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EMBEDDING ℓ_1 INTO THE PROJECTIVE TENSOR PRODUCT OF BANACH SPACES

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Abstract. Let X and Y be Banach spaces such that X has an unconditional basis. Then $X\hat{\otimes}Y$, the projective tensor product of X and Y, contains no copy of ℓ_1 if and only if both X and Y contain no copy of ℓ_1 and each continuous linear operator from X to Y^* is compact.

1. Introduction

In 1991, G. Emmanuele [3] showed that if Banach spaces X and Y contain no copy of ℓ_1 then their projective tensor product $X\hat{\otimes}Y$ contains no copy of ℓ_1 provided (*) each continuous linear operator from X to Y^* is compact. In this paper, we will use Rosenthal's ℓ_1-theorem (see [1, p.201]) and sequence space techniques to show that the condition (*) in Emmanuele's result is not only sufficient but also necessary in case one of X and Y has an unconditional basis.

For a Banach space X, let X^* denote its topological dual and B_X denote its closed unit ball. For Banach spaces X and Y, let $\mathcal{L}(X, Y)$ denote the space of all continuous linear operators from X to Y with its operator norm $\|\cdot\|$; and let $X\hat{\otimes}Y$ denote the completion of the tensor product $X \otimes Y$ with respect to the projective tensor norm. It is known that the dual of $X\hat{\otimes}Y$ is isometrically isomorphic to $\mathcal{L}(X, Y^*)$ (see [2, p. 230]). For a Banach space with a basis $\{e_n\}_1^\infty$, let $\{e_n^*\}_1^\infty$ be the biorthogonal functionals associated to the basis $\{e_n\}_1^\infty$, i.e.,

$$e_i^*(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

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Lemma 1. *Let X be a Banach space with an unconditional basis $\{e_n\}_1^\infty$. Then a bounded subset M of X is relatively compact if and only if*

$$\lim_n \sup \left\{ \left\| \sum_{i=n}^{\infty} e_i^*(x) e_i \right\| : x \in M \right\} = 0. \tag{1}$$

Proof. First suppose that M is relatively compact. If (1) does not hold, then noting that $\lim_n \| \sum_{i=n}^\infty e_i^*(x)e_i\| = 0$ for each $x \in X$, there exist an $\varepsilon_0 > 0$, a subsequence $n_1 < m_1 < n_2 < m_2 < \cdots$, and a sequence $\{x_k\}_1^\infty$ in M such that

$$\left\| \sum_{i=n_k}^{\infty} e_i^*(x_k) e_i \right\| \geq \varepsilon_0 \,, \qquad k = 1, 2, \cdots$$

and

$$\left\| \sum_{i=m}^{\infty} e_i^*(x_k) e_i \right\| \leq \varepsilon_0/2 \,, \qquad m > m_k,\ k = 1, 2, \cdots .$$

Let K be the unconditional basis constant of $\{e_n\}_1^\infty$. Then for each $k, j \in \mathbb{N}$ with $k > j$,

$$\begin{aligned} K \cdot \|x_k - x_j\| \;&\geq \left\| \sum_{i=n_k}^{\infty} e_i^*(x_k - x_j) e_i \right\| \geq \left\| \sum_{i=n_k}^{\infty} e_i^*(x_k) e_i \right\| - \left\| \sum_{i=n_k}^{\infty} e_i^*(x_j) e_i \right\| \\ &\geq \varepsilon_0 - \varepsilon_0/2 = \varepsilon_0/2. \end{aligned}$$

Therefore the sequence $\{x_k\}_1^\infty$ has no limit points in X, which shows that M is not relatively compact. This contradiction shows that (1) holds.

Next suppose that (1) holds. Pick a sequence $\{x_m\}_1^\infty$ in M. Since M is bounded, $\sup_m |e_i^*(x_m)| < \infty$ for each $i \in \mathbb{N}$. By diagonal method, we can find a subsequence $\{x_{m_k}\}_1^\infty$ of $\{x_m\}_1^\infty$ such that

$$\lim_k e_i^*(x_{m_k}) \text{ exists for each } i \in \mathbb{N}. \tag{2}$$

For each $\varepsilon > 0$, there exists by (1) an $n_0 \in \mathbb{N}$ such that

$$\left\| \sum_{i=n_0+1}^{\infty} e_i^*(x) e_i \right\| \leq \varepsilon/4 \,, \qquad \forall x \in M.$$

