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SHARP MAXIMAL INEQUALITIES AND COMMUTATORS ON MORREY SPACES WITH NON-DOUBLING MEASURES

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Abstract. In this paper, we establish a sharp maximal inequality for Morrey spaces with a Radon measure μ satisfying certain growth condition, which is not necessarily doubling. As an application, we obtain the boundedness of commutators generated by singular integral or fractional integrals with RBMO functions in Morrey spaces.

1. Introduction

The aim of this paper is to establish the sharp maximal inequality for the Morrey spaces with non-doubling measures. This inequality will be applied to obtain the boundedness of the commutators.

Throughout this paper μ will be a (positive) Radon measure on $\mathbf{R}^d$ satisfying the growth condition:

$$\mu(B(x,l)) \le C_0\, l^n \text{ for all } x \in \operatorname{supp}(\mu) \text{ and } l > 0, \tag{1}$$

where C_0 and n, $0 < n \le d$, are some fixed numbers. We do not assume that μ is doubling.

Recently the measure with growth condition has been shed light on because we can recover the Calderón-Zygmund theory. Nazarov, Treil and Volberg developed the theory of the singular integrals for the measures with growth condition to investigate the analytic capacity on the complex plane [9, 10]. X. Tolsa showed that the analytic capacity is subadditive and that it is bi-Lipschitz invariant [17, 18]. The research, which was started from their pioneering works using the modified maximal operator,

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has been developed in many ways: García-Cuerva and Eduardo Gatto defined a potential operator for the measures with growth condition [4]. X. Tolsa defined for the growth measures RBMO (regular bounded mean oscillation) space, the Hardy space $H^1(\mu)$ and the Littlewood-Paley decomposition [14, 16]. He also gave his $H^1(\mu)$ space in terms of the grand maximal operator [15]. Chen and Sawyer have modified the definition of RBMO to investigate the commutator of the potential operator and RBMO [1]. Deng, Han and Yang have defined the Besov-space and the Triebel-Lizorkin space for the growth measures [5, 6]. Hu, Meng and Yang also considered the multilinear operator [7, 8]. The authors also defined a Morrey space for non-doubling measures [13]. Some definitions are recalled later.

We denote by M the Hardy-Littlewood maximal operator and by $M^\sharp$ the sharp maximal operator. Then the sharp maximal inequality is the one of the form:

$$\|Mf \mid L^p(\mathbf{R}^d)\| \le C\,\|M^\sharp f \mid L^p(\mathbf{R}^d)\|, \quad 1 < p < \infty,$$

which was firstly introduced in [2]. It is well-known that this inequality does not hold without some integrability assumption. Indeed, let us remark that if we take $f \equiv 1$ then the inequality fails. So one assumes that $\min(1, Mf) \in L^p(\mathbf{R}^d)$ or that $f \in L^q(\mathbf{R}^d)$ for some q, $1 \le q \le p$. In this paper we will also discuss the integrability assumptions in terms of the Morrey spaces. Before stating our main result, we fix some notation and define some terminologies.

By "cube" $Q \subset \mathbf{R}^d$ we mean a compact cube whose edges are parallel to the coordinate axes. Its side length will be denoted by $\ell(Q)$. For $c > 0$, $c\,Q$ will denote a cube concentric to Q with its sidelength $c\,\ell(Q)$. The set of all cubes $Q \subset \mathbf{R}^d$ with positive μ-measure will be denoted by $\mathcal{Q}(\mu)$. We recall the definition of the Morrey spaces with non-doubling measures.

Let $k > 1$ and $1 \le q \le p < \infty$. We define a Morrey space $\mathcal{M}^p_q(k,\mu)$ as

$$\mathcal{M}^p_q(k,\mu) := \left\{ f \in L^q_{loc}(\mu) \mid \|f \mid \mathcal{M}^p_q(k,\mu)\| < \infty \right\},$$

where the norm $\|f \mid \mathcal{M}^p_q(k,\mu)\|$ is given by

$$\|f \mid \mathcal{M}^p_q(k,\mu)\| := \sup_{Q \in \mathcal{Q}(\mu)} \mu(k\,Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f|^q \, d\mu \right)^{\frac{1}{q}}. \tag{2}$$

By using Hölder's inequality to (2), it is easy to see that

$$L^p(\mu) = \mathcal{M}^p_p(k,\mu) \subset \mathcal{M}^p_{q_1}(k,\mu) \subset \mathcal{M}^p_{q_2}(k,\mu) \tag{3}$$

for $1 \le q_2 \le q_1 \le p < \infty$. The definition of the spaces is independent of the constant $k > 1$. The norms for different choices of $k > 1$ are equivalent. More precisely, for $k_1 > k_2 > 1$, we have (see [13])

$$(4) \qquad \|f \mid \mathcal{M}_q^p(k_1,\mu)\| \le \|f \mid \mathcal{M}_q^p(k_2,\mu)\| \le C_d \left(\frac{k_1-1}{k_2-1}\right)^d \|f \mid \mathcal{M}_q^p(k_1,\mu)\|.$$

Nevertheless, for definiteness, we will assume $k = 2$ in the definition and denote $\mathcal{M}_q^p(2,\mu)$ by $\mathcal{M}_q^p(\mu)$.

Our BMO here is a RBMO (regular bounded mean oscillation) introduced by X. Tolsa [14] which are the suitable substitutes for the classical BMO space. For the definition and its various equivalent norms we refer to [14, Lemma 2.10]. We list one of them.

Definition 1.1. [14, Sections 2.2 and 2.3]

(1) Given two cubes $Q, R \in \mathcal{Q}(\mu)$ with $Q \subset R$, set $K_{Q,R} := 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^kQ)}{\ell(2^kQ)^n}$, where $N_{Q,R}$ is the least integer $k \ge 1$ such that $2^kQ \supset R$.

(2) Q is called a doubling cube if $\mu(2Q) \le 2^{d+1}\mu(Q)$. One denotes by $\mathcal{Q}(\mu,2)$ the set of all doubling cubes.

(3) Given $Q \in \mathcal{Q}(\mu)$, set Q^* as the smallest doubling cube R of the form $R = 2^jQ$ with $j \in \mathbf{N}_0 := \{0\} \cup \mathbf{N}$.

(4) $f \in L^1_{loc}(\mu)$ is said to belong to RBMO if it satisfies

$$\sup_{Q \in \mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |f(y) - m_{Q^*}(f)|\, d\mu(y) + \sup_{\substack{Q \subset R \\ Q,R \in \mathcal{Q}(\mu,2)}} \frac{|m_Q(f) - m_R(f)|}{K_{Q,R}} < \infty,$$

where $m_Q(f) := \frac{1}{\mu(Q)} \int_Q f\, d\mu$. One denotes this quantity by $\|f\|_*$.

