedness and compactness of the operators $T_g, L_g : H^\infty \to \mathcal{B}_0^\alpha$. For this purpose, we need the following lemma (when $\alpha = 1$ and in the setting of the unit disk, the lemma was proved in [27], for general case in the unit ball, the proof is similar and will be omitted).

Lemma 4.3. *A closed set K in $\mathcal{B}_0^\alpha$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}(1-|z|^2)^\alpha|\mathcal{R}f(z)| = 0. \tag{23}$$

Theorem 4.5. *Suppose that g is a holomorphic function on B and $\alpha > 0$. Then the following statements are equivalent:*

(i) $T_g : H^\infty \to \mathcal{B}_0^\alpha$ *is bounded;*

(ii) $T_g : H^\infty \to \mathcal{B}_0^\alpha$ *is compact;*

(iii) $\lim_{|z|\to 1}(1-|z|^2)^\alpha|\mathcal{R}g(z)| = 0$.

Proof. $(iii) \Rightarrow (ii)$. In view of Lemma 4.3, we know that $T_g : H^\infty \to \mathcal{B}_0^\alpha$ is compact if and only if

$$\lim_{|z|\to 1}\sup_{\|f\|_\infty\le 1}(1-|z|^2)^\alpha|\mathcal{R}(T_g f)(z)| = 0.$$

We have

$$(1-|z|^2)^\alpha|\mathcal{R}(T_g f)(z)| = (1-|z|^2)^\alpha|\mathcal{R}g(z)||f(z)| \le \|f\|_\infty(1-|z|^2)^\alpha|\mathcal{R}g(z)|.$$

Taking the supremum over all $f \in H^\infty$ such that $\|f\|_\infty \le 1$, then letting $|z| \to 1$ in the obtained inequality we see that condition (iii) implies the compactness of the operator $T_g : H^\infty \to \mathcal{B}_0^\alpha$.

$(ii) \Rightarrow (i)$. It is obvious.

$(i) \Rightarrow (iii)$. Since $f(z) \equiv 1,\ z \in B$ is a bounded function, then (i) implies that $T_g(1) \in \mathcal{B}_0^\alpha$. Hence

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha|\mathcal{R}g(z)| = \lim_{|z|\to 1}(1-|z|^2)^\alpha|\mathcal{R}(T_g)(1)(z)| = 0,$$

as desired.

Theorem 4.6. *Suppose that g is a holomorphic function on B and $\alpha > 0$. Then the following statements are equivalent:*

(i) $L_g : H^\infty \to \mathcal{B}_0^\alpha$ *is bounded;*

(ii) $L_g : H^\infty \to \mathcal{B}_0^\alpha$ *is compact;*

(iii)

$$\begin{cases} \lim_{|z|\to 1}(1-|z|^2)^{\alpha-1}|g(z)| = 0, & \alpha > 1; \\ g \equiv 0, & \alpha \in (0,1]. \end{cases} \tag{24}$$

Proof. $(iii) \Rightarrow (ii)$. We have

$$\begin{aligned}(1-|z|^2)^\alpha|\mathcal{R}(L_g f)(z)| &= (1-|z|^2)^\alpha|g(z)||\mathcal{R}f(z)| \\ &\le C\|f\|_\infty(1-|z|^2)^{\alpha-1}|g(z)|.\end{aligned} \tag{25}$$

From (24) and (25) and by Lemma 4.3 it follows that $L_g : H^\infty \to \mathcal{B}_0^\alpha$ is compact for the case $\alpha > 1$. If $\alpha \in (0,1]$, the implication is obvious.

$(ii) \Rightarrow (iii)$. Assume now that $L_g : H^\infty \to \mathcal{B}_0^\alpha$ is compact. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|z_k| \to 1$ as $k \to \infty$ and let the sequence $(f_k)_{\in\mathbb{N}}$ be defined by (22). Then $\sup_{k\in\mathbb{N}}\|f_k\|_{H^\infty} \le 2$ and f_k converges to zero on compacts of B as $k \to \infty$, which by the compactness of the operator $L_g : H^\infty \to \mathcal{B}_0^\alpha$ implies that $\|L_g f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$.

