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MAJORIZED PROOF AND IMPROVEMENT OF THE DISCRETE STEFFENSEN'S INEQUALITY

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Abstract. We enlarge two weak majorization relations of the vectors to strong majorization relations of the vectors. An improvement of the discrete Steffensen's inequalities is established by the related propositions in the theory of majorization.

1. INTRODUCTION

Let $\{x_i\}_{i=1}^n$ be a nonincreasing finite sequence of nonnegative real numbers, and let $\{y_i\}_{i=1}^n$ be a finite sequence of real numbers such that for every i, $0 \leqslant y_i \leqslant 1$. Let $k_1, k_2 \in \{1, 2, \cdots, n\}$ be such that $k_2 \leqslant \sum_{i=1}^n y_i \leqslant k_1$. Then

$$\sum\nolimits_{i=n-k_2+1}^{n} x_i \leqslant \sum\nolimits_{i=1}^{n} x_i y_i \leqslant \sum\nolimits_{i=1}^{k_1} x_i. \tag{1}$$

The inequality (1) is called the discrete Steffensen's inequality. It was first given in [1] and then cited repeatedly in [2-6]. Recently, a new proof which is very simple and clear is given in [7]. The purpose of this note is to establish an improved Steffensen's inequality by means of the theory of majorization. The following definitions and lemmas will be used:

Definition. [8] Let x= ($x_1,\ldots,x_n)$ and y = ($y_1,\ldots,y_n)$ be two real n-tuples. Then x is said to be majorized by y (in symbols $x \prec y$) if

(i) $\sum\limits_{i=1}^{k} x_{[i]} \leqslant \sum\limits_{i=1}^{k} y_{[i]} \; for \; k = 1, 2, ..., n-1,$

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(ii) $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$,

where $x_{[1]} \geqslant x_{[2]} \geqslant \cdots \geqslant x_{[n]}$ and $y_{[1]} \geqslant y_{[2]} \geqslant \cdots \geqslant y_{[n]}$ are components of x and y rearranged in descending order. And x is said to be weakly submajorized by y (written $x \prec_w y)$ if

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]} \ , \ k = 1, 2, \ldots , n.$$

And x is said to be weakly supermajorized by y (written $x \prec^w y)$ if

$$\sum_{i=1}^{k} x_{(i)} \geqslant \sum_{i=1}^{k} y_{(i)} \ , \ k = 1, 2, \ldots , n,$$

where $x_{(1)} \leqslant x_{(2)} \leqslant \cdots \leqslant x_{(n)}$ and $y_{(1)} \leqslant y_{(2)} \leqslant \cdots \leqslant y_{(n)}$ are components of x and y rearranged in increasing order.

Relatively, the majorization is also said to be the strong majorization.

Lemma 1. [9, pp. 122-123] *Let $x, \ y \in R^n$,*

(a) *if $x \prec_w y$, then*

$$(x, \ x_{n+1}) \prec (y, \ y_{n+1}),$$

where $x_{n+1} = \min\{x_1, x_2, \cdots, x_n, \ y_1, y_2, \cdots, y_n\}, \ y_{n+1} = \sum_{i=1}^{n+1} x_i - \sum_{i=1}^{n} y_i$.

(b) *if $x \prec^w y$, then*

$$(x_0, x \ ) \prec (y_0, y \ ),$$

where $x_0 = \max\{x_1, x_2, \cdots, x_n, \ y_1, y_2, \cdots, y_n\}, \ y_0 = \sum_{i=0}^{n} x_i - \sum_{i=1}^{n} y_i$.

Lemma 2. [8, p. 12] ***Lemma 2.*** *For $x, \ y \in R^n$, we have*

$$\sum\nolimits_{i=1}^{n} x_{[i]} y_{(i)} \leqslant \sum\nolimits_{i=1}^{n} x_i y_i \leqslant \sum\nolimits_{i=1}^{n} x_{[i]} y_{[i]}.$$

Lemma 3. [8, p. 15] *For $x, \ y \in R^n$, we have*

(a) $x \prec y \Leftrightarrow \sum_{i=1}^{n} x_{[i]} u_{[i]} \leqslant \sum_{i=1}^{n} y_{[i]} u_{[i]}, \ \forall u \in R^n$,

(b) $x \prec y \Leftrightarrow \sum_{i=1}^{n} x_{(i)} u_{[i]} \geqslant \sum_{i=1}^{n} y_{(i)} u_{[i]}, \ \forall u \in R^n$,

(c) $x \prec y \Leftrightarrow \sum_{i=1}^{n} x_{[i]} u_{(i)} \geqslant \sum_{i=1}^{n} y_{[i]} u_{(i)}, \ \forall u \in R^n$.