By the growth condition (1) there are a lot of big doubling cubes. Precisely speaking, given any cube $Q \in \mathcal{Q}(\mu)$, we can find $j \in \mathbf{N}$ with $2^jQ \in \mathcal{Q}(\mu,2)$. Meanwhile, for μ-a.e. $x \in \mathbf{R}^d$, there exists a sequence of doubling cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ as $k \to \infty$. So we can say that there are a lot of small doubling cubes, too. (See [14].)

For $f \in L^1_{loc}(\mu)$ we define two maximal operators due to Tolsa (see [14]): The sharp maximal operator $M^\sharp f(x)$ is defined as

$$M^\sharp f(x) := \sup_{x \in Q \in \mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |f(y) - m_{Q^*}(f)|\, d\mu(y) + \sup_{\substack{x \in Q \subset R \\ Q,R \in \mathcal{Q}(\mu,2)}} \frac{|m_Q(f) - m_R(f)|}{K_{Q,R}}$$

and $Nf(x)$ is defined as $Nf(x) := \sup_{x \in Q \in \mathcal{Q}(\mu,2)} m_Q(|f|)$. The following proposition is a sharp maximal inequality of $L^p(\mu)$ for these operators.

Proposition 1.2. [14 p. 124]

(1) *Suppose that* $f \in L^1_{loc}(\mu)$. *Then, for* μ*-a.e.* $x \in \mathbf{R}^d$, *we have* $|f(x)| \le Nf(x)$.

(2) *Suppose that* $1 < p < \infty$. *We assume that* $\min(1, Nf) \in L^p(\mu)$ *when* $\mu(\mathbf{R}^d) = \infty$ *and that* $f \in L^1(\mu)$ *and* $\int_{\mathbf{R}^d} f \, d\mu = 0$ *when* $\mu(\mathbf{R}^d) < \infty$. *Then there exists a constant* $C > 0$ *independent on* f *such that* $\|Nf \,|\, L^p(\mu)\| \le C \, \|M^\sharp f \,|\, L^p(\mu)\|$.

Now we state our main results on the sharp maximal inequality for the Morrey space $\mathcal{M}^p_q(\mu)$.

Theorem 1.3. *Suppose that* $1 < q \le p < \infty$. *Then, for any* $f \in L^1_{loc}(\mu)$, *there exists a constant* $C > 0$ *independent on* f *such that*

$$\|Nf \,|\, \mathcal{M}^p_q(\mu)\| \le C \left(\|M^\sharp f \,|\, \mathcal{M}^p_q(\mu)\| + \|f \,|\, \mathcal{M}^p_1(\mu)\| \right).$$

It is known that $N : \mathcal{M}^p_q(\mu) \to \mathcal{M}^p_q(\mu)$ is a bounded operator (c.f. [13]).

Notice that we can use Theorem 1.3 for any locally integrable function f. This is an advantage of this new sharp maximal inequality. In showing the $L^p(\mu)$-boundedness of some linear operator T one often has to assume that T is $L^p(\mu)$-bounded on the set of bounded functions with compact support. Combining with the following theorem, we can recover the usual sharp maximal inequality with an even weaker and unified assumption.

Theorem 1.4. *Suppose that* $1 \le q \le p < \infty$ *and there exist an increasing sequence of concentric doubling cubes* $I_0 \subset I_1 \subset \ldots \subset I_k \subset \ldots$ *such that*

$$\lim_{k \to \infty} m_{I_k}(f) = 0 \text{ and } \bigcup_k I_k = \mathbf{R}^d. \tag{5}$$

Then there exists a constant $C > 0$ *independent on* f *such that*

$$\|f \,|\, \mathcal{M}^p_1(\mu)\| \le C \, \|M^\sharp f \,|\, \mathcal{M}^p_q(\mu)\|.$$

Corollary 1.5. *Suppose that* $1 < q \le p < \infty$ *and that the sequence of cubes* $\{I_k\}$ *satisfies the above conditions. Then there exists a constant* $C > 0$ *independent on* f *such that*

$$\|Nf \,|\, \mathcal{M}^p_q(\mu)\| \le C \, \|M^\sharp f \,|\, \mathcal{M}^p_q(\mu)\|.$$

As for this kind of approach in the case of $L^p(\mathbf{R}^d)$, N. Fujii obtained a result in a different context [3].

Remark 1.6. It would be interesting to restate Theorem 1.3 in the case of the Lebesgue space $L^p(\mathbf{R}^d)$. Notice that if $\mu = dx$ then $M^\sharp f(x)$ is equivalent to the usual one in [2]. Applying our result with $\mu = dx$ and $1 < p = q < \infty$, we have a norm equivalence

$$\|f \mid L^p(\mathbf{R}^d)\| \approx \left(\|M^\sharp f \mid L^p(\mathbf{R}^d)\| + \sup_{Q \subset \mathbf{R}^d} |Q|^{\frac{1}{p}-1} \int_Q |f|\, dx \right) \tag{6}$$

for all $f \in L^1_{loc}(\mathbf{R}^d)$.

As an application of Theorem 1.3 we obtain the boundedness of commutators.

A commutator is an operator of the form $[a, T]f(x) = a(x)Tf(x) - T(af)(x)$, where a is a function and T is a bounded operator. As was shown in the classical results, $[a, T]$ is bounded from $L^p(\mathbf{R}^d)$ to $L^p(\mathbf{R}^d)$ if $a \in$BMO and T is a Calderón-Zygmund operator and is bounded from $L^p(\mathbf{R}^d)$ to $L^q(\mathbf{R}^d)$ if $a \in$BMO and T is a fractional integral operator, where p and q are a suitable pair. Fazio and Ragusa [11] extended these results to the classical Morrey spaces. Our results and precise definitions will be given later (Section 4).

2. Preliminaries

The letter C will be used for constants that may change from one occurrence to another. Constants with subscripts, such as C_1, C_2, do not change in different occurrences. We will assume that the large constant C_0 in (1) has been chosen so that the following estimate holds :

$$\mu(Q) \le C_0 \ell(Q)^n \text{ for all } Q \in \mathcal{Q}(\mu).$$

Lemma 2.1. *The following properties hold* :

(1) *Let* $Q \in \mathcal{Q}(\mu)$ *and* $j \in \mathbf{N}$. *Then we have* $K_{Q,2^jQ} \le 1 + C_0 j$.

(2) *Let* $Q \subset R \in \mathcal{Q}(\mu)$ *be concentric cubes such that there are no doubling cubes of the form* 2^jQ, $j \ge 0$, *with* $Q \subset 2^jQ \subset R$. *Then we have* $K_{Q,R} \le 1 + 2\,C_0$.

(3) *Let* $Q \in \mathcal{Q}(\mu)$ *and* $\alpha > 0$. *Suppose that, for some* $c > 0$,

$$\alpha \le \mu(2^jQ) \le c\,\alpha, \quad j = 0, 1, \ldots, J.$$

Then we have $K_{Q,2^JQ} \le 1 + c\,C_0\,c_n$, *where* $c_n := \sum_{j=0}^{\infty} 2^{-nj}$.

