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WEIGHTED L^p BOUNDEDNESS FOR PARAMETRIZED LITTLEWOOD-PALEY OPERATORS

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Abstract. In this paper the authors established some sharp estimates for a class of the parametrized Littlewood-Paley operators. Using the result the authors give the weighted L^p bounds of the parametrized area integral and Littlewood-Paley g_λ^* function.

1. Introduction

Let $\Omega(x)$ be homogeneous of degree zero on $\mathbb{R}^n$ with $\Omega \in L^1(S^{n-1})$, where S^{n-1} denotes the unit sphere of $\mathbb{R}^n (n \geq 2)$ equipped with Lebesgue measure $d\sigma = d\sigma(x')$. Moreover, Ω satisfies

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0. \tag{1.1}$$

Then for $\rho > 0$ the parametrized Marcinkiewicz integral μ_Ω^ρ, area integral $\mu_{\Omega,S}^\rho$ and the Littlewood-Paley function $\mu_\lambda^{*,\rho}$ are defined respectively by

$$\mu_\Omega^\rho(f)(x) = \left(\int_0^\infty \Big| \int_{|y-x|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)dy \Big|^2 \frac{dt}{t^{1+2\rho}} \right)^{1/2},$$

$$\mu_{\Omega,S}^\rho(f)(x) = \left(\iint_{\Gamma(x)} \left| \frac{1}{t^\rho} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}$$

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and

$$\mu_\lambda^{*,\rho}(f)(x)=\left(\iint_{\mathbb{R}^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f(z)dz\right|^2\frac{dydt}{t^{n+1}}\right)^{1/2},$$

where $\Gamma(x)=\{(y,t)\in\mathbb{R}^{n+1}_+ : |x-y|<t\}$. We will denote simply μ^1_Ω, $\mu^1_{\Omega,S}$ and $\mu_\lambda^{*,1}$ by μ_Ω, μ_S and μ_λ^*, respectively. On the other hand, we will show that the parametrized Littlewood-Paley operators above are essentially the vector valued Calderón-Zygmund singular operators. In fact, we define the Hilbert spaces as follows.

$$\mathcal{H}=\left\{h:\|h\|=\left(\int_0^\infty\frac{|h(t)|^2}{t}dt\right)^{1/2}\right\},$$

$$\mathcal{H}_1=\left\{u:\ \ \|u\|_{\mathcal{H}_1}=\left(\iint_{\mathbb{R}^{n+1}_+}|u(y,t)|^2\chi_{|y|<1}(y)\frac{dydt}{t}\right)^{1/2}<\infty\right\},$$

and

$$\mathcal{H}_2=\left\{u:\ \ \|u\|_{\mathcal{H}_2}=\left(\iint_{\mathbb{R}^{n+1}_+}\left(\frac{1}{1+|y|}\right)^{\lambda n}|u(y,t)|^2\frac{dydt}{t}\right)^{1/2}<\infty,\quad \lambda>1\right\},$$

where $h(t)$ and $u(y,t)$ are measurable functions on $\mathbb{R}_+$ and $\mathbb{R}^{n+1}_+$, respectively. Denote B the unit ball in $\mathbb{R}^n$ and take $\varphi(x)=\Omega(x)|x|^{-n+\rho}\chi_B(x)$, if let

$$\begin{aligned}F(f)(x,y,t)&=\int_{\mathbb{R}^n}t^{-n}\varphi\left(\frac{x-z}{t}-y\right)f(z)dz,\quad\text{and}\\ G(f)(x,t)&=\int_{\mathbb{R}^n}t^{-n}\varphi\left(\frac{x-y}{t}\right)f(y)dy,\end{aligned}\tag{1.2}$$

then we have

(i) $\mu^\rho_\Omega(f)(x)=\|G(f)(x,\cdot)\|_{\mathcal{H}}$;

(ii) $\mu^\rho_{\Omega,S}(f)(x)=\|F(f)(x,\cdot,\cdot)\|_{\mathcal{H}_1}$;

(iii) $\mu_\lambda^{*,\rho}(f)(x)=\|F(f)(x,\cdot,\cdot)\|_{\mathcal{H}_2}$,

respectively.

The Marcinkiewicz integral μ_Ω was defined first by Stein [6]. Stein proved that μ_Ω is of weak type (1,1) and type (p,p) $(1<p\le 2)$ for $\Omega\in\text{Lip}_\alpha(S^{n-1})$ $(0<\alpha\le 1)$. In 1960, Hörmander [H] defined and gave the L^p $(1<p<\infty)$ bounds of the parametrized Marcinkiewicz integral μ^ρ_Ω $(\rho>0)$ for $\Omega\in\text{Lip}_\alpha(S^{n-1})$ $(0<\alpha\le 1)$. The parametrized Littlewood-Paley operators $\mu_\lambda^{*,\rho}$ and $\mu^\rho_{\Omega,S}$ were discussed by

Sakamoto and Yabuta [7] in 1999. In [7], the authors studied $L^p\,(1<p<\infty)$ boundedness for Ω satisfies the Lip_α condition.

On the other hand, Torchinsky and Wang [8] first considered the weighted L^p boundedness of μ_Ω. To state Torchinsky and Wang's result, we first give some definitions. For $p>0$ and $f\in L^1_{loc}(\mathbb{R}^n)$, let

$$M_pf(x)=\sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q|f(y)|^pdy\right)^{1/p}$$

where and what in following, Q is a cube with sides parallel to the coordinate axes. The generalized sharp function $M_p^\sharp f$ is given by

$$M_p^\sharp f(x)=\sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q|f(y)-f_Q|^pdy\right)^{1/p},$$

where f_Q is the average of f over Q; that is, $f_Q=\frac{1}{|Q|}\int_Q f(y)dy$. We simply denote $M_1f=Mf$, $M_1^\sharp f=M^\sharp f$.

Definition 1. A nonnegative locally integrable function $w(x)$ on $\mathbb{R}^n$ is said to be in A_p, if there exists a constant $C>0$ such that for every cube $Q\subset\mathbb{R}^n$

$$\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1}\le C \quad \text{for} \quad 1<p<\infty,$$

and for a.e., $x\in\mathbb{R}^n$ and $Q\ni x$

$$\frac{1}{|Q|}\int_Q w(y)dy\le Cw(x), \quad \text{for} \quad p=1.$$

Torchinsky and Wang's result is as follows.

Theorem A. ([8]) *Suppose* $\Omega\in\mathrm{Lip}_\alpha(0<\alpha\le 1)$, $1<p<\infty$ *and* $\omega\in A_p(1<p<\infty)$, *then there is a constant* $c_p(\omega)$, *independent of* f, *such that* $\|\mu_\Omega(f)\|_{p,\omega}\le c_p(\omega)\|f\|_{p,\omega}$.

In 1999, Ding, Fan and Pan [2] improved the above result. They gave the following weighted L^p boundedness of μ_Ω and μ_λ^*, $\mu_{\Omega,S}$:

Theorem B. ([2]) *Suppose that* $\Omega\in L^q(S^{n-1})\,(q>1)$ *satisfying* (1.1). *If* p,q *and* ω *satisfy one of the following conditions.*

(a) $q'<p<\infty$ *and* $\omega\in A_{p/q'}$,

(b) $1<p<q$ *and* $\omega^{1-p'}\in A_{p'/q'}$,

(c) $1 < p < \infty$ *and* $\omega^{q'} \in A_p$.

Then $\|\mu_\Omega(f)\|_{p,\omega} \le C\|f\|_{p,\omega}$, *where constant* C *is independent of* f.

Theorem C. ([2]) *Suppose that* $\Omega \in L^q(S^{n-1})(q > 1)$ *satisfying* (1.1). *If* p, q *and* ω *satisfy one of the following conditions.*

(a) $\max\{q', 2\} = \eta < p < \infty$ *and* $\omega \in A_{p/\eta}$,

(b) $2 < p < q$ *and* $\omega^{1-(p/2)'} \in A_{p'/q'}$,

(c) $2 \le p < \infty$ *and* $\omega^{q'} \in A_{p/2}$.

Then $\|\mu^*_\lambda\|_{p,\omega} \le C\|f\|_{p,\omega}$ *and* $\|\mu_{\Omega,S}\|_{p,\omega} \le C\|f\|_{p,\omega}$, *where constant* C *is independent of* f.

For general $\rho > 0$, in 2002, Ding, Lu and Yabuta gave the following L^p result with rough kernel.

Theorem D. ([4]) *Suppose that* $\Omega \in L\log^+L(S^{n-1})$ *satisfies* (1.1). *Then for* $\rho > 0$ *and* $2 \le p < \infty$, $\|\mu^\rho_\Omega(f)\|_{L^p} \le C_{n,p,\rho}\|f\|_{L^p}$,$\|\mu^\rho_{\Omega,S}(f)\|_{L^p} \le C_{n,p,\rho}\|f\|_{L^p}$ *and* $\|\mu^{*,\rho}_\lambda(f)\|_{L^p} \le C_{n,p,\rho}\|f\|_{L^p}$.