Thus

$$\begin{aligned}(1-|z_k|^2)^{\alpha-1}|g(z_k)||z_k|^2 &= (1-|z_k|^2)^\alpha|g(z_k)||\mathcal{R}f_k(z_k)| \\ &\le \sup_{z\in B}(1-|z|^2)^\alpha|\mathcal{R}(I_g f_k)(z)| \\ &= \|I_g f_k\|_{\mathcal{B}^\alpha} \to 0,\end{aligned} \tag{26}$$

as $k \to \infty$, and as a consequence we have that $\lim_{|z|\to 1}(1-|z|^2)^{\alpha-1}|g(z)| = 0$, when $\alpha > 1$.

For the case $\alpha \in (0,1]$, using the fact that the sequence $(1-|z_n|^2)^{\alpha-1}$ is bounded below by one, (26) and the maximum modulus principle we obtain $g(z) \equiv 0,\ z \in B$, as desired.

Since the implication $(ii) \Rightarrow (i)$ is obvious, we need only prove that $(i) \Rightarrow (iii)$. Assume to the contrary that there is a sequence $(z^{(k)})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}|z^{(k)}| = 1$ and

$$(1-|z^{(k)}|^2)^{\alpha-1}|g(z^{(k)})| \geq \varepsilon_0 > 0.$$

Without loss of generality we may assume that $(z^{(k)}) \to (1, 0, \ldots, 0)$ as $k \to \infty$ and that $(1-|z^{(k)}|^2) \geq \frac{1}{2}(1-|z_1^{(k)}|^2)$.

We may also assume that the sequence $(z_1^{(k)})_{k\in\mathbb{N}}$ is an interpolating sequence on the unit disk, that is, there exists a $\delta > 0$ such that

$$\inf_{k\in\mathbb{N}} \prod_{m\neq k} \frac{|z_1^{(k)} - z_1^{(m)}|}{|1-\overline{z_1^{(m)}}z_1^{(k)}|} > \delta > 0.$$

It is well known that then the Blaschke product

$$b(z) = \prod_{m=1}^{\infty} \frac{z - z_1^{(m)}}{1-\overline{z_1^{(m)}}z}$$

converges uniformly on compacts and that it is a holomorphic function on the unit disk. We have

$$(1-|z_1^{(k)}|^2)|b'(z_1^{(k)})| = \prod_{m\neq k} \frac{|z_1^{(k)} - z_1^{(m)}|}{|1-\overline{z_1^{(m)}}z_1^{(k)}|} \geq \delta > 0.$$

Hence, with $f(z) = b(z_1)$, and for sufficiently large k, we have

$$\begin{aligned} &(1-|z^{(k)}|^2)^{\alpha}|L_g(b)(z^{(k)})| \\ &= (1-|z^{(k)}|^2)^{\alpha}|g(z^{(k)})|\,|\mathcal{R}\,b(z^{(k)})| \\ &\geq \frac{1}{2}(1-|z^{(k)}|^2)^{\alpha-1}|g(z^{(k)})|\,(1-|z_1^{(k)}|^2)|z_1^{(k)}b'(z_1^{(k)})| \\ &\geq \frac{1}{4}\delta\varepsilon_0 > 0. \end{aligned}$$

Since $L_g(b) \in \mathcal{B}_0^{\alpha}$, it follows that

$$\lim_{k\to\infty}(1-|z^{(k)}|^2)^{\alpha}|L_g(b)(z^{(k)})| = 0,$$

which is a contradiction. Hence for each $\alpha > 0$ it holds

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha-1}|g(z)| = 0.$$

From this and by the maximum modulus principle it follows that $g(z) \equiv 0$ for the case $\alpha \in (0, 1]$.

4.3. The boundedness and compactness of $T_g, L_g : H(p,p,\phi) \to \mathcal{B}^\alpha$

In this section, we consider the boundedness and compactness of T_g and L_g from $H(p,p,\phi)$ into $\mathcal{B}^\alpha$. For this purpose, we need some auxiliary results incorporated in the following lemmas.