Proof. The assertion (1) is clear. We prove (2) firstly. Putting $N = N_{Q,R}$, we shall estimate $K_{Q,R}$. The growth condition (1) implies $d-n \geq 0$ and the assumption and the definition of the doubling cubes imply $2^{d+1}\mu(2^jQ) \leq \mu(2^{j+1}Q)$. These observations yield

$$K_{Q,R} \leq 1 + \frac{\mu(2^N Q)}{\ell(2^N Q)^n} \sum_{j=1}^{N} \left(2^{n-d-1}\right)^{N-j} \leq 1 + 2C_0.$$

Next we prove (3). It follows by assumption that

$$K_{Q,2^JQ} \leq 1+\sum_{j=0}^{J} \frac{\mu(2^jQ)}{\ell(2^jQ)^n} \leq 1+c\,\frac{\alpha}{\ell(Q)^n} \sum_{j=0}^{J} 2^{-nj} \leq 1+c\,C_0\,c_n. \qquad \blacksquare$$

The following lemmas will be needed in Section 4.

Lemma 2.2. *Suppose that* $1<q\leq p<\infty$, $0\leq\alpha < n$ *and* $1/s=1/p-\alpha/n > 0$.

(1) *For all* $f \in \mathcal{M}^p_q(\mu)$, $a \in$*RBMO*, $Q \in \mathcal{Q}(\mu)$ *and* $x \in Q$, *we have*

$$\int_{\mathbf{R}^d\setminus 2Q} \frac{|(m_{Q^*}(a) - a(y))\, f(y)|}{|x-y|^{n-\alpha}}\, d\mu(y) \leq C\, \ell(Q)^{-\frac{n}{s}} \|a\|_* \|f \mid \mathcal{M}^p_q(\mu)\|.$$

(2) *For all* $f \in \mathcal{M}^p_q(\mu)$, $Q \in \mathcal{Q}(\mu)$ *and* $x \in Q$, *we have*

$$\int_{\mathbf{R}^d\setminus 2Q} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, d\mu(y) \leq C\, \ell(Q)^{-\frac{n}{s}} \|f \mid \mathcal{M}^p_q(\mu)\|.$$

To prove this lemma we need the John-Nirenberg lemma for RBMO due to Tolsa.

Lemma 2.3. [14, Corollary 3.5]

(1) *Let* $a \in$*RBMO. For any cube* $Q \in \mathcal{Q}(\mu)$, *we have*

$$\mu\left\{x \in Q \mid |a(x) - m_{Q^*}(a)| > \lambda\right\} \leq C\,\mu\left(\frac{3}{2}Q\right) \exp\left(-\frac{C'\lambda}{\|a\|_*}\right), \quad \lambda > 0.$$

(2) *For all* $1 \leq r < \infty$, *the following norm is equivalent to* $\|a\|_*$.

$$\sup_{Q\in\mathcal{Q}(\mu)} \left(\frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |a(y) - m_{Q^*}(a)|^r\, d\mu(y)\right)^{\frac{1}{r}} + \sup_{\substack{Q\subset R\\ Q,R\in\mathcal{Q}(\mu,2)}} \frac{|m_Q(a) - m_R(a)|}{K_{Q,R}}.$$

Proof of Lemma 2.2. We will tackle the first assertion, the second one being similar. An elementary calculation yields $\displaystyle\int_0^\infty \frac{\chi_{B(x,l)}(y)}{l^n} l^{\alpha-1}\,dl = \int_{|x-y|}^\infty l^{\alpha-n-1}\,dl = \frac{C}{|x-y|^{n-\alpha}}$, where χ_A is the indicator function of a set $A \subset \mathbf{R}^d$.

This and Fubini's theorem lead us to

$$\begin{aligned}
&\int_{\mathbf{R}^d\setminus 2Q} \frac{|(m_{Q^*}(a)-a(y))\,f(y)|}{|x-y|^{n-\alpha}}\,d\mu(y)\\
&\le \int_{\mathbf{R}^d\setminus B(x,\ell(Q)/2)} \frac{|(m_{Q^*}(a)-a(y))\,f(y)|}{|x-y|^{n-\alpha}}\,d\mu(y)\\
&= C\int_{\mathbf{R}^d\setminus B(x,\ell(Q)/2)} \left(\int_0^\infty \chi_{B(x,l)}|(m_{Q^*}(a)-a(y))\,f(y)|\cdot l^{\alpha-n-1}\,dl\right)d\mu(y)\\
&= C\int_0^\infty \left(l^{\alpha-n-1}\int_{B(x,l)\setminus B(x,\ell(Q)/2)} |(m_{Q^*}(a)-a(y))\,f(y)|\,d\mu(y)\right)dl\\
&\le C\int_{\ell(Q)/2}^\infty \left\{ l^{\alpha-n-1}\left(\int_{B(x,l)} |a(y)-m_{Q^*}(a)|^{q'}\,d\mu(y)\right)^{\frac{1}{q'}}\right.\\
&\qquad \left.\cdot\left\{\int_{B(x,l)} |f(y)|^q\,d\mu(y)\right)^{\frac{1}{q}}\right\}dl,
\end{aligned}$$

where $\dfrac{1}{q'}+\dfrac{1}{q}=1$. It follows from the growth condition (1) that

$$(l^n)^{\frac{1}{p}-\frac{1}{q}}\left(\int_{B(x,l)}|f|^q\,d\mu\right)^{\frac{1}{q}} \le C\,\|f\mid\mathcal{M}^p_q(\mu)\|.$$

By using this estimate we have

$$\begin{aligned}
&\int_{\mathbf{R}^d\setminus 2Q} \frac{|(m_{Q^*}(a)-a(y))\,f(y)|}{|x-y|^{n-\alpha}}\,d\mu(y)\\
&\le\ C\,\|f|\mathcal{M}^p_q(\mu)\|\int_{\ell(Q)/2}^\infty \left(l^{-\frac{n}{s}-1}\left(\frac{1}{l^n}\int_{B(x,l)}|a(y)-m_{Q^*}(a)|^{q'}\,d\mu(y)\right)^{\frac{1}{q'}}\right)dl.
\end{aligned}$$

We shall estimate the right-hand side of this inequality by using Lemma 2.3 (2).