Comparing with the weighted boundedness of μ_Ω, an interesting question arises, that is, if the operators μ^ρ_Ω, $\mu^\rho_{\Omega,S}$ and $\mu^{*,\rho}_\lambda$ satisfy the similar weighted boundedness as μ_Ω. The main purpose of this paper is to give a positive answer to this problem. By establishing some sharp estimates, we give the weighted L^p boundedness of these operators. Let us first give a definition.

Definition 2. Let $\Omega(x') \in L^q(S^{n-1})$, $q \ge 1$. Then the integral modulus $\omega_q(\delta)$ of continuity of order q of Ω is defined by

$$\omega_q(\delta) = \sup_{\|\gamma\| \le \delta} \left(\int_{S^{n-1}} |\Omega(\gamma x') - \Omega(x')|^q d\sigma(x') \right)^{1/q},$$

where γ denotes a rotation on S^{n-1} and $\|\gamma\| = \sup_{x' \in S^{n-1}} |\gamma x' - x'|$. The function Ω is said to satisfy the L^q-Dini condition, if

$$\int_0^1 \frac{\omega_q(\delta)}{\delta} d\delta < \infty. \tag{1.3}$$

Our results are as follows.

Theorem 1. *Let* $\Omega \in L^2(S^{n-1})$ *satisfing* (1.1) *and the following condition*

$$\int_0^1 \frac{\omega_2(\delta)}{\delta}(1 + |\log \delta|)^\sigma d\delta < \infty, \quad \textit{for} \quad \sigma > 1. \tag{1.4}$$

Then for $\rho > n/2$, $\lambda > 2$ *and* $f \in L^p(\mathbb{R}^n)(1 < p < \infty)$,

$$M^\sharp(\mu^\rho_{\Omega,S} f)(x) \leq C_p M_p f(x) \quad \text{for all} \quad x \in \mathbb{R}^n \tag{1.5}$$

and

$$M^\sharp(\mu^{*,\rho}_\lambda f)(x) \leq C_p M_p f(x) \quad \text{for all} \quad x \in \mathbb{R}^n, \tag{1.6}$$

where C_p *is independent of* f.

As a corollary of Theorem 1, we get the following weighted boundedness of $\mu^\rho_{\Omega,S}$ and $\mu^{*,\rho}_\lambda$:

Theorem 2. *Let* Ω *satisfies the same condition as in Theorem 1 and* $\omega \in A_p$. *Then for* $\rho > n/2, \lambda > 2$ *and* $f \in L^p(\mathbb{R}^n)(1 < p < \infty)$, *there is a constant C, independent of* f, *such that*

$$\|\mu^\rho_{\Omega,S}(f)\|_{p,\omega} \leq C\|f\|_{p,\omega}, \quad \|\mu^{*,\rho}_\lambda(f)\|_{p,\omega} \leq C\|f\|_{p,\omega}.$$

Remark 1. Theorem 2 doesn't hold for $0 < \rho \leq n/2$ and $1 \leq p \leq 2n/(n+2\rho)$ if $n \geq 3$. This is can be seen by the counterexample in [7].

Remark 2. As is pointed out in [3], (1.4) is weaker than Lip$_\alpha$ condition. Moreover, combining the idea of proving Theorem 1 with the similar steps as in [8], we may get the weighted boundedness of μ^ρ_Ω. We omit the details here.

2. Proof of Theorem 1

We need the following Lemma.

Lemma 2.1. *Suppose that* $\rho > 0$, Ω *is homogeneous of degree zero and satisfies the* L^2*-Dini condition. If there exists a constant* $0 < \theta < 1/2$ *such that* $|x| < \theta R$, *then we have the folloing inequality*

$$\left(\int_{R<|y|<2R}\left|\frac{\Omega(y-x)}{|y-x|^{n-\rho}}-\frac{\Omega(y)}{|y|^{n-\rho}}\right|^2 dy\right)^{1/2} \leq CR^{n/2-(n-\rho)}\left\{\frac{|x|}{R}\int_{|x|/2R}^{|x|/R}\frac{\omega_2(\delta)}{\delta}d\delta\right\},$$

where the constant $C > 0$ *is independent of* R *and* x.

See [1] for the case $0 < \rho < n$ and the proof is trivial for the case $\rho \geq n$.

Now Let us turn to the proof of Theorem 1. First we want to prove (1.5)

$$M^\sharp(\mu^\rho_{\Omega,S} f)(x) \leq C_p M_p f(x) \quad \text{for all} \quad x \in \mathbb{R}^n.$$

Given $x \in \mathbb{R}^n$, let $Q = Q(\bar{x}, r_0)$ be a cube centered at $\bar{x}$, half side length r_0 and contains x. Denote Q^* be a ball with center at $\bar{x}$ and radius $r = 2\sqrt{n}r_0$. Set

$$f = f\chi_{8Q^*} + f(1-\chi_{8Q^*}) =: f_1 + f_2.$$

Then by the $L^p(p>1)$ bounds of the operator $\mu^{\rho}_{\Omega,S}$ (By Theorem D for the case $2 \le p < \infty$ and see [5] for the case $1 < p < 2$),

$$\int_Q \mu^{\rho}_{\Omega,S}(f_1)^p(u)du \le \int_{\mathbb{R}^n} \mu^{\rho}_{\Omega,S}(f_1)^p(u)du \le C_p \int_{\mathbb{R}^n} |f_1(u)|^p du \le C_p \int_{8Q^*} |f(u)|^p du,$$

so

$$\frac{1}{|Q|}\int_Q \mu^{\rho}_{\Omega,S}(f_1)(u)du \le \left(\frac{1}{Q}\int_Q \mu^{\rho}_{\Omega,S}(f_1)^p(u)du\right)^{1/p} \le C_p M_p f(x). \tag{2.1}$$

In Q, we can find a point $x_0 \in Q$ such that $\mu^{\rho}_{\Omega,S}(f_2)(x_0) < \infty$. In fact, since $f \in L^p$, and $\mu^{\rho}_{\Omega,S}(f)$ is L^p bounded, so

$$\int_Q |\mu^{\rho}_{\Omega,S}(f_2)(u)|^p du \le \int_{R^n} |\mu^{\rho}_{\Omega,S}(f_2)(u)|^p du \le C\int_{R^n} |f_2(u)|^p du \le C\int_{R^n} |f(u)|^p du.$$

This shows that $\mu^{\rho}_{\Omega,S}(f_2)(u) < \infty$ a.e. on Q, so except a subset E with measure zero, for all $u \in Q\backslash E$, $\mu^{\rho}_{\Omega,S}(f_2)(u) < \infty$. Hence we can take $x_0 \in Q\backslash E$.

On the other hand, by (2.1) we get $\mu^{\rho}_{\Omega,S}(f_1)(u) < \infty$ a.e. on Q. Given any point $v \in Q\backslash E$, We now consider $I = |\mu^{\rho}_{\Omega,S}(f_2)(x_0) - \mu^{\rho}_{\Omega,S}(f_2)(v)|$. Since

$$I = \big|\|F(f_2)(x_0,\cdot,\cdot)\|_{\mathcal{H}_1} - \|F(f_2)(v,\cdot,\cdot)\|_{\mathcal{H}_1}\big| \le \|F(f_2)(x_0,\cdot,\cdot) - F(f_2)(v,\cdot,\cdot)\|_{\mathcal{H}_1},$$

where $F(f)(x,y,t) = \int_{\mathbb{R}^n} t^{-n}\phi(\frac{x-z}{t}-y)f(z)dz$ and $\varphi(x) = \frac{\Omega(x)}{|x|^{n-\rho}}\chi_{\{|x|<1\}}$, we have

$$\begin{aligned}
I &\le \left(\int_0^\infty \int_{|y|<1} \left|\int t^{-n}\big(\varphi(\frac{x_0-z}{t}-y) - \varphi(\frac{v-z}{t}-y)\big) f_2(z)dz\right|^2 \frac{dydt}{t}\right)^{1/2} \\
&\le \left(\int_0^\infty \int_{|y|<1} \left|\int_{\substack{|\frac{x_0-z}{t}-y|<1 \\ |\frac{v-z}{t}-y|>1}} t^{-n}\varphi(\frac{x_0-z}{t}-y) f_2(z)dz\right|^2 \frac{dydt}{t}\right)^{1/2} \\
&\quad + \left(\int_0^\infty \int_{|y|<1} \left|\int_{\substack{|\frac{x_0-z}{t}-y|>1 \\ |\frac{v-z}{t}-y|<1}} t^{-n}\varphi(\frac{v-z}{t}-y) f_2(z)dz\right|^2 \frac{dydt}{t}\right)^{1/2} \\
&\quad + \left(\int_0^\infty \int_{|y|<1} \left|\int_{\substack{|\frac{x_0-z}{t}-y|<1 \\ |\frac{v-z}{t}-y|<1}} t^{-n}\big(\varphi(\frac{x_0-z}{t}-y) - \varphi(\frac{v-z}{t}-y)\big) f_2(z)dz\right|^2 \frac{dydt}{t}\right)^{1/2}.
\end{aligned}$$