Moreover, there exists by (2) a $k_0 \in \mathbb{N}$ such that for each $k, j > k_0$,

$$\left| e_i^*(x_{m_k} - x_{m_j}) \right| < \varepsilon/2cn_0 \,, \qquad i = 1, 2, \cdots, n_0,$$

where $c = \sup_n \|e_n\| < \infty$. Thus for each $k, j > k_0$,

$$\begin{aligned}\|x_{m_k} - x_{m_j}\| &= \left\|\sum_{i=1}^{\infty} e_i^*(x_{m_k} - x_{m_j})e_i\right\| \\ &\le c\sum_{i=1}^{n_0} \left|e_i^*(x_{m_k} - x_{m_j})\right| + \left\|\sum_{i=n_0+1}^{\infty} e_i^*(x_{m_k})e_i\right\| + \left\|\sum_{i=n_0+1}^{\infty} e_i^*(x_{m_j})e_i\right\| \\ &\le \varepsilon/2 + \varepsilon/4 + \varepsilon/4 = \varepsilon.\end{aligned}$$

Therefore $\{x_{m_k}\}_1^\infty$ is a Cauchy sequence in X, and hence it has a limit point in X. This shows that M is relatively compact. ■

Lemma 2. *Let X and Y be Banach spaces such that X has an unconditionally shrinking basis $\{e_n\}_1^\infty$. For a continuous linear operator T from X to Y, let $y_n = Te_n$ for each $n \in \mathbb{N}$. Then T is compact if and only if*

$$\limsup_n \left\{ \left\|\sum_{i=n}^{\infty} e_i^*(x)y_i\right\|_Y : x \in B_X \right\} = 0.$$

Proof. Since $\{e_n\}_1^\infty$ is an unconditionally shrinking basis of X, $\{e_n^*\}_1^\infty$ is an unconditional basis of X^*. Let T^* be the adjoint operator of T. Then for each $y^* \in Y^*$, $T^*(y^*) = \sum_{n=1}^\infty y^*(y_n)e_n^*$. Thus $\{T^*(y^*) : y^* \in B_{Y^*}\} = \{\sum_{n=1}^\infty y^*(y_n)e_n^* : y^* \in B_{Y^*}\}$. Note that for each $n \in \mathbb{N}$,

$$\sup \left\{ \left\|\sum_{i=n}^{\infty} e_i^*(x)y_i\right\|_Y : x \in B_X \right\} = \sup \left\{ \left\|\sum_{i=n}^{\infty} y^*(y_i)e_i^*\right\|_{X^*} : y^* \in B_{Y^*} \right\}.$$

By Lemma 1, T is compact if and only if T^* is compact if and only if

$$\limsup_n \left\{ \left\|\sum_{i=n}^{\infty} y^*(y_i)e_i^*\right\|_{X^*} : y^* \in B_{Y^*} \right\} = 0$$

if and only if

$$\limsup_n \left\{ \left\|\sum_{i=n}^{\infty} e_i^*(x)y_i\right\|_Y : x \in B_X \right\} = 0. \qquad ■$$

Lemma 3. *Let X and Y be Banach spaces such that X has an unconditional basis $\{e_n\}_1^\infty$. For a continuous linear operator T from X to Y^*, let $y_n^* = Te_n$ for each $n \in \mathbb{N}$. Define*

$$\begin{aligned} I_T : \quad X\hat{\otimes}Y &\longrightarrow \ell_1 \\ z &\longmapsto \left(\textstyle\sum_{k=1}^\infty e_n^*(x_k)\cdot y_n^*(y_k)\right)_n , \end{aligned}$$

where z *admits a representation* $z = \sum_{k=1}^{\infty} x_k \otimes y_k$. *Then* I_T *is a well-defined continuous linear operator.*

Proof. Let $z \in X \hat{\otimes} Y$ and $s = (s_n)_n \in \ell_\infty$. For each $\varepsilon > 0$, z admits a representation

$$z = \sum_{k=1}^{\infty} x_k \otimes y_k$$

such that

$$\sum_{k=1}^{\infty} \|x_k\| \cdot \|y_k\| \le \|z\|_{X \hat{\otimes} Y} + \varepsilon.$$

Let

$$u_k = \sum_{n=1}^{\infty} s_n e_n^*(x_k) e_n \,, \qquad k = 1, 2, \cdots .$$

Then by [4, p.19, Proposition 1.c.7], $u_k \in X$ for each $k \in \mathbb{N}$ and

$$\|u_k\| \le 2K \cdot \|s\|_{\ell_\infty} \cdot \|x_k\|, \qquad k = 1, 2, \cdots ,$$

where K is the unconditional basis constant for $\{e_n\}_1^\infty$. Thus

$$\begin{aligned} |\langle s, I_T(z)\rangle| &= \left| \sum_{n=1}^{\infty} s_n \sum_{k=1}^{\infty} e_n^*(x_k) \cdot y_n^*(y_k) \right| = \left| \sum_{k=1}^{\infty} \langle T(u_k), y_k\rangle \right| \\ &= \left| \langle \sum_{k=1}^{\infty} u_k \otimes y_k, T\rangle \right| \le \|T\| \cdot \left\| \sum_{k=1}^{\infty} u_k \otimes y_k \right\|_{X \hat{\otimes} Y} \\ &\le \|T\| \cdot \sum_{k=1}^{\infty} \|u_k\| \cdot \|y_k\| \le 2K \|s\|_{\ell_\infty} \cdot \|T\| \cdot \sum_{k=1}^{\infty} \|x_k\| \cdot \|y_k\| \\ &\le 2K \|s\|_{\ell_\infty} \cdot \|T\| \cdot (\|z\|_{X \hat{\otimes} Y} + \varepsilon). \end{aligned}$$