Let k be the least integer satisfying $2^kQ \supset B(x,l)$. Then we have by the growth condition

$$\begin{aligned}
&\left(\frac{1}{l^n}\int_{B(x,l)} |a(y)-m_{Q^*}(a)|^{q'}\,d\mu(y)\right)^{\frac{1}{q'}}\\
&\le C\left(\frac{1}{\mu\left(\frac{3}{2}2^kQ\right)}\int_{2^kQ} |a(y)-m_{Q^*}(a)|^{q'}\,d\mu(y)\right)^{\frac{1}{q'}}\\
&\le C\left(\left(\frac{1}{\mu\left(\frac{3}{2}2^kQ\right)}\int_{2^kQ} |a(y)-m_{(2^kQ)^*}(a)|^{q'}\,d\mu(y)\right)^{\frac{1}{q'}}\right.\\
&\qquad \left. +|m_{(2^kQ)^*}(a)-m_{Q^*}(a)|\right)\\
&\le C\,K_{Q^*,(2^kQ)^*}\|a\|_*.
\end{aligned}$$

It follows from Lemma 2.1 (1) and (2) that $K_{Q^*,(2^kQ)^*} \le C\,(1+k) \le C\left(1+\log\frac{l}{\ell(Q)/2}\right)$. Thus, we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^d\setminus 2Q} \frac{|(m_{Q^*}(a)-a(y))\,f(y)|}{|x-y|^{n-\alpha}}\,d\mu(y)\\
&\le C\,\|a\|_*\|f\,|\,\mathcal{M}^p_q(\mu)\| \int_{\ell(Q)/2}^{\infty} l^{-\frac{n}{s}-1}\left(1+\log\frac{l}{\ell(Q)/2}\right)\,dl\\
&\le C\,\ell(Q)^{-\frac{n}{s}}\|a\|_*\|f\,|\,\mathcal{M}^p_q(\mu)\|.
\end{aligned}$$

This is what we desired. ■

For $f \in L^1_{loc}(\mu)$, $\kappa > 1$ and $0 \le \alpha < n$, a fractional maximal operator $M^\alpha_\kappa f(x)$ is defined as

$$M^\alpha_\kappa f(x) := \sup_{x\in Q\in\mathcal{Q}(\mu)} \frac{1}{\mu(\kappa Q)^{1-\frac{\alpha}{n}}}\int_Q |f|\,d\mu.$$

We will denote M^0_κ by M_κ. As for the boundedness of this operator on the Morrey spaces, the following lemma is known.

Lemma 2.4. [13] *Suppose that* $\kappa > 1$, $0 < \alpha < n$, $1 < q \le p < \infty$, $1 < t \le s < \infty$, $1/s = 1/p-\alpha/n$ *and* $t/s = q/p$. *Then we have*

$$\|M^\alpha_\kappa f\,|\,\mathcal{M}^s_t(\mu)\| \le C\,\|f\,|\,\mathcal{M}^p_q(\mu)\|.$$

3. Proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3

In this section we shall prove Theorem 1.3 by using a good-λ inequality for the

Morrey spaces. For this purpose we may assume (at least) $f \in \mathcal{M}^p_1(\mu)$.

Let $Q_0 \in \mathcal{Q}(\mu)$ and $f \in \mathcal{M}^p_1(\mu)$. For the time being we shall fix them. We define $\mathcal{Q}_0$ and $\mathcal{Q}_1$ as

$$\mathcal{Q}_0 : = \{R \in \mathcal{Q}(\mu, 2) \,|\, R \text{ meets } Q_0 \text{ and is not contained in } 8Q_0\},$$
$$\mathcal{Q}_1 : = \{R \in \mathcal{Q}(\mu, 2) \,|\, R \text{ meets } Q_0 \text{ and is contained in } 8Q_0\}.$$

We also define Λ as $\Lambda := \Lambda_{Q_0}(f) = \sup_{R \in \mathcal{Q}_0} m_R(|f|)$, which will be a key to our arguments. Since we have assumed that $f \in \mathcal{M}^p_1(\mu)$, we observe $\mu(Q_0)^{\frac{1}{p}}\Lambda \le C \, \|f \,|\, \mathcal{M}^p_1(\mu)\| < \infty$.

In fact, we have that if $R \in \mathcal{Q}_0$, then $R \in \mathcal{Q}(\mu, 2)$, $2R \supset Q_0$ and, hence, $\mu(Q_0) \le C\,\mu(R)$. This implies that $\mu(Q_0)^{\frac{1}{p}}\, m_R(|f|) \le C\,\mu(R)^{\frac{1}{p}-1} \int_R |f|\, d\mu$ for all $R \in \mathcal{Q}_0$ and hence

$$\mu(Q_0)^{\frac{1}{p}}\Lambda \le C\,\|f \,|\, \mathcal{M}^p_1(\mu)\| < \infty. \tag{7}$$

Lemma 3.1. *Suppose that $\lambda > \Lambda$. Then, for all $\varepsilon > 0$, there exists $C_1 > 0$ such that for any sufficiently small $\delta > 0$ we have*

$$\mu\{x \in Q_0 \,|\, Nf(x) > (1+\varepsilon)\lambda,\, M^\sharp f(x) \le \delta\lambda\} \le \frac{C_1\delta}{\varepsilon}\,\mu\{x \in 8Q_0 \,|\, Nf(x) > \lambda\}.$$

The proof is standard and similar to those of Tolsa [14] except for the argument involved with Λ. Fix $\varepsilon > 0$ and choose $\delta > 0$ sufficiently small. We set

$$E_\lambda := \{x \in Q_0 | Nf(x) > (1+\varepsilon)\lambda, M^\sharp f(x) \le \delta\lambda\} \text{ and } \Omega_\lambda := \{x \in 8Q_0 | Nf(x) > \lambda\}.$$

For all $x \in E_\lambda$, we can select a doubling cube $Q_x \ni x$ that satisfies $Q_x \in \mathcal{Q}_1$ and $m_{Q_x}(|f|) > (1+\varepsilon/2)\lambda$. By replacing larger one, if necessary, we may assume that $m_Q(|f|) < (1+\varepsilon/2)\lambda$ for any cube Q with $2Q_x \subset Q \in \mathcal{Q}_1$. Let $S_x = (4Q_x)^*$. We claim that if δ is small enough we have $m_{S_x}(|f|) > \lambda$. Indeed, using Lemma 2.1 we see that $K_{Q_x, S_x} \le C$ and noting $M^\sharp|f|(x) \le C_2 M^\sharp f(x)$ (For the proof of this estimate we refer to [14] Remark 6.1), we obtain that

$$m_{S_x}(|f|) \ge m_{Q_x}(|f|) - |m_{Q_x}(|f|) - m_{S_x}(|f|)| \ge (1+\varepsilon/2)\lambda - C\,C_3\delta\lambda > \lambda.$$

Thus, we have

$$S_x \in \mathcal{Q}_1 \text{ and } (1+\varepsilon/2)\lambda > m_{S_x}(|f|) > \lambda. \tag{8}$$

In particular, $S_x \subset \Omega_\lambda$ for all $x \in E_\lambda$ and $\sup_{x \in E_\lambda} \ell(S_x) < \infty$.