Using the transform $y \to \frac{x_0-y\prime}{t}$ (we still use y instead y'), then

$$I \le \Big(\int_0^\infty \int_{|x_0-y|<t} \Big| \int_{\substack{|y-z|<t \\ |v-x_0+y-z|>t}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f_2(z)dz \Big|^2 \frac{dydt}{t^{n+2\rho+1}} \Big)^{1/2}$$

$$(2.2) \qquad + \Big(\int_0^\infty \int_{|x_0-y|<t} \Big| \int_{\substack{|y-z|>t \\ |v-x_0+y-z|<t}} \frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}} f_2(z)dz \Big|^2 \frac{dydt}{t^{n+2\rho+1}} \Big)^{1/2}$$

$$\Big(\int_0^\infty \int_{|x_0-y|<t} \Big| \int_{\substack{|y-z|<t \\ |v-x_0 y-z|<t}} \Big(\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(v-x_0 y-z)}{|v-x_0 y-z|^{n-\rho}} \Big) f_2(z)dz \Big|^2 \frac{dydt}{t^{n+2\rho+1}} \Big)^{1/2}$$

$$:= I_1 + I_2 + I_3.$$

As for I_1, by the Minkowski inequality we get

$$(2.3) \qquad I_1 \le C \int_{(8Q^*)^c} |f(z)| \Big[\Big(\iint_{\substack{y\in 2Q^* \\ |y-z|<t \\ |x_0-y|<t \\ |v-x_0+y-z|>t}} + \iint_{\substack{y\in (2Q^*)^c \\ |y-z|<t \\ |x_0-y|<t \\ |v-x_0+y-z|>t}} \Big) \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \Big]^{1/2} dz \le I_{1.1} + I_{1.2},$$

where

$$I_{1.1} = C \int_{(8Q^*)^c} |f(z)| \Big(\iint_{\substack{y\in 2Q^* \\ |y-z|<t \\ |x_0-y|<t \\ |v-x_0+y-z|>t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \Big)^{1/2} dz$$

and

$$I_{1.2} = C \int_{(8Q^*)^c} |f(z)| \Big(\iint_{\substack{y\in (2Q^*)^c \\ |y-z|<t \\ |x_0-y|<t \\ |v-x_0+y-z|>t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \Big)^{1/2} dz.$$

As for $I_{1.1}$, take $0 < \varepsilon < \min\{1/2, (\lambda-2)n/2, \rho-n/2, \sigma-1\}$(we always restrict ε satisfies this in the whole section). Since $y \in 2Q^*$, $z \in (8Q^*)^c$, $|y-z| \sim |x_0-z| \sim |v-x_0+y-z|$, so

$$\begin{aligned} & I_{1.1} \\ & \le C \int_{(8Q^*)^c} |f(z)| \Big(\int_{y\in 2Q^*} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \int_{|y-z|<t<|v-x_0+y-z|} \frac{dt}{t^{n+2\rho+1}} dy \Big)^{1/2} dz \\ & \le C \int_{(8Q^*)^c} |f(z)| \Big(\int_{y\in 2Q^*} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \Big| \frac{1}{|v-x_0 y-z|^{n+2\rho}} - \frac{1}{|y-z|^{n+2\rho}} \Big| dy \Big)^{1/2} dz \end{aligned}$$

$$
\begin{aligned}
(2.4)\qquad &\le C\int_{(8Q^*)^c}|f(z)|\Big(\int_{y\in 2Q^*}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{r}{|y-z|^{2\rho-n+1-2\varepsilon}}\frac{1}{|z-x_0|^{2n+2\varepsilon}}dy\Big)^{1/2}dz\\
&\le C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^{n\varepsilon}}\Big(\int_{y\in 2Q^*}\frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}}dy\Big)^{1/2}dz\\
&\le C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^{n\varepsilon}}\Big(\int_{|y-z|>6r}\frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}}dy\Big)^{1/2}dz\\
&\le Cr^{\varepsilon}\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}dz\\
&\le Cr^{\varepsilon}\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x|^{n+\varepsilon}}dz\le C_pM_p(f)(x).
\end{aligned}
$$

Now we give the estimate of $I_{1.2}$.

$$
\begin{aligned}
I_{1.2}\le C&\int_{(8Q^*)^c}|f(z)|\Big(\iint_{\substack{y\in(2Q^*)^c,|y-z|<t,\\ 2|y-z|\ge|z-x_0|,|x_0-y|<t\\ |v-x_0+y-z|>t}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}dz\\
&+C\int_{(8Q^*)^c}|f(z)|\Big(\iint_{\substack{y\in(2Q^*)^c,\ |y-z|<t,\\ 2|y-z|<|z-x_0|,\ |x_0-y|<t\\ \|v-x_0+y-z|>t}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}dz\\
&:=I_{1.2'}+I_{1.2''}.
\end{aligned}
$$

First we give the estimate of $I_{1.2'}$.

$$
\begin{aligned}
I_{1.2'}\ &\le C\int_{(8Q^*)^c}|f(z)|\Big(\int_{\substack{y\in(2Q^*)^c\\ 2|y-z|\ge|z-x_0|}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\int_{|y-z|<t<|v-x_0+y-z|}\frac{dt}{t^{n+2\rho+1}}dy\Big)^{1/2}dz\\
&\le C\int_{(8Q^*)^c}|f(z)|\Big(\int_{\substack{y\in(2Q^*)^c\\ 2|y-z|\ge\ |z-x_0|}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{r}{|y-z|^{n+2\rho+1}}dy\Big)^{1/2}dz\\
(2.5)\qquad &\le C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}\Big(\int_{\substack{y\in(2Q^*)^c\\ 2|y-z|\ge|z-x_0|}}\frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}}dy\Big)^{1/2}dz\\
&\le C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}\Big(\int_{2|y-z|\ge|z-x_0|>4r}\frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}}dy\Big)^{1/2}dz\\
&\le C_pM_p(f)(x).
\end{aligned}
$$

The estimate of $I_{1.2''}$ is more complicated.

$$
\begin{aligned}
I_{1.2''}\le C&\int_{(8Q^*)^c}|f(z)|\Big(\iint_{\substack{y\in(2Q^*)^c,|y-z|<t,\\ 2|y-z|<|z-x_0|,\ |x_0-y|<t\\ |v-x_0+y-z|>t,|y-z|<2r}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}dz\\
(2.6)\qquad &+C\int_{(8Q^*)^c}|f(z)|\Big(\iint_{\substack{y\in(2Q^*)^c,\ |y-z|<t,\\ 2|y-z|<|z-x_0|,\ |x_0-y|<t\\ |v-x_0+y-z|>t,|y-z|\ge 2r}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}dz\\
&:=I^1_{1.2''}+I^2_{1.2''}.
\end{aligned}
$$

For $I^1_{1.2''}$, since $|y-x_0| \geq |z-x_0| - |y-z| > |z-x_0| - 2r$, so $\frac{1}{|y-x_0|} < \frac{1}{|z-x_0|-2r}$ and

$$\begin{aligned}
I^1_{1.2''} &\leq C\int_{(8Q^*)^c}|f(z)|\Bigg(\int_{\substack{|z-x_0|-2r<|y-x_0|\\|y-z|<2r}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\int_{|y-x_0|}^{\infty}\frac{dt}{t^{n+2\rho+1}}dy\Bigg)^{1/2}dz\\
&\leq C\int_{(8Q^*)^c}\frac{|f(z)|}{(|z-x_0|-2r)^{n/2+\rho}}\Bigg(\int_{|y-z|<2r}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}dy\Bigg)^{1/2}dz\\
&\leq Cr^{\rho-n/2}\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^{n/2+\rho}}dz\\
&\leq C_pM_p(f)(x).
\end{aligned}\tag{2.7}$$

For $I^2_{1.2''}$, note that $t > |y-x_0| > |z-x_0| - |y-z| > |z-x_0|/2$ and $|y-z| \sim |v-x_0+y-z|$ so

$$\begin{aligned}
I^2_{1.2''} &\leq C\int_{(8Q^*)^c}|f(z)|\Bigg(\int_{\substack{y\in(2Q^*)^c\\2|y-z|<|z-x_0|\\|y-z|\geq 2r}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\int_{|y-z|<t<|v-x_0+y-z|}\\
&\qquad\frac{dt}{t^{2\rho-n+1-2\varepsilon}}\times\frac{1}{(|z-x_0|/2)^{2n+2\varepsilon}}dy\Bigg)^{1/2}dz\\
&\leq C\int_{(8Q^*)^c}\frac{|f(z)|}{(|z-x_0|/2)^{n+\varepsilon}}\\
&\qquad\Bigg(\int_{|y-z|\geq 2r}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{r}{|y-z|^{2\rho-n+1-2\varepsilon}}dy\Bigg)^{1/2}dz\\
&\leq C_pM_p(f)(x).
\end{aligned}\tag{2.8}$$

Similarly as we deal with I_1, we can obtain $I_2 \leq C_pM_p(f)(x)$.