Letting $\varepsilon \longrightarrow 0$,

$$|\langle s, I_T(z)\rangle| \le 2K \|s\|_{\ell_\infty} \cdot \|T\| \cdot \|z\|_{X \hat{\otimes} Y}.$$

Therefore I_T is well-defined and continuous. ∎

Rosenthal's ℓ_1-theorem ([1, p. 201]).

A Banach space X contains no copy of ℓ_1 if and only if each bounded sequence in X has a weakly Cauchy subsequence.

Theorem 4. *Let* X *and* Y *be Banach spaces such that* X *has an unconditional basis. Then* $X \hat{\otimes} Y$, *the projective tensor product of* X *and* Y, *contains no copy of*

ℓ_1 if and only if both X and Y contain no copy of ℓ_1 and each continuous linear operator from X to Y^* is compact.

Proof. First let us suppose that both X and Y contain no copy of ℓ_1 and each continuous linear operator from X to Y^* is compact. Let $\{e_n\}_1^\infty$ be an unconditional basis of X. By [4, p. 21, Theorem 1.c.9], $\{e_n\}_1^\infty$ is also a shrinking basis. Let $\{z_n\}_1^\infty$ be a bounded sequence in $X\hat{\otimes}Y$, and let z_n admit representations

$$z_n = \sum_{k=1}^{\infty} x_{k,n} \otimes y_{k,n} \qquad n = 1, 2, \cdots$$

such that

$$\sum_{k=1}^{\infty} \|x_{k,n}\| \cdot \|y_{k,n}\| \le \|z_n\|_{X\hat{\otimes}Y} + 1, \qquad n = 1, 2, \cdots .$$

Denote $M = \sup_n \|z_n\|_{X\hat{\otimes}Y} < \infty$ and $c = \sup_n \|e_n^*\| < \infty$. Then for each $i, n \in \mathbb{N}$,

$$\left\|\sum_{k=1}^{\infty} e_i^*(x_{k,n}) y_{k,n}\right\|_Y \le c \cdot \sum_{k=1}^{\infty} \|x_{k,n}\| \cdot \|y_{k,n}\| \le c\left(\|z_n\|_{X\hat{\otimes}Y} + 1\right) \le c(M+1).$$

Thus for each $i \in \mathbb{N}$, $\{\sum_{k=1}^\infty e_i^*(x_{k,n}) y_{k,n}\}_{n=1}^\infty$ is a bounded sequence in Y. By Rosenthal's ℓ_1-theorem, using diagonal method, there exists a subsequence of $\{\sum_{k=1}^\infty e_i^*(x_{k,n}) y_{k,n}\}_{n=1}^\infty$, without loss of generality, say itself $\{\sum_{k=1}^\infty e_i^*(x_{k,n}) y_{k,n}\}_{n=1}^\infty$, which is coordinate-wisely weakly Cauchy sequence, i.e.,

$$\text{(3)} \quad \text{weak-}\lim_{m,n} \left(\sum_{k=1}^{\infty} e_i^*(x_{k,m}) y_{k,m} - \sum_{k=1}^{\infty} e_i^*(x_{k,n}) y_{k,n}\right) = 0, \qquad i = 1, 2, \cdots .$$

Now for each $T \in (X\hat{\otimes}Y)^* = \mathcal{L}(X, Y^*)$, let $y_n^* = Te_n$ for each $n \in \mathbb{N}$. By hypothesis, T is compact. For each $\varepsilon > 0$, there exists, by Lemma 2, an $l \in \mathbb{N}$ such that

$$\sup\left\{\left\|\sum_{i=l+1}^{\infty} e_i^*(x) y_i^*\right\|_{Y^*} : x \in B_X\right\} \le \varepsilon/4M .$$

Define $T_l : X \longrightarrow Y^*$ by $T_l(x) = \sum_{i=l+1}^\infty e_i^*(x) y_i^*$ for each $x \in X$. Then $\|T_l\| \le \varepsilon/4M$. From (3), there exists an $n_0 \in \mathbb{N}$ such that for each $m, n > n_0$,

$$\left|y_i^* \left(\sum_{k=1}^{\infty} e_i^*(x_{k,m}) y_{k,m} - \sum_{k=1}^{\infty} e_i^*(x_{k,n}) y_{k,n}\right)\right| \le \varepsilon/2l , \qquad i = 1, 2, \cdots, l.$$