Lemma 4.4. *Suppose that $0 < p < \infty$ and ϕ is normal on $[0, 1)$. If $f \in H(p,p,\phi)$, then*

$$|f(z)| \le C\frac{\|f\|_{H(p,p,\phi)}}{\phi(|z|)(1-|z|^2)^{n/p}}. \qquad z \in B. \tag{27}$$

Proof. For $0 < r < 1$ and $z \in B$, by the subharmonicity of $|f(z)|^p$ and the normality of ϕ, we obtain

$$\begin{aligned} |f(z)|^p &\le \frac{C}{(1-|z|^2)^{n+1}}\int_{D(z,r)}|f(w)|^p dv(w) \\ &\le \frac{C}{(1-|z|^2)^n\phi^p(|z|)}\int_{D(z,r)}|f(w)|^p\frac{\phi^p(|w|)}{1-|w|}dv(w) \\ &\le \frac{C}{(1-|z|^2)^n\phi^p(|z|)}\int_B\frac{\phi^p(|w|)}{1-|w|}|f(w)|^p dv(w) \\ &\le \frac{C\|f\|^p_{H(p,p,\phi)}}{(1-|z|^2)^n\phi^p(|z|)}, \end{aligned}$$

from which the desired result follows.

Lemma 4.5. ([17, Theorem 2]) *Suppose that $0 < p < \infty$ and ϕ is normal on $[0, 1)$. Then for $f \in H(B)$,*

$$\|f\|^p_{H(p,p,\phi)} \asymp |f(0)|^p + \int_B|\mathcal{R}f(z)|^p(1-|z|^2)^p\frac{\phi^p(|z|)}{1-|z|}dv(z).$$

Lemma 4.6. *Let $0 < p < \infty$ and ϕ is normal on $[0,1)$. If $f \in H(p,p,\phi)$ and $z \in B$, then*

$$|\mathcal{R}f(z)| \le C\frac{\|f\|_{H(p,p,\phi)}}{\phi(|z|)(1-|z|^2)^{n/p+1}}, \quad (z \in B).$$

Proof. By the subharmonicity of $|\mathcal{R}f(z)|^p$, normality of $\phi(r)$, and by Lemma 4.5, similar to the proof of Lemma 4.4, we obtain the desired result.

Now, we are in a position to formulate and prove the main results of this section.

Theorem 4.7. *Suppose that g is a holomorphic function on B, $0 < p < \infty, \alpha > 0$ and ϕ is normal on $[0,1)$. Then $T_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is bounded if and only if*

$$\sup_{z\in B}\frac{(1-|z|^2)^{\alpha-n/p}}{\phi(|z|)}|\mathcal{R}g(z)| < \infty.$$

Proof. Let $f \in H(p,p,\phi)$. Then by Lemma 4.4,

$$\begin{aligned}\|T_g f\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^\alpha|\mathcal{R}(T_g f)(z)| = \sup_{z\in B}(1-|z|^2)^\alpha|f(z)||\mathcal{R}g(z)| \\ &\le C\|f\|_{H(p,p,\phi)}\sup_{z\in B}\frac{(1-|z|^2)^\alpha}{\phi(|z|)(1-|z|^2)^{n/p}}|\mathcal{R}g(z)|.\end{aligned}$$

Therefore (28) implies that $T_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is bounded.

Conversely, suppose T_g is a bounded operator from $H(p,p,\phi)$ to $\mathcal{B}^\alpha$. For $w \in B$, set

$$f_w(z) = \frac{(1-|w|^2)^{t+1}}{\phi(|w|)(1-\langle z,w\rangle)^{n/p+t+1}}. \tag{29}$$

It is easy to see that

$$f_w(w) = \frac{1}{\phi(|w|)(1-|w|^2)^{n/p}}, \quad |\mathcal{R}f_w(w)| = \frac{|w|^2}{\phi(|w|)(1-|w|^2)^{n/p+1}}. \tag{30}$$