So we only need to give the estimate of I_3. Apply the Minkowski inequality to I_3 and divide the region by $|y-z| \geq 8r$,$|y-z| < 8r$, we get

$$\begin{aligned}
I_3 &\leq C\int_{(8Q^*)^c}|f(z)|\Bigg(\iint_{\substack{y\in(2Q^*)^c,\ |y-z|<t\\|y-x_0|<t,|v-x_0+y-z|<t\\|y-z|<8r}}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\\
&\qquad-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2}dz\\
&\qquad+C\int_{(8Q^*)^c}|f(z)|\Bigg(\iint_{\substack{y\in(2Q^*)^c,\ |y-z|<t\\|y-x_0|<t,|v-x_0+y-z|<t\\|y-z|\geq 8r}}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}\\
&\qquad-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2}dz\\
&:= I_{3,1}+I_{3,2}.
\end{aligned}\tag{2.9}$$

It is easy to see that when $z \in (8Q^*)^c$, $|y-z| < 8r$, $|v-x_0+y-z| \leq |v-x_0|+8r \leq$

$9r$ and $|y-x_0| \sim |z-x_0|$. Then

$$I_{3,1} \le C\int_{(8Q^*)^c}|f(z)|\Bigg(\iint_{\substack{y\in(2Q^*)^c,\quad |y-z|<t\\ |y-x_0|<t,\, |y-z|<8r\\ |v-x_0+y-z|<t}}\Bigg(\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}+\frac{|\Omega(v-x_0+y-z)|^2}{|v-x_0+y-z|^{2n-2\rho}}\Bigg)$$

$$\times\frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2}dz$$

$$\le C\int_{(8Q^*)^c}|f(z)|\Bigg(\int_{\substack{y\in(2Q^*)^c,\, |y-z|<8r\\ |v-x_0+y-z|<9r}}\Bigg(\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}+\frac{|\Omega(v-x_0+y-z)|^2}{|v-x_0+y-z|^{2n-2\rho}}\Bigg)$$

$$\times\int_{|y-x_0|}^{\infty}\frac{dt}{t^{n+2\rho+1}}dy\Bigg)^{1/2}dz$$

$$\le C\int_{(8Q^*)^c}|f(z)|\Bigg(\int_{\substack{y\in(2Q^*)^c,\, |y-z|<8r\\ |v-x_0+y-z|<9r}}\Bigg(\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}+\frac{|\Omega(v-x_0+y-z)|^2}{|v-x_0+y-z|^{2n-2\rho}}\Bigg) \tag{2.10}$$

$$\times\frac{1}{|z-x_0|^{n+2\rho}}dy\Bigg)^{1/2}dz$$

$$\le C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^{n/2+\rho}}\Bigg(\int_{|y-z|<8r}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}dy\Bigg)^{1/2}dz$$

$$+C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^{n/2+\rho}}\Bigg(\int_{|v-x_0+y-z|<9r}\frac{|\Omega(v-x_0+y-z)|^2}{|v-x_0+y-z|^{2n-2\rho}}dy\Bigg)^{1/2}dz$$

$$\le C_pM_p(f)(x).$$

Now, we will give the estimate of $I_{3.2}$.

Note that $|z-x_0|<|x_0-y|+|y-z|<2t$, so $t>|z-x_0|/2$. Since $|y-z|/r>8$, Integration by part, one can easily get

$$\int_{|y-z|}^{\infty}\frac{(\log\frac{t}{r})^{2+2\varepsilon}}{t^{2\rho-n+1}}dt\le C\frac{(\log\frac{|y-z|}{r})^{2+2\varepsilon}}{|y-z|^{2\rho-n}}.$$

Then by Lemma 2.1, we have

$$I_{3.2}\le C\int_{(8Q^*)^c}|f(z)|\Bigg(\iint_{\substack{y\in(2Q^*)^c,\quad |y-z|<t\\ |y-x_0|<t,\,|v-x_0+y-z|<t\\ |y-z|\ge 8r,\,t>|z-x_0|/2}}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2$$

$$\times\frac{(\log\frac{t}{r})^{2+2\varepsilon}dt}{t^{2\rho-n+1}t^{2n}(\log\frac{t}{r})^{2+2\varepsilon}}dy\Bigg)^{1/2}dz$$

$$\le C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^n(\log\frac{|z-x_0|}{r})^{1+\varepsilon}}\Bigg(\iint_{\substack{y\in(2Q^*)^c,\quad |y-z|<t\\ |y-x_0|<t,\,|v-x_0+y-z|<t\\ |y-z|\ge 8r,\,t>|z-x_0|/2}}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$

$$-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2\frac{(\log\frac{t}{r})^{2+2\varepsilon}dt}{t^{2\rho-n+1}}dy\Bigg)^{1/2}dz$$

$$\le C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^n(\log\frac{|z-x_0|}{r})^{1+\varepsilon}}\Bigg(\int_{|y-z|\ge 8r}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$

$$- \frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2 \Big(\int_{|y-z|<t} \frac{(\log \frac{t}{r})^{2+2\varepsilon}}{t^{2\rho-n+1}} dt \Big) dy \Big)^{1/2} dz$$

$$\leq C \int_{(8Q^*)^c} \frac{|f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{r})^{1+\varepsilon}} \Big(\int_{|y-z|\geq 8r} \Big| \frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$

$$- \frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2 \frac{(\log \frac{|y-z|}{r})^{2+2\varepsilon}}{|y-z|^{2\rho-n}} dy \Big)^{1/2} dz$$

$$\leq C \int_{(8Q^*)^c} \frac{|f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{r})^{1+\varepsilon}} \Big(\sum_{j=3}^{\infty} \int_{2^j r\leq |y-z|<2^{j+1}r} \Big| \frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$

$$- \frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2 \frac{(\log \frac{|y-z|}{r})^{2+2\varepsilon}}{|y-z|^{2\rho-n}} dy \Big)^{1/2} dz$$

$$\leq C \int_{(8Q^*)^c} \frac{|f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{r})^{1+\varepsilon}} \sum_{j=3}^{\infty} \frac{(\log \frac{2^{j+1}r}{r})^{1+\varepsilon}}{(2^j r)^{\rho-n/2}} \Big(\int_{2^j r\leq |y-z|<2^{j+1}r} \Big| \frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$

$$- \frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2 dy \Big)^{1/2} dz$$

$$\leq C \int_{(8Q^*)^c} \frac{|f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{r})^{1+\varepsilon}} \sum_{j=3}^{\infty} \frac{(j+1)^{1+\varepsilon}}{(2^j r)^{\rho-n/2}} (2^j r)^{n/2-(n-\rho)}$$

$$\Big\{ \frac{|v-x_0|}{2^j r} + \int_{\frac{|v-x_0|}{2^{j+1}r}}^{\frac{|v-x_0|}{2^j r}} \frac{\omega_2(\delta)}{\delta} d\delta \Big\} dz$$

$$\leq C \int_{(8Q^*)^c} \frac{|f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{r})^{1+\varepsilon}} \sum_{j=3}^{\infty} (j+1)^{1+\varepsilon}$$

$$\Big\{ \frac{1}{2^j} + \frac{1}{(1+j\log 2)^{\sigma}} \int_{\frac{|v-x_0|}{2^{j+1}r}}^{\frac{|v-x_0|}{2^j r}} \frac{\omega_2(\delta)}{\delta} (1+\log\delta)^{\sigma} d\delta \Big\} dz$$

$$\leq C \int_{(8Q^*)^c} \frac{|f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{r})^{1+\varepsilon}} dz.$$

Note that

$$\int_{(8Q^*)^c} \frac{|f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{r})^{1+\varepsilon}} dz$$

$$\leq \sum_{k=3}^{\infty} \int_{2^k r\leq |z-x_0|<2^{k+1}r} \frac{|f(z)|}{|z-x_0|^n (\log \frac{|z-x_0|}{r})^{1+\varepsilon}} dz$$

$$\leq C \sum_{k=3}^{\infty} \frac{1}{(k\log 2)^{1+\varepsilon}} \frac{1}{(2^{k+1}r)^n} \int_{|z-x_0|<2^{k+1}r} |f(z)| dz$$

$$\leq C \sum_{k=3}^{\infty} \frac{1}{(k\log 2)^{1+\varepsilon}} \frac{1}{(2^{k+1}r)^n} \int_{|z-x|<2^{k+2}r} |f(z)| dz$$

$$\leq C_p M_p(f)(x),$$

we get

$$I_{3.2} \le C_p M_p(f)(x). \tag{2.11}$$

Hence add up (2.2)-(2.11), we obtain

$$|\mu^{\rho}_{\Omega,S}(f_2)(x_0) - \mu^{\rho}_{\Omega,S}(f_2)(v)| < C_p M_p(f)(x) \ \text{ for all } x \in \mathbb{R}^n \ ,$$

therefore

$$\begin{aligned} &\frac{1}{|Q|}\int_Q \big|\mu^{\rho}_{\Omega,S}(f_2)(x_0) - \mu^{\rho}_{\Omega,S}(f_2)(v)\big| dv \\ &\quad = \frac{1}{|Q|}\int_{Q\setminus E} \big|\mu^{\rho}_{\Omega,S}(f_2)(x_0) - \mu^{\rho}_{\Omega,S}(f_2)(v)\big| dv \le C_p M_p(f)(x). \end{aligned} \tag{2.12}$$