Thus for each $m, n > n_0$,

$$\begin{aligned}
|\langle z_m - z_n, T\rangle| &= \left|\sum_{k=1}^{\infty}\langle Tx_{k,m}, y_{k,m}\rangle - \sum_{k=1}^{\infty}\langle Tx_{k,n}, y_{k,n}\rangle\right| \\
&= \left|\sum_{k=1}^{\infty}\langle\sum_{i=1}^{\infty} e_i^*(x_{k,m})y_i^*, y_{k,m}\rangle - \sum_{k=1}^{\infty}\langle\sum_{i=1}^{\infty} e_i^*(x_{k,n})y_i^*, y_{k,n}\rangle\right| \\
&= \left|\sum_{i=1}^{\infty}\left(\sum_{k=1}^{\infty} e_i^*(x_{k,m})\cdot y_i^*(y_{k,m}) - \sum_{k=1}^{\infty} e_i^*(x_{k,n})\cdot y_i^*(y_{k,n})\right)\right| \\
&\le \sum_{i=1}^{l}\left|y_i^*\left(\sum_{k=1}^{\infty} e_i^*(x_{k,m})y_{k,m} - \sum_{k=1}^{\infty} e_i^*(x_{k,n})y_{k,n}\right)\right| \\
&\quad + \left|\sum_{i=l+1}^{\infty}\sum_{k=1}^{\infty} e_i^*(x_{k,m})\cdot y_i^*(y_{k,m})\right| + \left|\sum_{i=l+1}^{\infty}\sum_{k=1}^{\infty} e_i^*(x_{k,n})\cdot y_i^*(y_{k,n})\right| \\
&\le \varepsilon/2 + |\langle z_m, T_l\rangle| + |\langle z_n, T_l\rangle| \\
&\le \varepsilon/2 + (\|z_m\|_{X\hat{\otimes}Y} + \|z_n\|_{X\hat{\otimes}Y})\cdot\|T_l\| \le \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Therefore, $\{z_n\}_1^\infty$ is a weakly Cauchy sequence in $X\hat{\otimes}Y$, and hence, by Rosenthal's ℓ_1-theorem again, $X\hat{\otimes}Y$ contains no copy of ℓ_1.

Next suppose that $X\hat{\otimes}Y$ contains no copy of ℓ_1. It is obvious that X and Y contain no copy of ℓ_1. Let $\{e_n\}_1^\infty$ be an unconditional basis of X. By [4, p.21, Theorem 1.c.9], $\{e_n\}_1^\infty$ is also a shrinking basis. Now for each $T \in (X\hat{\otimes}Y)^* = \mathcal{L}(X, Y^*)$, let $y_n^* = Te_n$ for each $n \in \mathbb{N}$. If T is not compact, by Lemma 2, there are $\varepsilon_0 > 0$, a subsequence $n_1 < n_2 < \cdots$, and a sequence $\{x_k\}_1^\infty$ in B_X such that

$$\left\|\sum_{i=n_k}^{\infty} e_i^*(x_k)y_i^*\right\|_{Y^*} > \varepsilon_0\,, \qquad k = 1, 2, \cdots.$$

Moreover, there exists a sequence $\{y_k\}_1^\infty$ in B_Y such that

$$\left|\sum_{i=n_k}^{\infty} e_i^*(x_k)y_i^*(y_k)\right| > \varepsilon_0\,, \qquad k = 1, 2, \cdots. \tag{4}$$

Let $z_k = x_k \otimes y_k, k = 1, 2, \cdots$. Then $z_k \in B_{X\hat{\otimes}Y}$ for each $k \in \mathbb{N}$. It follows from Rosenthal's ℓ_1-theorem that $\{z_k\}_1^\infty$ has a subsequence, without loss of generality, say itself, which is weakly Cauchy. By Lemma 3, $\{I_T(z_k)\}_1^\infty$ is a weakly Cauchy sequence in ℓ_1, and hence relatively weakly sequentially compact. Thanks to the Schur property, it is a relatively sequentially compact subset of ℓ_1. Thus there exists an $m \in \mathbb{N}$ such that

$$\sum_{i=m}^{\infty} |I_T(z_k)_i| = \sum_{i=m}^{\infty} |e_i^*(x_k)y_i^*(y_k)| < \varepsilon_0\,, \qquad k = 1, 2, \cdots. \tag{5}$$

Pick an $n_k > m$. Then from (4) and (5),

$$\varepsilon_0 < \left| \sum_{i=n_k}^{\infty} e_i^*(x_k)y_i^*(y_k) \right| \le \sum_{i=m}^{\infty} |e_i^*(x_k)y_i^*(y_k)| < \varepsilon_0.$$

Contradiction. This shows that T must be compact. ∎

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