By Besicovitch's covering lemma there exists a countable subset $\{x_j\}_{j \in J} \subset E_\lambda$ such that

$$E_\lambda \subset \bigcup_{j \in J} S_{x_j} \text{ and } \sum_{j \in J} \chi_{S_{x_j}} \le C_4 \chi_{\Omega_\lambda}. \tag{9}$$

To simplify the notation, we write $S_j = S_{x_j}$ and $Q_j = Q_{x_j}$. Now we claim the following:

Claim 3.2. *If δ is small enough, then we have*

$$\mu(S_j \cap E_\lambda) \le \frac{C\delta}{\varepsilon}\, \mu(S_j) \text{ for all } j \in J.$$

Accepting the claim, we finish the proof of the lemma. By using this claim and (9) we have

$$\mu(E_\lambda) \le \sum_{j \in J} \mu(S_j \cap E_\lambda) \le \frac{C\delta}{\varepsilon} \sum_{j \in J} \mu(S_j) \le \frac{C\, C_4 \delta}{\varepsilon}\, \mu(\Omega_\lambda).$$

Thus, the proof is over modulo the claim.

Proof of Claim 3.2. Let $y \in S_j \cap E_\lambda$. There exists a doubling cube $R_y \ni y$ that satisfies $R_y \in \mathcal{Q}_1$ and $m_{R_y}(|f|) > (1+\varepsilon)\lambda$. If $\ell(R_y) > \frac{1}{8}\ell(S_j)$, then we have $64R_y \supset S_j \supset Q_j$ and $K_{R_y,(64R_y)^*} \le C$. Hence,

$$\begin{aligned} &(1+\varepsilon/2)\lambda > m_{(64R_y)^*}(|f|) \\ &\ge m_{R_y}(|f|) - |m_{(64R_y)^*}(|f|) - m_{R_y}(|f|)| \ge (1+\varepsilon)\lambda - C\, C_2 \delta\lambda. \end{aligned}$$

Hence, if $\delta < \dfrac{\varepsilon}{2C\, C_2} = C_3\varepsilon$, we have $\ell(R_y) \le \frac{1}{8}\ell(S_j)$. Thus, if $\delta < C_3\varepsilon$, we have

$$N\left(\chi_{\frac{5}{4}S_j} f\right)(y) > (1+\varepsilon)\lambda \text{ for all } y \in S_j \cap E_\lambda.$$

From (8) we obtain that $N\left(\chi_{\frac{5}{4}S_j}(f - m_{S_j}(f))\right)(y) > \varepsilon\lambda/2$ for all $y \in S_j \cap E_\lambda$. An application of the the weak-$(1,1)$ boundedness of N yields

$$\begin{aligned} &\mu(S_j \cap E_\lambda) \\ &\le \mu\left\{y \in \mathbf{R}^d \mid N\left(\chi_{\frac{5}{4}S_j}(f - m_{S_j}(f))\right)(y) > \varepsilon\lambda/2\right\} \\ &\le \frac{C}{\varepsilon\lambda} \int_{\frac{5}{4}S_j} |f(y) - m_{S_j}(f)|\, d\mu(y). \end{aligned}$$

Noting that

$$\frac{1}{\mu\left(\frac{15}{8}S_j\right)}\int_{\frac{5}{4}S_j}|f(y)-m_{S_j}(f)|\,d\mu(y)$$
$$\leq \frac{1}{\mu\left(\frac{15}{8}S_j\right)}\int_{\frac{5}{4}S_j}|f(y)-m_{\left(\frac{5}{4}S_j\right)^*}(f)|\,d\mu(y)+\left|m_{\left(\frac{5}{4}S_j\right)^*}(f)-m_{S_j}(f)\right| \leq C\,\delta\lambda,$$

we see $\mu(S_j\cap E_\lambda)\leq \dfrac{C\delta}{\varepsilon}\,\mu(2S_j)\leq \dfrac{C\delta}{\varepsilon}\,\mu(S_j)$. ∎

Proof of Theorem 1.3. It is clear that instead of considering $\|Nf\,|\,\mathcal{M}^p_q(\mu)\|$ directly, we have only to estimate

$$\|Nf\,|\,\mathcal{M}^p_q(32,\mu)\|_L := \sup_{\substack{Q\in\mathcal{Q}(\mu)\\ \ell(Q)\leq L}} \mu(32Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_0^L q\lambda^{q-1}\mu\{x\in Q\,|\,Nf(x)>\lambda\}\,d\lambda\right)^{\frac{1}{q}}, \quad L\gg 1,$$

with constants independent on L. Note that this quantity is finite because of the growth condition (1).

Let $Q_0\in\mathcal{Q}(\mu)$ and $\ell(Q_0)\leq L$. Let $\Lambda=\Lambda_{Q_0}(f)$ be the quantity defined in the previous lemma. We will estimate

$$\mu(32Q_0)^{\frac{1}{p}-\frac{1}{q}}\left(\int_0^L q\lambda^{q-1}\mu\{x\in Q_0\,|\,Nf(x)>\lambda\}\,d\lambda\right)^{\frac{1}{q}}$$

according as $L\geq 2\Lambda$ or not.

Suppose first that $2\Lambda\geq L$. In this case we have by (7)

$$\mu(32Q_0)^{\frac{1}{p}-\frac{1}{q}}\left(\int_0^L q\lambda^{q-1}\mu\{x\in Q_0\,|\,Nf(x)>\lambda\}\,d\lambda\right)^{\frac{1}{q}}$$
$$\leq \mu(32Q_0)^{\frac{1}{p}-\frac{1}{q}}\left(\int_0^{2\Lambda} q\lambda^{q-1}\mu(Q_0)\,d\lambda\right)^{\frac{1}{q}}$$
$$\leq 2\mu(32Q_0)^{\frac{1}{p}-\frac{1}{q}}\mu(Q_0)^{\frac{1}{q}}\Lambda\leq 2\mu(Q_0)^{\frac{1}{p}}\Lambda\leq C\|f\,|\,\mathcal{M}^p_1\|.$$

Suppose instead that we have $L\geq 2\Lambda$. In this case we will separate the integral.

$$\frac{1}{2}\left(\int_0^L q\lambda^{q-1}\mu\{x\in Q_0\,|\,Nf(x)>\lambda\}\,d\lambda\right)^{\frac{1}{q}}$$
$$= \left(\int_0^{L/2} q\lambda^{q-1}\mu\{x\in Q_0\,|\,Nf(x)>2\lambda\}\,d\lambda\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&\le \mu(Q_0)^{\frac{1}{q}}\,\Lambda + \left(\int_\Lambda^{L/2} q\lambda^{q-1}\mu\{x\in Q_0 \,|\, Nf(x)>2\lambda\}\,d\lambda\right)^{\frac{1}{q}}\\
&\le \mu(Q_0)^{\frac{1}{q}}\,\Lambda + \left(\int_\Lambda^{L/2} q\lambda^{q-1}\mu\{x\in Q_0 \,|\, Nf(x)>2\lambda,\, M^\sharp f(x)\le\delta\lambda\}\,d\lambda\right)^{\frac{1}{q}}\\
&+ \left(\int_\Lambda^{L/2} q\lambda^{q-1}\mu\{x\in Q_0 \,|\, Nf(x)>2\lambda,\, M^\sharp f(x)>\delta\lambda\}\,d\lambda\right)^{\frac{1}{q}}.
\end{aligned}
$$