For any $x \in Q\setminus E$, we have

$$\begin{aligned} &\big|\mu^{\rho}_{\Omega,S}(f_1+f_2)(v) - \mu^{\rho}_{\Omega,S}(f_2)(x_0)\big| \\ &\quad \le \big|\mu^{\rho}_{\Omega,S}(f_1+f_2)(v) - \mu^{\rho}_{\Omega,S}(f_2)(v)\big| + \big|\mu^{\rho}_{\Omega,S}(f_2)(v) - \mu^{\rho}_{\Omega,S}(f_2)(x_0)\big| \\ &\quad = \big|\|F(f_1+f_2)(v,\cdot,\cdot)\|_{\mathcal{H}_1} - \|F(f_2)(v,\cdot,\cdot)\|_{\mathcal{H}_1}\big| + \big|\mu^{\rho}_{\Omega,S}(f_2)(v) - \mu^{\rho}_{\Omega,S}(f_2)(x_0)\big| \\ &\quad \le \|F(f_1+f_2)(v,\cdot,\cdot) - F(f_2)(v,\cdot,\cdot)\|_{\mathcal{H}_1} + \big|\mu^{\rho}_{\Omega,S}(f_2)(v) - \mu^{\rho}_{\Omega,S}(f_2)(x_0)\big| \\ &\quad = \mu^{\rho}_{\Omega,S}(f_1)(v) + \big|\mu^{\rho}_{\Omega,S}(f_2)(v) - \mu^{\rho}_{\Omega,S}(f_2)(x_0)\big|. \end{aligned}$$

Finally, by (2.1) and (2.12) and the above inequality yields

$$\begin{aligned} &\frac{1}{|Q|}\int_Q |\mu^{\rho}_{\Omega,S}(f)(v) - \mu^{\rho}_{\Omega,S}(f_2)(x_0)| dv \\ &\quad \le \frac{1}{|Q|}\int_Q \mu^{\rho}_{\Omega,S}(f_1)(v) dv + \frac{1}{|Q|}\int_{Q\setminus E} |\mu^{\rho}_{\Omega,S}(f_2)(v) - \mu^{\rho}_{\Omega,S}(f_2)(x_0)| dv \\ &\quad \le C_p M_p(f)(x). \end{aligned}$$

Recall that $M^{\sharp}$ is defined by

$$M^{\sharp}(f)(x) = \sup_{x\in Q} \frac{1}{Q}\int_Q |f(y) - f_Q| dy \approx \sup_{x\in Q} \inf_c \frac{1}{Q}\int_Q |f(y) - c| dy.$$

So we just take $c = \mu^{\rho}_{\Omega,S}(f_2)(x_0)$ and (1.5) follows from the above inequality.

Below we will give the proof of (1.6) for $\mu^{*,\rho}_{\lambda}$. Given $x \in \mathbb{R}^n$, let $Q, \bar{x}, r_0$,Q^*, r be the same as before, also set

$$f = f\chi_{8Q^*} + f(1-\chi_{8Q^*}) =: f_1 + f_2.$$

Then using the L^p-boundness of $\mu_\lambda^{*,\rho}(1<p<\infty)$, we have

$$\int_Q \mu_\lambda^{*,\rho}(f_1)^p(u)du \le \int_{\mathbb{R}^n} \mu_\lambda^{*,\rho}(f_1)^p(u)du \le C_p \int_{\mathbb{R}^n} |f_1(u)|^p du \le C_p \int_{8Q^*} |f(u)|^p du.$$

So

$$\frac{1}{|Q|}\int_Q \mu_\lambda^{*,\rho}(f_1)(u)du \le \left(\frac{1}{Q}\int_Q \mu_\lambda^{*,\rho}(f_1)^p(u)du\right)^{1/p} \le C_p M_p f(x). \tag{2.13}$$

By the same reason as we show in the beginning of the Proof for $\mu_{\Omega,S}^\rho$, there exists a measurable set E with measure zero such that $\mu_\lambda^{*,\rho}(f_2)(x)<\infty$ for any $x\in Q\backslash E$. Now we fixed one point $x_0\in Q\backslash E$ and for any $v\in Q\backslash E$, we consider $J=|\mu_\lambda^{*,\rho}(f_2)(x_0)-\mu_\lambda^{*,\rho}(f_2)(v)|$. Since

$$J=\big|\|F(f_2)(x_0,\cdot,\cdot)\|_{\mathcal{H}_2}-\|F(f_2)(v,\cdot,\cdot)\|_{\mathcal{H}_2}\big| \le \|F(f_2)(x_0,\cdot,\cdot)-F(f_2)(v,\cdot,\cdot)\|_{\mathcal{H}_2},$$

we have

$$\begin{aligned} J \le \Big(\int_0^\infty \big(\int_{|y|<1}+\int_{|y|\ge 1}\big)\big(\frac{1}{1+|y|}\big)^{\lambda n}\Big|\int t^{-n}\big[\varphi(\frac{x_0-z}{t}-y)\\ -\varphi(\frac{v-z}{t}-y)\big]f_2(z)dz\Big|^2\frac{dydt}{t}\Big)^{1/2} \le J_1+J_2, \end{aligned} \tag{2.14}$$

where

$$J_1=\Big(\int_0^\infty\int_{|y|<1}\big(\frac{1}{1+|y|}\big)^{\lambda n}\Big|\int t^{-n}\big[\varphi(\frac{x_0-z}{t}-y)-\varphi(\frac{v-z}{t}-y)\big]f_2(z)dz\Big|^2\frac{dydt}{t}\Big)^{1/2}$$

and

$$J_2=\Big(\int_0^\infty\int_{|y|\ge 1}\big(\frac{1}{1+|y|}\big)^{\lambda n}\Big|\int t^{-n}\big[\varphi(\frac{x_0-z}{t}-y)-\varphi(\frac{v-z}{t}-y)\big]f_2(z)dz\Big|^2\frac{dydt}{t}\Big)^{1/2}.$$

Since $\big(\frac{1}{1+|y|}\big)^{\lambda n}\le 1$, then $J_1\le I_1+I_2+I_3$, by the proof for the operator $\mu_{\Omega,S}^\rho$ before, we get

$$J_1 \le C_p M_p(f)(x).$$

$$\begin{aligned} J_2 \le &\Big(\int_0^\infty\int_{|y|\ge 1}\big(\frac{1}{1+|y|}\big)^{\lambda n}\Big|\int_{\substack{|\frac{x_0-z}{t}-y|<1\\ |\frac{v-z}{t}-y|>1}} t^{-n}\varphi(\frac{x_0-z}{t}-y)f_2(z)dz\Big|^2\frac{dydt}{t}\Big)^{1/2}\\ &+\Big(\int_0^\infty\int_{|y|\ge 1}\big(\frac{1}{1+|y|}\big)^{\lambda n}\Big|\int_{\substack{|\frac{x_0-z}{t}-y|>1\\ |\frac{v-z}{t}-y|<1}} t^{-n}\varphi(\frac{v-z}{t}-y)f_2(z)dz\Big|^2\frac{dydt}{t}\Big)^{1/2}\\ &+\Big(\int_0^\infty\int_{|y|\ge 1}\big(\frac{1}{1+|y|}\big)^{\lambda n}\Big|\int_{\substack{|\frac{x_0-z}{t}-y|<1\\ |\frac{v-z}{t}-y|<1}} t^{-n}\big[\varphi(\frac{x_0-z}{t}-y)-\varphi(\frac{v-z}{t}-y)\big]\\ &\times f_2(z)dz\Big|^2\frac{dydt}{t}\Big)^{1/2}. \end{aligned} \tag{2.15}$$