By [17], we know that

$$M_p(f_w, r) \le C\frac{(1-|w|^2)^{t+1}}{\phi(|w|)(1-r|w|)^{t+1}}.$$

Since ϕ is normal and applying Lemma 3.3, we have that

$$
\begin{aligned}
\|f_w\|_{p,p,\phi}^p &= \int_0^1 M_p^p(f_w,r)\frac{\phi^p(r)}{1-r}r^{2n-1}dr \\
&\le \int_0^1 \frac{(1-|w|^2)^{p(t+1)}}{\phi^p(|w|)(1-r|w|)^{p(t+1)}}\frac{\phi^p(r)}{1-r}dr \\
&\le C\Big[\int_0^{|w|} \frac{(1-|w|^2)^{p(t+1)}}{\phi^p(|w|)(1-r|w|)^{p(t+1)}}\frac{\phi^p(r)}{1-r}dr \\
&\qquad + \int_{|w|}^1 \frac{(1-|w|^2)^{p(t+1)}}{\phi^p(|w|)(1-r|w|)^{p(t+1)}}\frac{\phi^p(r)}{1-r}dr\Big] \\
&\le C\Big[\frac{(1-|w|^2)^{p(t+1)}}{\phi^p(|w|)}\frac{\phi^p(|w|)}{(1-|w|^2)^{pt}}\int_0^{|w|}\frac{(1-r)^{pt-1}}{(1-r|w|)^{p(t+1)}}dr \\
&\qquad + \frac{(1-|w|^2)^{p(t+1)}}{\phi^p(|w|)}\frac{\phi^p(|w|)}{(1-|w|^2)^{ps}}\int_{|w|}^1\frac{(1-r)^{ps-1}}{(1-r|w|)^{p(t+1)}}dr\Big] \\
&\le C.
\end{aligned} \tag{31}
$$

Therefore $f_w \in H(p,p,\phi)$, and moreover $\sup_{w\in B}\|f_w\|_{H(p,p,\phi)} \le C$. Hence

$$
\begin{aligned}
(1-|w|^2)^\alpha|f_w(w)\mathcal{R}g(w)| &\le \sup_{z\in B}(1-|z|^2)^\alpha|f_w(z)\mathcal{R}g(z)| \\
&\le \|T_g f_w\|_{\mathcal{B}^\alpha} \le C\|T_g\|_{H(p,p,\phi)\to\mathcal{B}^\alpha},
\end{aligned}
$$

i.e., we obtain (28), as desired.

Theorem 4.8. *Suppose that g is a holomorphic function on B, $0<p<\infty$, $\alpha>0$ and ϕ is normal on $[0,1)$. Then $L_g : H(p,p,\phi)\to\mathcal{B}^\alpha$ is bounded if and only if*

$$
\sup_{z\in B}\frac{(1-|z|^2)^{\alpha-n/p-1}}{\phi(|z|)}|g(z)|<\infty. \tag{32}
$$

Proof. Assume that (32) holds and let $f\in H(p,p,\phi)$, then by Lemma 4.6, we have

$$
\begin{aligned}
(1-|z|^2)^\alpha|\mathcal{R}(L_g f)(z)| &= (1-|z|^2)^\alpha|\mathcal{R}f(z)||g(z)| \\
&\le C\frac{\|f\|_{H(p,p,\phi)}}{\phi(|z|)(1-|z|^2)^{n/p+1}}|g(z)|(1-|z|^2)^\alpha \\
&\le C\|f\|_{H(p,p,\phi)}\frac{(1-|z|^2)^{\alpha-n/p-1}}{\phi(|z|)}|g(z)|.
\end{aligned}
$$

Taking the supremum over $z \in B$, it follows that $L_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is bounded.

Conversely, assume that $L_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is bounded. For $w \in B$, let $f_w(z)$ be defined by (29), then by (19) and (30), we have

$$\begin{aligned}
&\frac{|w|^4}{\phi^2(|w|)(1-|w|^2)^{2(n/p+1)}}|g(w)|^2 = |\mathcal{R}f_w(w)g(w)|^2 \\
\le\ & \frac{C}{(1-|w|^2)^{n+1}}\int_{D(w,r)}|\mathcal{R}f_w(z)|^2|g(z)|^2 dv(z) \\
=\ & \frac{C}{(1-|w|^2)^{n+1}}\int_{D(w,r)}|\mathcal{R}f_w(z)|^2|g(z)|^2(1-|z|^2)^{2\alpha}\frac{1}{(1-|z|^2)^{2\alpha}}dv(z) \\
\le\ & C\int_{D(w,r)}\frac{dv(z)}{(1-|z|^2)^{2\alpha+n+1}}\sup_{z\in D(w,r)}(1-|z|^2)^{2\alpha}|\mathcal{R}f_w(z)|^2|g(z)|^2 \\
\le\ & \frac{C}{(1-|w|^2)^{2\alpha}}\|L_g f_w\|^2_{\mathcal{B}^\alpha}.
\end{aligned}$$

Therefore

$$\frac{(1-|w|^2)^\alpha|w|^2}{\phi(|w|)(1-|w|^2)^{n/p+1}}|g(w)| \le C\|L_g f_w\|_{\mathcal{B}^\alpha} \le C\|L_g\|_{H^\infty\to\mathcal{B}^\alpha}.$$

Similar to the proof of Theorem 4.2, we obtain that (32) holds, as desired.