Using Lemma 3.1 with $\varepsilon=1$ and $\delta>0$ sufficiently small, we see that

$$
\begin{aligned}
&\frac{1}{2}\left(\int_0^L q\lambda^{q-1}\mu\{x\in Q_0 \,|\, Nf(x)>\lambda\}\,d\lambda\right)^{\frac{1}{q}}\\
&\le \mu(Q_0)^{\frac{1}{q}}\,\Lambda + \left(C_1\delta\int_0^L q\lambda^{q-1}\mu\{x\in 8Q_0 \,|\, Nf(x)>\lambda\}\,d\lambda\right)^{\frac{1}{q}}\\
&\quad + \left(\int_0^\infty q\lambda^{q-1}\mu\{x\in Q_0 \,|\, M^\sharp f(x)>\delta\lambda\}\,d\lambda\right)^{\frac{1}{q}}\\
&= \mu(Q_0)^{\frac{1}{q}}\Lambda + \left(C_1\delta\int_0^L q\lambda^{q-1}\mu\{x\in 8Q_0 \,|\, Nf(x)>\lambda\}\,d\lambda\right)^{\frac{1}{q}}\\
&\quad + \frac{1}{\delta}\left(\int_{Q_0}\left(M^\sharp f\right)^q\,d\mu\right)^{\frac{1}{q}}.
\end{aligned}
$$

Hence, we have obtained the following estimate

$$
\begin{aligned}
&\frac{1}{2}\left(\int_0^L q\lambda^{q-1}\mu\{x\in Q_0 \,|\, Nf(x)>\lambda\}\,d\lambda\right)^{\frac{1}{q}}\\
&\le \mu(Q_0)^{\frac{1}{q}}\,\Lambda + C_1{}^{\frac{1}{q}}\delta^{\frac{1}{q}}\left(\int_0^L q\lambda^{q-1}\mu\{x\in 8Q_0 \,|\, Nf(x)>\lambda\}\,d\lambda\right)^{\frac{1}{q}}\\
&\quad + \tfrac{1}{\delta}\left(\int_{Q_0}\left(M^\sharp f\right)^q\,d\mu\right)^{\frac{1}{q}}.
\end{aligned}
\tag{10}
$$

Divide equally $8Q_0$ into the 16^d cubes $Q_1, Q_2, \ldots, Q_{16^d}$ with their sidelength equal to $\ell(Q_0)/2$. Noting that $32Q_j \subset 32Q_0$, we have

$$
\mu(32Q_0)^{\frac{1}{p}-\frac{1}{q}} \le \mu(32Q_j)^{\frac{1}{p}-\frac{1}{q}}, \text{ for all } j=1,2,\ldots,16^d. \tag{11}
$$

We have also

$$
\mu(32Q_0)^{\frac{1}{p}-\frac{1}{q}} \le \mu(Q_0)^{\frac{1}{p}-\frac{1}{q}}. \tag{12}
$$

Multiplying $\mu(32Q_0)^{\frac{1}{p}-\frac{1}{q}}$ to the both sides of (10) and using (11) and (12), we obtain that

$$\frac{1}{2}\,\mu(32Q_0)^{\frac{1}{p}-\frac{1}{q}}\left(\int_0^L q\lambda^{q-1}\mu\{x\in Q_0\,|\,Nf(x)>\lambda\}\,d\lambda\right)^{\frac{1}{q}}$$
$$\le\ \mu(Q_0)^{\frac{1}{p}}\,\Lambda+16^d(C_1\delta)^{\frac{1}{q}}\|Nf\,|\,\mathcal{M}^p_q(32,\mu)\|_L+\delta^{-1}\|M^\sharp f\,|\,\mathcal{M}^p_q(32,\mu)\|.$$

Lastly, as we have seen in (7), the estimate $\mu(Q_0)^{\frac{1}{p}}\,\Lambda\le C\,\|f\,|\,\mathcal{M}^p_1(\mu)\|$ holds.

Choosing δ small enough, we obtain that

$$\|Nf\,|\,\mathcal{M}^p_q(32,\mu)\|_L\le C\,\left(\|M^\sharp f\,|\,\mathcal{M}^p_q(32,\mu)\|+\|f\,|\,\mathcal{M}^p_1(\mu)\|\right).$$

This proves the theorem. ■

3.2 Proof of Theorem 1.4

In this section we prove Theorem 1.4. Let $R\in\mathcal{Q}(\mu)$. We shall estimate $\mu(2R)^{\frac{1}{p}-1}\int_R|f|\,d\mu$. It follows by Lemma 2.1 and Hölder's inequality that

$$\begin{aligned}
&\mu(2R)^{\frac{1}{p}-1}\int_R|f|\,d\mu\\
&\le\ \mu(2R)^{\frac{1}{p}-1}\int_{\frac{3}{2}R}\left(\frac{1}{\mu\left(\frac{3}{2}R\right)}\int_R|f(y)-m_{R^*}(f)|\,d\mu(y)+|m_{R^*}(f)-m_{(2R)^*}(f)|\right)d\mu\\
&\quad+\mu(2R)^{\frac{1}{p}}\,|m_{(2R)^*}(f)|\\
&\le C\,\mu(2R)^{\frac{1}{p}-1}\int_{\frac{3}{2}R}\left(\frac{1}{\mu\left(\frac{3}{2}R\right)}\int_R|f(y)-m_{R^*}(f)|d\mu(y)+\frac{|m_{R^*}(f)-m_{(2R)^*}(f)|}{K_{R^*,(2R)^*}}\right)d\mu\\
&\quad+\mu(2R)^{\frac{1}{p}}\,|m_{(2R)^*}(f)|\\
&\le C\,\|M^\sharp f\,|\,\mathcal{M}^p_q(4/3,\mu)\|+\mu(2R)^{\frac{1}{p}}\,|m_{(2R)^*}(f)|.
\end{aligned}$$

So we shall concentrate ourselves on estimating the second term :

$$\mu(2R)^{\frac{1}{p}}\,|m_{(2R)^*}(f)|. \tag{13}$$

We choose a cube inductively. Let $R_0=(2R)^*$ and $R_j=(2R_{j-1})^*$, $j=1,2,\ldots$. Let d be the distance between the center of I_0 and that of R. We select $K_0\in\mathbf{N}$ so big that $\ell(R_{K_0})\ge 2d$ and there exists some I_{K_1} such that $R_{K_0}\subset I_{K_1}$, $R_{K_0+1}\not\subset I_{K_1}$ and

$$\mu(2R)^{\frac{1}{p}}|m_{I_{K_1}}(f)|\le\|M^\sharp f\,|\,\mathcal{M}^p_q(\mu)\|.$$