Using the transform $y \to \frac{x_0-y'}{t}$ again (we still use y instead y'), we have

$$
\begin{aligned}
J_2 \le & \Big(\int_0^\infty \int_{|x_0-y|\ge t} (\frac{t}{t+|x_0-y|})^{\lambda n} \\
& \Big|\int_{\substack{|y-z|<t\\ |v-x_0+y-z|>t}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f_2(z)dz\Big|^2 \frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2} \\
& + \Big(\int_0^\infty \int_{|x_0-y|\ge t} (\frac{t}{t+|x_0-y|})^{\lambda n} \\
& \Big|\int_{\substack{|y-z|>t\\ |v-x_0+y-z|<t}} \frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}} f_2(z)dz\Big|^2 \frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2} \\
& + \Big(\int_0^\infty \int_{|x_0-y|\ge t} (\frac{t}{t+|x_0-y|})^{\lambda n} \\
& \Big|\int_{\substack{|y-z|<t\\ |v-x_0+y-z|<t}} (\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}) \\
& \times f_2(z)dz\Big|^2 \frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2} \\
:= & L_1 + L_2 + L_3.
\end{aligned} \tag{2.16}
$$

Now we consider L_1, we claim that $y \in (2Q^*)^c$, otherwise if $y \in 2Q^*$ then $t \le |x_0-y| < 4r$, but $z \in (8Q^*)^c, t > |y-z| \ge 6r$. Thus by the Minskowski inequality we have

$$
\begin{aligned}
L_1 \le & \Big(\int_0^\infty \int_{\substack{|x_0-y|\ge t\\ y\in(2Q^*)^c}} (\frac{t}{t+|x_0-y|})^{\lambda n} \\
& \Big|\int_{\substack{|y-z|<t\\ |v-x_0+y-z|>t}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f_2(z)dz\Big|^2 \frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2} \\
\le & C\int_{(8Q^*)^c} |f(z)| \Big(\int_0^\infty \int_{\substack{|x_0-y|\ge t, y\in(2Q^*)^c\\ |y-z|<t, |v-x_0+y-z|>t}} \\
& \Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2} dz \\
\le & L_{1.1} + L_{1.2},
\end{aligned} \tag{2.17}
$$

where

$$
\begin{aligned}
L_{1.1} = & C\int_{(8Q^*)^c} |f(z)| \Big(\int_0^\infty \int_{\substack{|x_0-y|\ge t, y\in(2Q^*)^c\\ |y-z|<8r, |y-z|<t\\ |v-x_0+y-z|>t}} \\
& \Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2} dz
\end{aligned}
$$

and

$$
L_{1.2} = C\int_{(8Q^*)^c} |f(z)| \Big(\int_0^\infty \int_{\substack{|x_0-y|\ge t, y\in(2Q^*)^c\\ |y-z|\ge 8r, |y-z|<t\\ |v-x_0+y-z|>t}}
$$

$$\left(\frac{t}{t+|x_0-y|}\right)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2}dz.$$

First we give the estimate of $L_{1.1}$. Since $|y-z|<8r, z\in(8Q^*)^c$, then $|y-x_0|\sim|z-x_0|$ and

$$\begin{aligned}
L_{1.1} &\le C\int_{(8Q^*)^c}|f(z)|\Bigg(\int_0^\infty\int_{\substack{|x_0-y|\ge t\\ y\in(2Q^*)^c\\ |y-z|<8r,|y-z|<t}}\Big(\frac{1}{t+|x_0-y|}\Big)^{2n+2\varepsilon}\\
&\quad \frac{t^{\lambda n-2n-2\varepsilon}t^{2n+2\varepsilon}}{(t+|x_0-y|)^{\lambda n-2n-2\varepsilon}}\times\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2}dz\\
&\le C\int_{(8Q^*)^c}|f(z)|\Bigg(\int_0^\infty\int_{\substack{|x_0-y|\ge t\\ y\in(2Q^*)^c\\ |y-z|<8r,|y-z|<t}}\frac{1}{|z-x_0|^{2n+2\varepsilon}}\frac{|\Omega(y-z)|^2}{|y-z|^{n-\varepsilon}}\frac{dydt}{t^{1-\varepsilon}}\Bigg)^{1/2}dz \qquad (2.18)\\
&\le C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon/2}}\Bigg(\int_{|y-z|<8r}\frac{|\Omega(y-z)|^2}{|z-x_0|^{\varepsilon}|y-z|^{n-\varepsilon}}\int_0^{|x_0-y|}\frac{1}{t^{1-\varepsilon}}dt\,dy\Bigg)^{1/2}dz\\
&\le C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon/2}}r^{\varepsilon/2}dz\le C_pM_p(f)(x).
\end{aligned}$$

As for $L_{1.2}$,

$$\begin{aligned}
L_{1.2} &\le C\int_{(8Q^*)^c}|f(z)|\Bigg(\int_0^\infty\int_{\substack{|x_0-y|\ge t,y\in(2Q^*)^c\\ |y-z|\ge 8r,|y-z|<t\\ |v-x_0+y-z|>t}}\\
&\quad \Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2}dz\\
&\le C\int_{(8Q^*)^c}|f(z)|\Bigg(\int_0^\infty\int_{\substack{|x_0-y|\ge t,y\in(2Q^*)^c\\ |y-z|\ge 8r,|y-z|<t\\ |v-x_0+y-z|>t,2|y-z|\ge|z-x_0|}}\\
&\quad \Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2}dz \qquad (2.19)\\
&\quad +C\int_{(8Q^*)^c}|f(z)|\Bigg(\int_0^\infty\int_{\substack{|x_0-y|\ge t,y\in(2Q^*)^c\\ |y-z|\ge 8r,|y-z|<t}\ |v-x_0+y-z|>t,2|y-z|<|z-x_0|}\\
&\quad \Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2}dz\\
&:= L_{1.2'}+L_{1.2''},
\end{aligned}$$

while

$$
\begin{aligned}
L_{1.2'} &\le C\int_{(8Q^*)^c}|f(z)|\Bigg(\int_{\substack{y\in(2Q^*)^c\\|y-z|\ge 8r\\2|y-z|\ge|z-x_0|}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\\
&\qquad\int_{|y-z|<t<|v-x_0+y-z|}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\times\frac{dt}{t^{n+2\rho+1}}dy\Bigg)^{1/2}dz\\
&\le C\int_{(8Q^*)^c}|f(z)|\Bigg(\int_{\substack{y\in(2Q^*)^c\\|y-z|\ge 8r\\2|y-z|\ge|z-x_0|}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\\
&\qquad\int_{|y-z|<t<|v-x_0+y-z|}\frac{dt}{t^{n+2\rho+1}}dy\Bigg)^{1/2}dz\\
&\le C\int_{(8Q^*)^c}|f(z)|\Bigg(\int_{\substack{y\in(2Q^*)^c\\|y-z|\ge 8r\\2|y-z|\ge|z-x_0|}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\\
&\qquad\left|\frac{1}{|y-z|^{n+2\rho}}-\frac{1}{|v-x_0+y-z|^{n+2\rho}}\right|dy\Bigg)^{1/2}dz \qquad (2.20)\\
&\le C\int_{(8Q^*)^c}|f(z)|\Bigg(\int_{\substack{y\in(2Q^*)^c\\|y-z|\ge 8r\\2|y-z|\ge|z-x_0|}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{r}{|y-z|^{n+2\rho+1}}dy\Bigg)^{1/2}dz\\
&\le C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}\Bigg(\int_{|y-z|\ge 8r}\frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}}dy\Bigg)^{1/2}dz\\
&\le C_p r^{\varepsilon}\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}dz\le C_pM_p(f)(x).
\end{aligned}
$$

Since $|x_0-y|\ge|z-x_0|-|y-z|>|z-x_0|/2$ and $|y-z|\sim|v-x_0+y-z|$, we have

$$
\begin{aligned}
L_{1.2''} &\le C\int_{(8Q^*)^c}|f(z)|\Bigg(\iint_{\substack{y\in(2Q^*)^c\\|y-z|\ge 8r\\|y-z|<t\\|x_0-y|>|z-x_0|/2\\|v-x_0+y-z|>t}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n-2n-2\varepsilon}\\
&\qquad\frac{t^{2n+2\varepsilon}}{(|z-x_0|/2)^{2n+2\varepsilon}}\times\frac{dydt}{t^{n+2\rho+1}}\Bigg)^{1/2}dz\\
&\le C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}\Bigg(\iint_{\substack{y\in(2Q^*)^c\\|y-z|\ge 8r\\|y-z|<t\\|v-x_0+y-z|>t}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{t^{2n+2\varepsilon}}{t^{n+2\rho+1}}dydt\Bigg)^{1/2}dz \qquad (2.21)\\
&\le C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}\Bigg(\int_{\substack{y\in(2Q^*)^c\\|y-z|\ge 8r}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\Big(\int_{|y-z|}^{|v-x_0+y-z|}\frac{dt}{t^{2\rho-n-2\varepsilon+1}}\Big)dy\Bigg)^{1/2}dz\\
&\le C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}\Bigg(\int_{\substack{y\in(2Q^*)^c\\|y-z|\ge 8r}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\Big|\frac{1}{|y-z|^{2\rho-n-2\varepsilon}}\\
&\qquad-\frac{1}{|v-x_0+y-z|^{2\rho-n-2\varepsilon}}\Big|\Big)dy\Bigg)^{1/2}dz
\end{aligned}
$$

$$\le C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}\Big(\int_{\substack{y\in(2Q^*)^c\\|y-z|\ge 8r}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{r}{|y-z|^{2\rho-n+1-2\varepsilon}})dy\Big)^{1/2}dz$$
$$\le C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}\Big(\int_{|y-z|\ge 8r}\frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}})dy\Big)^{1/2}dz$$
$$\le C_pM_p(f)(x).$$