Theorem 4.9. *Suppose that g is a holomorphic function on B, $0 < p < \infty, \alpha \ge 0$ and ϕ is normal on $[0,1)$. Then $T_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is compact if and only if*

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^{\alpha-n/p}}{\phi(|z|)}|\mathcal{R}g(z)| = 0. \tag{33}$$

Proof. Assume that (33) holds and that $(f_k)_{k\in\mathbb{N}}$ is a sequence in $H(p,p,\phi)$ such that $\sup_{k\in\mathbb{N}}\|f_k\|_{H(p,p,\phi)} \le L$ and $f_k \to 0$ uniformly on compact subsets of B as $k \to \infty$. By the assumption, for every $\varepsilon > 0$, there is a constant $\delta \in (0,1)$, such that

$$\frac{(1-|z|^2)^\alpha|\mathcal{R}g(z)|}{\phi(|z|)(1-|z|^2)^{n/p}} < \varepsilon/L$$

whenever $\delta < |z| < 1$.

Let $K = \{z \in B : |z| \le \delta\}$ (K is a compact subsect of B) and ϕ be a normal function, then by Lemma 4.4 we have

$$\begin{aligned}\|T_gf_k\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^\alpha|\mathcal{R}(T_gf_k)(z)|\\ &\le \sup_{z\in K}(1-|z|^2)^\alpha|\mathcal{R}g(z)f_k(z)| + C\sup_{z\in B\setminus K}\frac{(1-|z|^2)^\alpha}{\phi(|z|)(1-|z|^2)^{n/p}}|\mathcal{R}g(z)|\|f_k\|_{H(p,p,\phi)}\\ &\le CM\sup_{z\in K}(1-|z|^2)^{n/p}|\phi(z)||f_k(z)| + C\varepsilon,\end{aligned}$$

where

$$M=\sup_{z\in B}\frac{(1-|z|^2)^\alpha}{\phi(|z|)(1-|z|^2)^{n/p}}|\mathcal{R}g(z)|<\infty.$$

By the assumption and Theorem 4.7 we obtain $\|T_gf_k\|_{\mathcal{B}^\alpha}\to 0$ as $k\to\infty$. Therefore, $T_g : H(p,p,\phi)\to\mathcal{B}^\alpha$ is compact.

Conversely, suppose $T_g : H(p,p,\phi)\to\mathcal{B}^\alpha$ is compact. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|z_k|\to 1$ as $k\to\infty$, and set

(34) $$f_k(z)=\frac{(1-|z_k|^2)^{t+1}}{\phi(|z_k|)(1-\langle z,z_k\rangle)^{n/p+t+1}},\quad k\in\mathbb{N}.$$

Then $f_k\in H(p,p,\phi)$, moreover $\sup_{k\in\mathbb{N}}\|f_k\|_{H(p,p,\phi)}<\infty$ and f_k converges to 0 uniformly on compact subsets of B as $k\to\infty$. Since T_g is compact, by Lemma 4.2 it follows that $\|T_gf_k\|_{\mathcal{B}^\alpha}\to 0$ as $k\to\infty$. From this and since

$$\begin{aligned}\|T_gf_k\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^\alpha|\mathcal{R}(T_gf_k)(z)| = \sup_{z\in B}(1-|z|^2)^\alpha|f_k(z)||\mathcal{R}g(z)|\\ &\ge \frac{(1-|z_k|^2)^\alpha|\mathcal{R}g(z_k)|}{\phi(|z_k|)(1-|z_k|^2)^{n/p}},\end{aligned}$$

for every $k\in\mathbb{N}$, we obtain (33), finishing the proof of the theorem.