This is possible since f is not identically equal to a nonzero constant by assumption. Then a simple geometric observation shows that $R_{K_0} \subset I_{K_1} \subset R_{K_0+3}$, and hence,

$$K_{R_{K_0},I_{K_1}} \leq K_{R_{K_0},R_{K_0+3}} \leq C. \tag{14}$$

We put for $i = 0, 1, \ldots$

$$J_i := \left\{ j \in \mathbf{N}_0 \cap [0, K_0] \,|\, 2^i \mu(2R) \leq \mu(R_j) < 2^{i+1}\mu(2R) \right\}.$$

Discarding all empty sets, we obtain a finite sequence of nonnegative integers $0 \leq i_1 < i_2 < \ldots < i_{K_2}$ such that

$$J_{i_k} \neq \emptyset, \quad k = 1, 2, \ldots, K_2 \text{ and that } J_l = \emptyset \text{ if } l \notin \{i_1, \ldots, i_{K_2}\}.$$

Set $a(i_k) := \min J_{i_k}$ and $b(i_k) := \max J_{i_k}$. Note that $b(i_{K_2}) = K_0$ by the definition. From Lemma 2.1 we see that

$$K_{R_{a(i_k)},R_{b(i_k)}} \leq C \text{ and } K_{R_{b(i_k)},R_{a(i_{k+1})}} \leq C.$$

This implies that

$$\begin{aligned}
&\mu(2R)^{\frac{1}{p}} \left(|m_{R_{a(i_k)}}(f) - m_{R_{b(i_k)}}(f)| + |m_{R_{b(i_k)}}(f) - m_{R_{a(i_{k+1})}}(f)| \right) \\
&\leq C\, 2^{-\frac{i_k}{p}} \mu(R_{a(i_k)})^{\frac{1}{p}-1} \\
&\quad \times \int_{R_{a(i_k)}} \left(\frac{|m_{R_{a(i_k)}}(f) - m_{R_{b(i_k)}}(f)|}{K_{R_{a(i_k)},R_{b(i_k)}}} + \frac{|m_{R_{b(i_k)}}(f) - m_{R_{a(i_{k+1})}}(f)|}{K_{R_{b(i_k)},R_{a(i_{k+1})}}} \right) d\mu \\
&\leq C\, 2^{-\frac{i_k}{p}} \mu(2R_{a(i_k)})^{\frac{1}{p}-\frac{1}{q}} \left(\int_{R_{a(i_k)}} (M^\sharp f)^q \, d\mu \right)^{\frac{1}{q}} \\
&\leq C\, 2^{-\frac{i_k}{p}} \|M^\sharp f \,|\, \mathcal{M}^p_q(\mu)\|.
\end{aligned}$$

From (14) we also have

$$\begin{aligned}
\mu(2R)^{\frac{1}{p}} &\left(|m_{R_{a(i_{K_2})}}(f) - m_{R_{K_0}}(f)| + |m_{R_{K_0}}(f) - m_{I_{K_1}}(f)| \right) \\
&\leq C\, 2^{-\frac{i_{K_2}}{p}} \|M^\sharp f \,|\, \mathcal{M}^p_q(\mu)\|.
\end{aligned}$$

Using the triangle inequality to (13), we deduce

$$
\begin{aligned}
&\mu(2R)^{\frac{1}{p}}\,|m_{(2R)^*}(f)| \\
&\le\ \mu(2R)^{\frac{1}{p}}\sum_{k=1}^{K_2-1}\Big(|m_{R_{a(i_k)}}(f)-m_{R_{b(i_k)}}(f)|+|m_{R_{b(i_k)}}(f)-m_{R_{a(i_{k+1})}}(f)|\Big)\\
&\quad+\mu(2R)^{\frac{1}{p}}\left\{\Big(|m_{R_{a(i_{K_2})}}(f)-m_{R_{K_0}}(f)|+|m_{R_{K_0}}(f)-m_{I_{K_1}}(f)|\Big)+|m_{I_{K_1}}(f)|\right\}\\
&\le\ C\sum_{k=1}^{K_2}\left(2^{-\frac{i_k}{p}}\,\|M^{\sharp}f\,|\,\mathcal{M}^p_q(\mu)\|\right)+\mu(2R)^{\frac{1}{p}}\,|m_{I_{K_1}}(f)|.
\end{aligned}
$$

Note that

$$\sum_{k=1}^{K_2}2^{-\frac{i_k}{p}}\le C,$$

since $0\le i_1<i_2<\ldots$. This and the above inequalities imply the desired inequality:

$$\mu(2R)^{\frac{1}{p}}\,|m_{(2R)^*}(f)|\le C\,\|M^{\sharp}f\,|\,\mathcal{M}^p_q(\mu)\|.$$

4. An Application to Commutators

Definitions and Known Results. In this section we list some definitions and known results needed to state our commutator theorems.

Definition 4.1. ([10] p. 466) The singular integral operator T is a bounded linear operator on $L^2(\mu)$ with a kernel function K that satisfies the following three properties :

(1) For some appropriate constant $C>0$, we have

$$|K(x,y)|\le\frac{C}{|x-y|^n}, \tag{15}$$

where n is a constant in the growth condition (1).

(2) There exist constants $\varepsilon>0$ and $C>0$ such that

$$|K(x,y)-K(z,y)|+|K(y,x)-K(y,z)|\le C\,\frac{|x-z|^{\varepsilon}}{|x-y|^{n+\varepsilon}}\ \text{if}\ |x-y|>2|x-z|. \tag{16}$$

(3) If f is a bounded measurable function with a compact support, then we have

$$Tf(x)=\int_{\mathbf{R}^d}K(x,y)f(y)\,d\mu(y)\ \text{for all}\ x\notin\operatorname{supp}(f). \tag{17}$$

Definition 4.2. [4, Definition 3.1] For α with $0 < \alpha < n$, we define a fractional integral operator I_α by

$$I_\alpha f(x) := \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} \, d\mu(y),$$

where n is a constant in the growth condition (1).

It is well-known that T is a bounded operator on $L^p(\mu)$ if $1 < p < \infty$ (see [10]) and I_α is a bounded operator from $L^p(\mu)$ to $L^q(\mu)$ if $1 < p < q \leq \infty$ and $1/q = 1/p - \alpha/n$ (see [4]). In [13] it is also proved that T is a bounded operator on $\mathcal{M}^p_q(\mu)$ if $1 < q \leq p < \infty$ and I_α is a bounded operator from $\mathcal{M}^p_q(\mu)$ to $\mathcal{M}^s_t(\mu)$ if

$$1 < q \leq p < \infty, \quad 1 < t \leq s < \infty, \quad 1/s = 1/p - \alpha/n \text{ and } t/s = q/p. \tag{18}$$

Now we recell the commutator results for these operators.