The estimate of L_2 is similar as L_1, and we get $L_2\le C_pM_p(f)(x)$. Finally, we deal the last part L_3. By the Minskowski inequality

$$L_3=\Big(\int_0^\infty\int_{|x_0-y|\ge t}(\frac{t}{t+|x_0-y|})^{\lambda n}\Big|\int_{\substack{|y-z|<t\\|v-x_0+y-z|<t}}$$
$$\Big(\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big)\times f_2(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}$$
$$\le C\int_{(8Q^*)^c}|f(z)|\Big(\iint_{\substack{y\in(2Q^*)^c,|y-z|<t\\|x_0-y|\ge t,|v-x_0+y-z|<t}}(\frac{t}{t+|x_0-y|})^{\lambda n}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$
$$-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}dz$$
$$\le C\int_{(8Q^*)^c}|f(z)|\Big(\iint_{\substack{y\in(2Q^*)^c,|y-z|<t\\|x_0-y|\ge t,|v-x_0+y-z|<t\\|y-z|\le 8r}}(\frac{t}{t+|x_0-y|})^{\lambda n}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$
$$-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}dz\tag{2.22}$$
$$+C\int_{(8Q^*)^c}|f(z)|\Big(\iint_{\substack{y\in(2Q^*)^c,|y-z|<t\\|x_0-y|\ge t,|v-x_0+y-z|<t\\|y-z|>8r}}(\frac{t}{t+|x_0-y|})^{\lambda n}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$
$$-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}dz$$
$$:=L_{3.1}+L_{3.2}.$$

Note that when $|y-z|<8r$, then $|v-x_0+y-z|<9r$, so

$$L_{3.1}\le C\int_{(8Q^*)^c}|f(z)|\Big(\iint_{\substack{y\in(2Q^*)^c,|y-z|<t\\|x_0-y|\ge t,|v-x_0+y-z|<t\\|y-z|\le 8r}}(\frac{t}{t+|x_0-y|})^{\lambda n}(\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}$$
$$+\frac{|\Omega(v-x_0+y-z)|^2}{|v-x_0+y-z|^{2n-2\rho}})\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}dz$$
$$\le C\int_{(8Q^*)^c}|f(z)|\Big(\iint_{\substack{y\in(2Q^*)^c\\|x_0-y|\ge t\\|y-z|\le 8r}}(\frac{t}{t+|x_0-y|})^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}dz$$
$$+C\int_{(8Q^*)^c}|f(z)|\Big(\iint_{\substack{y\in(2Q^*)^c\\|x_0-y|\ge t\\|v-x_0+y-z|<9r}}(\frac{t}{t+|x_0-y|})^{\lambda n}\frac{|\Omega(v-x_0+y-z)|^2}{|v-x_0+y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}dz$$
$$:=L_{3.1'}+L_{3.1''}.$$

Using the same methods and steps as we deal $L_{1.1}$, we easily have $L_{3.1'} \leq C_p M_p(f)(x)$, $L_{3.1''} \leq C_p M_p(f)(x)$, thus

$$L_{3.1} \leq C_p M_p(f)(x). \tag{2.23}$$

The estimate of $L_{3.2}$ is more complicate. we also divide the region by $2|y-z| \geq |z-x_0|$ and $2|y-z| < |z-x_0|$, hence

$$\begin{aligned} L_{3.2} \leq {} & C\int_{(8Q^*)^c} |f(z)|\Big(\iint_{\substack{y\in(2Q^*)^c,|y-z|<t\\ |x_0-y|\geq t,|v-x_0+y-z|<t\\ |y-z|>8r,2|y-z|\geq|z-x_0|}} (\frac{t}{t+|x_0-y|})^{\lambda n}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} \\ & -\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}dz \\ & +C\int_{(8Q^*)^c} |f(z)|\Big(\iint_{\substack{y\in(2Q^*)^c,|y-z|<t\\ |x_0-y|\geq t,|v-x_0+y-z|<t\\ |y-z|>8r,2|y-z|<|z-x_0|}} (\frac{t}{t+|x_0-y|})^{\lambda n}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} \\ & -\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}dz \\ := {} & L_{3.2'}+L_{3.2''}. \end{aligned} \tag{2.24}$$

Since $t > |y-z| > |z-x_0|/2$, so we have

$$\begin{aligned} L_{3.2'} \leq {} & C\int_{(8Q^*)^c} |f(z)|\Big(\iint_{\substack{y\in(2Q^*)^c\\ |y-z|>8r\\ t\geq|z-x_0|/2}} (\frac{t}{t+|x_0-y|})^{\lambda n}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2 \\ & \frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}dz \\ \leq {} & C\int_{(8Q^*)^c} |f(z)|\Big(\int_{\substack{y\in(2Q^*)^c\\ |y-z|>8r}} \Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2 \\ & \times\Big(\int_{\max\{|y-z|,|z-x_0|/2\}}^{\infty}\frac{dt}{t^{n+2\rho+1}}\Big)dy\Big)^{1/2}dz \\ \leq {} & C\int_{(8Q^*)^c} |f(z)|\Big(\int_{\substack{y\in(2Q^*)^c\\ |y-z|>8r}} \Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2 \\ & \times\Big(\int_{\max\{|y-z|,|z-x_0|/2\}}^{\infty}\frac{(\log\frac{t}{r})^{2+2\varepsilon}dt}{t^{n+2\rho+1}(\log\frac{t}{r})^{2+2\varepsilon}}\Big)dy\Big)^{1/2}dz \\ \leq {} & C\int_{(8Q^*)^c} |f(z)|\Big(\int_{\substack{y\in(2Q^*)^c\\ |y-z|>8r}} \Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2 \\ & \times\Big(\int_{\max\{|y-z|,|z-x_0|/2\}}^{\infty}\frac{(\log\frac{t}{r})^{2+2\varepsilon}dt}{t^{2\rho-n+1}|z-x_0|^n(\log\frac{|z-x_0|}{r})^{2+2\varepsilon}}\Big)dy\Big)^{1/2}dz \end{aligned}$$

$$\leq C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^n(\log\frac{|z-x_0|}{r})^{1+\varepsilon}}\Big(\int_{\substack{y\in(2Q^*)^c\\|y-z|>8r}}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2$$

$$\times\Big(\int_{\max\{|y-z|,|z-x_0|/2\}}^{\infty}\frac{(\log\frac{t}{r})^{2+2\varepsilon}dt}{t^{2\rho-n+1}}\Big)dy\Big)^{1/2}dz$$

$$\leq C\int_{(8Q^*)^c}\frac{|f(z)|}{|z-x_0|^n(\log\frac{|z-x_0|}{r})^{1+\varepsilon}}\Big(\int_{\substack{y\in(2Q^*)^c\\|y-z|>8r}}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2$$

$$\times\Big(\int_{|y-z|}^{\infty}\frac{(\log\frac{t}{r})^{2+2\varepsilon}dt}{t^{2\rho-n+1}}\Big)dy\Big)^{1/2}dz.$$

By the estimate of $I_{3.2}$ in this section, we get

$$L_{3.2'}\leq C_pM_p(f)(x). \tag{2.25}$$

For $L_{3.2''}$, Denote $C(\varepsilon)=e^{(2+2\varepsilon)/\varepsilon}$. Since $2|y-z|<|z-x_0|$, then $|x_0-y|>|z-x_0|-|y-z|\geq|z-x_0|/2$. Thus

$$L_{3.2''}\leq C\int_{(8Q^*)^c}|f(z)|\Big(\iint_{\substack{y\in(2Q^*)^c,|y-z|>8r,|x_0-y|\geq t\\|x_0-y|>|z-x_0|/2,|y-z|<t}}\big(\frac{t}{t+|x_0-y|}\big)^{\lambda n}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$

$$-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}dz$$

$$\leq C\int_{(8Q^*)^c}|f(z)|\Big(\iint_{\substack{y\in(2Q^*)^c,|y-z|>8r,|x_0-y|\geq t\\|x_0-y|>|z-x_0|/2,|y-z|<t}}$$

$$\frac{t^{\lambda n}(\log\frac{t+|x_0-y|+8C(\varepsilon)r}{r})^{2+2\varepsilon}}{(t+|x_0-y|)^{\lambda n-2n+2n}(\log\frac{t+|x_0-y|+8C(\varepsilon)r}{r})^{2+2\varepsilon}}$$

$$\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}dz$$

$$\leq C\int_{(8Q^*)^c}\frac{|f(z)|}{(|z-x_0|)^n(\log\frac{|z-x_0|/2}{r})^{1+\varepsilon}}\Big(\iint_{\substack{y\in(2Q^*)^c,|y-z|>8r\\|x_0-y|\geq t,|y-z|<t}}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$

$$-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2\frac{t^{\lambda n}(\log\frac{t+|x_0-y|+8C(\varepsilon)r}{r})^{2+2\varepsilon}}{(t+|x_0-y|)^{\lambda n-2n}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{1/2}dz$$

$$\leq C\int_{(8Q^*)^c}\frac{|f(z)|}{(|z-x_0|)^n(\log\frac{|z-x_0|/2}{r})^{1+\varepsilon}}\Big(\int_{\substack{y\in(2Q^*)^c\\|y-z|>8r}}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$

$$-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2\Big(\int_{|y-z|}^{|x_0-y|}\frac{t^{\lambda n}(\log\frac{t+|x_0-y|+8C(\varepsilon)r}{r})^{2+2\varepsilon}}{(t+|x_0-y|)^{\lambda n-2n}}\frac{dt}{t^{n+2\rho+1}}\Big)dy\Big)^{1/2}dz.$$