Theorem 4.10. *Suppose that g is a holomorphic function on B, $0<p<\infty,\alpha>0$ and ϕ is normal on $[0,1)$. Then $L_g : H(p,p,\phi)\to\mathcal{B}^\alpha$ is compact if and only if*

(35) $$\lim_{|z|\to 1}\frac{(1-|z|^2)^{\alpha-n/p-1}}{\phi(|z|)}|g(z)|=0.$$

Proof. Proof. Assume that the condition (35) holds, and $(f_k)_{k\in\mathbb{N}}$ is a sequence in $H(p,p,\phi)$ with $\sup_{k\in\mathbb{N}}\|f_k\|_{H(p,p,\phi)}\le K$ and $f_k\to 0$ uniformly on compact subsets of B as $k\to\infty$. From (35) we have that for every $\varepsilon>0$, there is a constant $\delta\in(0,1)$, such that

(36) $$\frac{(1-|z|^2)^\alpha|g(z)|}{\phi(|z|)(1-|z|^2)^{1+n/p}}<\varepsilon.$$

whenever $\delta < |z| < 1$. Introducing the set $E = \{z \in B : |z| \le \delta\}$, similar to the proof of Theorem 4.9, we can prove that $\|L_g f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$. Therefore, $L_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is compact.

Conversely, suppose $L_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is compact. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|z_k| \to 1$ as $k \to \infty$, and $(f_k)_{k\in\mathbb{N}}$ is defined by (34). Then $\sup_{k\in\mathbb{N}} \|f_k\|_{H(p,p,\phi)} \le C$ and f_k converges to 0 uniformly on compact subsets of B as $k \to \infty$. Since L_g is compact, we have $\|L_g f_k\|_{\mathcal{B}^\alpha} \to 0$ as $k \to \infty$. Thus

$$\begin{aligned} \|L_g f_k\|_{\mathcal{B}^\alpha} &= \sup_{z\in B}(1-|z|^2)^\alpha |\mathcal{R}(L_g f_k)(z)| \\ &\ge (n/p+t+1)\frac{(1-|z_k|^2)^\alpha |g(z_k)|\,|z_k|^2}{\phi(|z_k|)(1-|z_k|^2)^{1+n/p}}, \end{aligned}$$

which implies (35).

By Theorems 4.7-4.10, we obtain the following corollary.

Corollary 4.1. *Suppose that g is a holomorphic function on B, $0 < p < \infty$ and $\alpha > 0$. Then*

(i) *$T_g : A^p \to \mathcal{B}^\alpha$ is bounded if and only if*

$$\sup_{z\in B}(1-|z|^2)^{\alpha-(n+1)/p}|\mathcal{R}g(z)| < \infty.$$

(ii) *$L_g : A^p \to \mathcal{B}^\alpha$ is bounded if and only if*

$$\sup_{z\in B}(1-|z|^2)^{\alpha-(n+1)/p-1}|g(z)| < \infty.$$

(iii) *$T_g : A^p \to \mathcal{B}^\alpha$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha-(n+1)/p}|\mathcal{R}g(z)| = 0.$$

(iv) *$L_g : A^p \to \mathcal{B}^\alpha$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha-(n+1)/p-1}|g(z)| = 0.$$

4.4. The boundedness and compactness of $T_g, L_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$

In this section, we characterize the boundedness and the compactness of $T_g, L_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$.

Theorem 4.11. *Suppose that g is a holomorphic function on B, $0 < p < \infty$, $\alpha > 0$ and ϕ is normal on $[0,1)$. Then the following statements hold.*

(*i*) $T_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$ *is bounded if and only if* $g \in \mathcal{B}_0^\alpha$ *and* $T_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ *is bounded;*

(*ii*) $T_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^{\alpha-n/p}}{\phi(|z|)} |\mathcal{R}g(z)| = 0. \tag{37}$$

Proof. (*i*). It is clear that $g \in \mathcal{B}_0^\alpha$ and $T_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is bounded if $T_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$ is bounded.

Conversely, assume that $T_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is bounded and $g \in \mathcal{B}_0^\alpha$. Then, for any polynomial $p(z)$, since $g \in \mathcal{B}_0^\alpha$ and

$$\begin{aligned}(1-|z|^2)^\alpha \big|\mathcal{R}(T_g p)(z)\big| &= (1-|z|^2)^\alpha |p(z)|\, |\mathcal{R}g(z)| \\ &\le (1-|z|^2)^\alpha |\mathcal{R}g(z)| \max_{z\in\mathbb{D}} |p(z)| \to 0,\end{aligned}$$

as $|z| \to 1$, we obtain that $T_g p \in \mathcal{B}_0^\alpha$. For any $f \in H(p,p,\phi)$, there exist a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that $\|f - p_k\|_{H(p,p,\phi)} \to 0$, as $k \to \infty$. Since $\mathcal{B}_0^\alpha$ is closed, we obtain

$$T_g f = \lim_{k\to\infty} T_g p_k \in \mathcal{B}_0^\alpha.$$

In addition, $T_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is bounded. Therefore $T_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$ is bounded.