3. Main Results and Proofs

Theorem 1. *Let* $\{y_i\}_{i=1}^n$ *be a finite sequence of real numbers such that* $0 \leqslant y_i \leqslant 1$ *for* $i = 1, 2, \cdots, n$, *and let* $k_1, k_2 \in \{1, 2, \cdots, n\}$ *be such that* $k_2 \leqslant \sum_{i=1}^n y_i \leqslant k_1$. *Then*

$$y = (y_1, y_2, \cdots, y_n) \prec^w \Big(\underbrace{0, \cdots, 0}_{n-k_2}, \underbrace{1, \cdots, 1}_{k_2}\Big) = z, \tag{2}$$

$$y = (y_1, y_2, \cdots, y_n) \prec_w \Big(\underbrace{1, \cdots, 1}_{k_1}, \underbrace{0, \cdots 0}_{n-k_1}\Big) = v. \tag{3}$$

Proof. Firstly, we prove that $y \prec^w z$ by the definition of the weak supermajorization. When $1 \leqslant k \leqslant n - k_2$, clearly, $\sum_{i=1}^k y_{(i)} \geqslant \sum_{i=1}^k z_{(i)} = 0$. When $n \geqslant k > n-k_2$, Using reduction to absurdity, we prove that $\sum_{i=1}^k y_{(i)} \geqslant \sum_{i=1}^k z_{(i)}$, namely, if there exist k $(n \geqslant k > n - k_2)$ such that $\sum_{i=1}^k y_{(i)} < \sum_{i=1}^k z_{(i)}$, then by $0 \leqslant y_i \leqslant 1, i = 1, \cdots, n$, we have

$$\sum_{i=1}^n y_{(i)} = \sum_{i=1}^k y_{(i)} + \sum_{i=k+1}^n y_{(i)} < k - (n - k_2) + (n - k) = k_2,$$

It contradicts with $k_2 \leqslant \sum_{i=1}^n y_i$.

Secondly, we prove that $y \prec_w v$ by the definition of the weak submajorizzation. Note that $0 \leqslant y_i \leqslant 1, i = 1, 2, \cdots, n$, when $1 \leqslant k \leqslant k_1$, we have $\sum_{i=1}^k y_{[i]} \leqslant k = \sum_{i=1}^k v_{[i]}$. When $k_1+1 \leqslant k \leqslant n$, we have $\sum_{i=1}^k y_{[i]} \leqslant \sum_{i=1}^n y_{[i]} \leqslant k_1 = \sum_{i=1}^k v_{[i]}$. This completes the proof of Theorem 1. ■

Theorem 2. *Let* $\{x_i\}_{i=1}^n$ *be a nonincreasing finite sequence of real numbers, and let* $\{y_i\}_{i=1}^n$ *be a finite sequence of real numbers such that for every* i, $0 \leqslant y_i \leqslant 1$. *Let* $k_1, k_2 \in \{1, 2, \cdots, n\}$ *be such that* $k_2 \leqslant \sum_{i=1}^n y_i \leqslant k_1$. *Then*

$$\begin{aligned} &\sum_{i=n-k_2+1}^n x_i + \left(\sum_{i=1}^n y_i - k_2\right) x_n \\ &\quad \leqslant \sum_{i=1}^n x_i y_i \leqslant \sum_{i=1}^{k_1} x_i - \left(k_1 - \sum_{i=1}^n y_i\right) x_n. \end{aligned} \tag{4}$$