Notice that the function $G(s)=\frac{(\log s)^{2+2\varepsilon}}{s^{\varepsilon}}$ is decreasing when $s>e^{(2+2\varepsilon)/\varepsilon}$ and

$$\frac{t+|x_0-y|+8C(\varepsilon)r}{r}\geq\frac{|y-z|+8C(\varepsilon)r}{r}>C(\varepsilon)=e^{(2+2\varepsilon)/\varepsilon},$$

Then

$$\frac{[\log(\frac{t+|x_0-y|+8C(\varepsilon)r}{r})]^{2+2\varepsilon}}{(\frac{t+|x_0-y|+8C(\varepsilon)r}{r})^{\varepsilon}} = G(\frac{t+|x_0-y|+8C(\varepsilon)r}{r})$$
$$\le G(\frac{|y-z|+8C(\varepsilon)r}{r}) = \frac{[\log(\frac{|y-z|+8C(\varepsilon)r}{r})]^{2+2\varepsilon}}{(\frac{|y-z|+8C(\varepsilon)r}{r})^{\varepsilon}}.$$

Notice that $t+|x_0-y| \sim t+|x_0-y|+8C(\varepsilon)r$ and $0<\varepsilon<\min\{1/2,(\lambda-2)n/2,\rho-n/2,\sigma-1\}$, then

$$\int_{|y-z|}^{|x_0-y|} \frac{t^{\lambda n}(\log\frac{t+|x_0-y|+8C(\varepsilon)r}{r})^{2+2\varepsilon}}{(t+|x_0-y|)^{\lambda n-2n}}\frac{dt}{t^{n+2\rho+1}}$$
$$= \int_{|y-z|}^{|x_0-y|} \frac{(\log\frac{t+|x_0-y|+8C(\varepsilon)r}{r})^{2+2\varepsilon}}{(t+|x_0-y|)^{\varepsilon}}\frac{t^{\lambda n}}{(t+|x_0-y|)^{\lambda n-2n-\varepsilon}}\frac{dt}{t^{n+2\rho+1}}$$
$$\le C\int_{|y-z|}^{\infty} \frac{[\log(\frac{|y-z|+8C(\varepsilon)r}{r})]^{2+2\varepsilon}}{(|y-z|+8C(\varepsilon)r)^{\varepsilon}}\frac{dt}{t^{2\rho-n+1-\varepsilon}}$$
$$\le C\frac{[\log(\frac{|y-z|+8C(\varepsilon)r}{r})]^{2+2\varepsilon}}{|y-z|^{2\rho-n}}.$$

Since $|y-z|>8r$, there exists an $\ell_0\ge 1$ such that $|y-z|+8C(\varepsilon)r\le 2^{\ell_0}|y-z|$. Hence

$$L_{3.2''} \le C\int_{(8Q^*)^c} \frac{|f(z)|}{(|z-x_0|)^n(\log\frac{|z-x_0|/2}{r})^{1+\varepsilon}}\Bigg(\int_{\substack{y\in(2Q^*)^c\\ |y-z|>8r}} \Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}$$
$$-\frac{\Omega(v-x_0+y-z)}{|v-x_0+y-z|^{n-\rho}}\Big|^2\frac{(\log\frac{2^{\ell_0}|y-z|}{r})^{2+2\varepsilon}}{|y-z|^{2\rho-n}}dy\Bigg)^{1/2}dz.$$

Using the same method of estimating $I_{3.2}$, we get

$$L_{3.2''} \le C_pM_p(f)(x). \tag{2.26}$$

Add up (2.13)-(2.26), we obtain

$$J\le J_1+J_2\le J_1+L_1+L_2+L_3\le C_pM_p(f)(x).$$

Hence

$$|\mu_\lambda^{*,\rho}(f_2)(x_0)-\mu_\lambda^{*,\rho}(f_2)(v)|<C_pM_p(f)(x) \quad \text{for all } x\in\mathbb{R}^n\ .$$

Therefore

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |\mu_\lambda^{*,\rho}(f_2)(x_0)-\mu_\lambda^{*,\rho}(f_2)(v)|dv \\
&= \frac{1}{|Q|}\int_{Q\setminus E} |\mu_\lambda^{*,\rho}(f_2)(x_0)-\mu_\lambda^{*,\rho}(f_2)(v)|d\omega \le C_p M_p(f)(x).
\end{aligned} \tag{2.27}$$

Since for any $v\in Q\setminus E$,

$$\begin{aligned}
&\left|\mu_\lambda^{*,\rho}(f_1+f_2)(v)-\mu_\lambda^{*,\rho}(f_2)(x_0)\right| \\
&\quad \le \left|\mu_\lambda^{*,\rho}(f_1+f_2)(v)-\mu_\lambda^{*,\rho}(f_2)(v)\right| + \left|\mu_\lambda^{*,\rho}(f_2)(v)-\mu_\lambda^{*,\rho}(f_2)(x_0)\right| \\
&\quad = \left|\|\phi_{t,y}(f_1+f_2)(v)\|_{\mathcal{H}_2}-\|\phi_{t,y}(f_2)(v)\|_{\mathcal{H}_2}\right| + \left|\mu_\lambda^{*,\rho}(f_2)(v)-\mu_\lambda^{*,\rho}(f_2)(x_0)\right| \\
&\quad \le \|\phi_{t,y}(f_1+f_2)(v)-\phi_{t,y}(f_2)(v)\|_{\mathcal{H}_2} + \left|\mu_\lambda^{*,\rho}(f_2)(v)-\mu_\lambda^{*,\rho}(f_2)(x_0)\right| \\
&\quad = \mu_\lambda^{*,\rho}(f_1)(v) + \left|\mu_\lambda^{*,\rho}(f_2)(v)-\mu_\lambda^{*,\rho}(f_2)(x_0)\right|.
\end{aligned}$$

By (2.3) and (2.27), we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |\mu_\lambda^{*,\rho}(f)(v)-\mu_\lambda^{*,\rho}(f_2)(x_0)|dv \\
&\le \frac{1}{|Q|}\int_Q \mu_\lambda^{*,\rho}(f_1)(v)dv + \frac{1}{|Q|}\int_{Q\setminus E} |\mu_\lambda^{*,\rho}(f_2)(v)-\mu_\lambda^{*,\rho}(f_2)(x_0)|dv \\
&\le C_p M_p(f)(x).
\end{aligned}$$

Take $c=\mu_\lambda^{*,\rho}(f_2)(x_0)$, (1.6) follows from the above inequality and the definition of $M^\sharp$ and the proof of Theorem 1 is finished.

3. Proof of Theorem 2

By the properties of A_p weights, for any $w\in A_p$, we can find $s>1$ such that $p/s>1$ and $w\in A_{p/s}$. Thus by (1.5) of Theorem 1, we get

$$\begin{aligned}
\int_{\mathbb{R}^n}[\mu_{\Omega,S}^\rho(f)(x)]^p w(x)dx &\le \int_{\mathbb{R}^n}[M_d(\mu_{\Omega,S}^\rho f)(x)]^p w(x)dx \\
&\le C\int_{\mathbb{R}^n}[M^\sharp(\mu_{\Omega,S}^\rho f)(x)]^p w(x)dx \\
&\le C_s^p\int_{\mathbb{R}^n}[M(|f|^s)(x)]^{p/s} w(x)dx \\
&\le C_s^p\int_{\mathbb{R}^n}[|f(x)|^s]^{p/s} w(x)dx \\
&= C_s^p\int_{\mathbb{R}^n}|f(x)|^p w(x)dx,
\end{aligned}$$

where M_d is the dyadic maximal operator.

Similarly, by (1.6) we have

$$\int_{\mathbb{R}^n} [\mu_\lambda^{*,\rho}(f)(x)]^p w(x)dx \leq C_s^p \int_{\mathbb{R}^n} |f(x)|^p w(x)dx.$$

Then we finish the proof of Theorem 2.

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