(*ii*). Sufficiency. From Lemma 4.3 it follows that $T_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$ is compact if and only if

$$\lim_{|z|\to 1} \sup_{\|f\|_{H(p,p,\phi)}\le 1} (1-|z|^2)^\alpha |\mathcal{R}(T_g f)(z)| = 0.$$

By Lemma 4.4, we have

$$\begin{aligned}(1-|z|^2)^\alpha |\mathcal{R}(T_g f)(z)| &= \frac{(1-|z|^2)^\alpha |\mathcal{R}g(z)|}{\phi(|z|)(1-|z|^2)^{n/p}} \phi(|z|)(1-|z|^2)^{n/p} |f(z)| \\ &\le C \frac{(1-|z|^2)^\alpha |\mathcal{R}g(z)|}{\phi(|z|)(1-|z|^2)^{n/p}} \|f\|_{H(p,p,\phi)}.\end{aligned}$$

Taking the supremum in the last inequality over the set $\{f \in H(B) : \|f\|_{H(p,p,\phi)} \le 1\}$ and letting $|z| \to 1$ the result follows.

Necessity. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in B such that $|z_k| \to 1$ as $k \to \infty$ and let $(f_k)_{k\in\mathbb{N}}$ be the sequence defined by (34). Then $\sup_{k\in\mathbb{N}} \|f_k\|_{H(p,p,\phi)} \le C$ and f_k converges to 0 uniformly on compact subsets of B as $k \to \infty$. Hence, by Lemma 4.2 it follows that $\lim_{k\to\infty} \|T_g(f_k)\|_{\mathcal{B}^\alpha} = 0$. On the other hand, we have

$$f_k(z_k) = \frac{1}{\phi(|z_k|)(1-|z_k|^2)^{n/p}}.$$

Since

$$\frac{(1-|z_k|^2)^\alpha|\mathcal{R}g(z_k)|}{\phi(|z_k|)(1-|z_k|^2)^{n/p}} \le \sup_{z\in B}(1-|z|^2)^\alpha|\mathcal{R}(T_gf_k)(z)| = \|T_g(f_k)\|_{\mathcal{B}^\alpha},$$

we have

$$\lim_{k\to\infty}\frac{(1-|z_k|^2)^\alpha|\mathcal{R}g(z_k)|}{\phi(|z_k|)(1-|z_k|^2)^{n/p}} = 0,$$

i.e. we obtain (37), finishing the proof of the theorem.

Theorem 4.12. *Suppose that g is a holomorphic function on B, $0 < p < \infty$, $\alpha > 0$ and ϕ is normal on $[0,1)$. Then the following statements hold.*

(i) *$L_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$ is bounded if and only if $L_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is bounded and*

$$\lim_{|z|\to 1}|g(z)|(1-|z|^2)^\alpha = 0;$$

(ii) *$L_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$ is compact if and only if*

$$\lim_{|z|\to 1}\frac{(1-|z|^2)^{\alpha-n/p-1}}{\phi(|z|)}|g(z)| = 0.$$

Proof. It can be deduced similarly to Theorem 4.11. We omit the details.

Corollary 4.2. *Suppose that g is a holomorphic function on B, $0 < p < \infty$ and $\alpha > 0$. Then*

(i) *$T_g : A^p \to \mathcal{B}_0^\alpha$ is bounded if and only if $g \in \mathcal{B}_0^\alpha$ and $T_g : A^p \to \mathcal{B}^\alpha$ is bounded;*

(ii) *$T_g : A^p \to \mathcal{B}_0^\alpha$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha-(n+1)/p}|\mathcal{R}g(z)| = 0;$$

(iii) *$L_g : A^p \to \mathcal{B}_0^\alpha$ is bounded if and only if $L_g : A^p \to \mathcal{B}^\alpha$ is bounded and*

$$\lim_{|z|\to 1}|g(z)|(1-|z|^2)^\alpha = 0;$$

(iv) *$L_g : A^p \to \mathcal{B}_0^\alpha$ is compact if and only if*

$$\lim_{|z|\to 1}(1-|z|^2)^{\alpha-(n+1)/p-1}|g(z)| = 0.$$

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