Proof. By Theorem 1, we have

$$y = (y_1, y_2, \cdots, y_n) \prec_w \left(\underbrace{1, \cdots, 1}_{k_1}, \underbrace{0, \cdots 0}_{n-k_1}\right) = v,$$

using Lemma 1 (a), we obtain

$$(y_1, y_2, \cdots, y_n, y_{n+1}) \prec \left(\underbrace{1, \cdots, 1}_{k_1}, \underbrace{0, \cdots, 0}_{n-k_1}, v_{n+1}\right),$$

where $y_{n+1} = \min\{y_1, y_2, \cdots, y_n, v_1, v_2, \cdots, v_n\}$, $v_{n+1} = \sum\limits_{i=1}^{n+1} y_i - \sum\limits_{i=1}^{n} v_i = \sum\limits_{i=1}^{n} y_i$ $+y_{n+1} - k_1$. It is clear that $y_{n+1} = 0$, and then $v_{n+1} \leqslant 0$.

Choosing $u = (x_1, x_2, \cdots, x_n,\ x_n)$, from Lemma 2 and Lemma 3 (a) we have

$$\sum\nolimits_{i=1}^{n} x_i y_i + x_n y_{n+1} \leqslant \sum\nolimits_{i=1}^{n} x_{[i]} y_{[i]} + x_n y_{n+1} \leqslant \sum\nolimits_{i=1}^{k_1} x_i + \left(\sum_{i=1}^{n} y_i + y_{n+1} - k_1\right) x_n,$$

hence

$$\sum\nolimits_{i=1}^{n} x_i y_i \leqslant \sum\nolimits_{i=1}^{k_1} x_i - \left(k_1 - \sum_{i=1}^{n} y_i\right) x_n.$$

Also, from Theorem 1 and Lemma 1 (b) we have

$$y = (y_1, y_2, \cdots, y_n) \prec^w \left(\underbrace{0, \cdots, 0}_{n-k_2}, \underbrace{1, \cdots, 1}_{k_2}\right) = z,$$

and

$$(y_1, y_2, \cdots, y_n, y_0) \prec \left(\underbrace{0, \cdots 0}_{n-k_2}, \underbrace{1, \cdots, 1}_{k_2}, z_0\right),$$

where $y_0 = \max\{y_1, y_2, \cdots, y_n, z_1, \cdots, z_n\}$, $z_0 = \sum\limits_{i=0}^{n} y_i - \sum\limits_{i=1}^{n} v_i = \sum\limits_{i=1}^{n} y_i + y_0 - k_2$. It is clear that $y_0 \geqslant 1$, and then $z_0 \geqslant 1$.

Choosing $u = (x_1, x_2, \cdots, x_n,\ x_n)$, from Lemma 2 and Lemma 3 (b) we have

$$\sum\nolimits_{i=1}^{n} y_i x_i + y_0 x_n \geqslant \sum\nolimits_{i=1}^{n} y_{(i)} x_{[i]} + y_0 x_n \geqslant \sum\nolimits_{i=n-k_2+1}^{n} x_i + \left(\sum_{i=1}^{n} y_i + y_0 - k_2\right) x_n,$$

thus

$$\sum\nolimits_{i=1}^{n} x_i y_i \geqslant \sum\nolimits_{i=n-k_2+1}^{n} x_i + \left(\sum_{i=1}^{n} y_i - k_2\right) x_n.$$

This completes the proof of Theorem 2. ■

As consequence, a refinement of the discrete Steffensen's inequality follows from Theorem 2 directly:

Corollary . *Let* $\{x_i\}_{i=1}^n$ *be a nonincreasing finite sequence of nonnegative real numbers, and let* $\{y_i\}_{i=1}^n$ *be a finite sequence of real numbers such that for every* i, $0 \leqslant y_i \leqslant 1$. *Let* $k_1, k_2 \in \{1, 2, \cdots, n\}$ *be such that* $k_2 \leqslant \sum_{i=1}^n y_i \leqslant k_1$. *Then*

$$\begin{aligned}\sum_{i=n-k_2+1}^n x_i \leqslant \sum_{i=n-k_2+1}^n x_i + \left(\sum_{i=1}^n y_i - k_2\right) x_n \leqslant \sum_{i=1}^n x_i y_i \\ \leqslant \sum_{i=1}^{k_1} x_i - \left(k_1 - \sum_{i=1}^n y_i\right) x_n \leqslant \sum_{i=1}^{k_1} x_i.\end{aligned} \tag{5}$$

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