Proposition 4.3. [14, Theorem 9.1] *Suppose that a ∈RBMO. Let $1 < p < \infty$ and T be a singular integral operator with associated kernel K. Then*

$$[a, T]f(x) := \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} (a(x) - a(y)) K(x,y) f(y) \, d\mu(y)$$

defines a bounded operator on $L^p(\mu)$.

Proposition 4.4. [1, Theorem 1][*Suppose that a ∈RBMO. If $0 < \alpha < n$, $1 < p < q \leq \infty$ and $1/q = 1/p - \alpha/n$, then*

$$[a, I_\alpha]f(x) := \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{(a(x) - a(y))}{|x-y|^{n-\alpha}} f(y) \, d\mu(y)$$

defines a bounded operator from $L^p(\mu)$ to $L^q(\mu)$.

Main Results. In this section we shall extend Propositions 4.3 and 4.4 to the Morrey spaces.

Theorem 4.5. *Suppose that a ∈RBMO. Let $1 < q \leq p < \infty$ and T be a singular integral operator with associated kernel K. Then*

$$[a, T]f(x) := \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} (a(x) - a(y)) K(x,y) f(y) \, d\mu(y)$$

can be extended to a bounded operator on $\mathcal{M}^p_q(\mu)$.

Theorem 4.6. *Suppose that* $a \in$*RBMO. If the parameters satisfy (18), then*

$$[a, I_\alpha]f(x) := \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{(a(x) - a(y))}{|x-y|^{n-\alpha}} f(y)\, d\mu(y)$$

can be extended to a bounded operator from $\mathcal{M}^p_q(\mu)$ *to* $\mathcal{M}^s_t(\mu)$.

In Appendix we consider another type of commutators. The proof of Theorem 4.5 will be somehow easier than that of Theorem 4.6. Firstly we will prove Theorem 4.6 and we add a remark to the proof of Theorem 4.5.

To prove the theorem we need the following pointwise estimate of $[a, I_\alpha]f$. The definition and the estimate are due to Chen and Sawyer [1].

Definition 4.7. [1, p. 1291] Let $0 \le \alpha < n$ and $Q \subset R \in \mathcal{Q}(\mu)$. Then we set

$$K^{(\alpha)}_{Q,R} := 1 + \sum_{j=1}^{N_{Q,R}} \left(\frac{\mu(2^j Q)}{\ell(2^j Q)^n} \right)^{1-\alpha/n}$$

and we define the corresponding sharp maximal operator by

$$\begin{aligned} M^{\sharp,\alpha} f(x) := &\sup_{x \in Q \in \mathcal{Q}(\mu)} \frac{1}{\mu\left(\frac{3}{2}Q\right)} \int_Q |f(y) - m_{Q^*}(f)|\, d\mu(y) \\ &+ \sup_{\substack{x \in Q \subset R \\ Q, R \in \mathcal{Q}(\mu, 2)}} \frac{|m_Q(f) - m_R(f)|}{K^{(\alpha)}_{Q,R}}. \end{aligned}$$

Let us remark that all the theorems on $M^\sharp$, especially Theorem 1.3, are still available even if we replace $M^\sharp$ by $M^{\sharp,\alpha}$.

Lemma 4.8. [1, p. 1293] *We have for almost* μ*-a.e.* $x \in \operatorname{supp}(\mu)$

$$(M^{\sharp,\alpha}[a, I_\alpha]f)(x) \le C\, \|a\|_* \left(M^\alpha_{(\frac{9}{8})} f(x) + (M_{(\frac{3}{2})} I_\alpha f)(x) + I_\alpha |f|(x) \right).$$

Proof of Theorem 4.6. Let $u > 1$ be an auxiliary constant such that $1/u = 1/q - \alpha/n$. Applying Theorem 1.3 with the boundedness of I_α and Lemmas 2.4 and 4.8, we have only to prove

$$\|[a, I_\alpha]f \,|\, \mathcal{M}^s_1(\mu)\| \le C\, \|f \,|\, \mathcal{M}^p_q(\mu)\|.$$

This can be reduced by (3) to

$$\|[a, I_\alpha]f \,|\, \mathcal{M}^s_u(\mu)\| \le C\, \|f \,|\, \mathcal{M}^p_q(\mu)\|.$$

Let us remark that $u \le t$.

Fix a cube $Q \in \mathcal{Q}(\mu)$. We decompose $f \in \mathcal{M}^p_q(\mu)$ according to $2Q$: We put $f_1 = \chi_{2Q} f$ and $f_2 = \chi_{(2Q)^c} f$. We shall estimate

$$\mu(4Q)^{\frac{1}{s}-\frac{1}{u}} \left(\int_Q |[a, I_\alpha] f(x)|^u \, d\mu(x) \right)^{\frac{1}{u}}.$$

Along this decomposition it suffices to estimate

$$\mu(4Q)^{\frac{1}{s}-\frac{1}{u}} \left(\int_Q |[a, I_\alpha] f_1(x)|^u \, d\mu(x) \right)^{\frac{1}{u}} \text{ and } \mu(4Q)^{\frac{1}{s}-\frac{1}{u}} \left(\int_Q |[a, I_\alpha] f_2(x)|^u \, d\mu(x) \right)^{\frac{1}{u}}$$

respectively.

The estimate of the first term is over by Proposition 4.4 :

$$\begin{aligned}
&\mu(4Q)^{\frac{1}{s}-\frac{1}{u}} \left(\int_Q |[a, I_\alpha] f_1(x)|^u \, d\mu(x) \right)^{\frac{1}{u}} \\
&\quad \le \mu(4Q)^{\frac{1}{s}-\frac{1}{u}} \left(\int_{\mathbf{R}^d} |[a, I_\alpha] f_1(x)|^u \, d\mu(x) \right)^{\frac{1}{u}} \\
&\quad \le C \, \mu(4Q)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{2Q} |f|^q \, d\mu \right)^{\frac{1}{q}} \\
&\quad \le C \, \| f \mid \mathcal{M}^p_q(\mu) \|.
\end{aligned}$$

So, we shall estimate the second term. We see that for $x \in Q$

$$\begin{aligned}
&|[a, I_\alpha] f_2(x)| \\
&\le \int_{\mathbf{R}^d \setminus 2Q} \frac{|(a(x) - a(y)) f(y)|}{|x-y|^{n-\alpha}} \, d\mu(y) \\
&\le C \left(\int_{\mathbf{R}^d \setminus 2Q} \frac{|(a(x) - m_{Q^*}(a)) \, f(y)|}{|z_Q - y|^{n-\alpha}} \, d\mu(y) \right. \\
&\qquad \left. + \int_{\mathbf{R}^d \setminus 2Q} \frac{|(m_{Q^*}(a) - a(y)) \, f(y)|}{|z_Q - y|^{n-\alpha}} \, d\mu(y) \right),
\end{aligned}$$

where z_Q is the center of Q.

The growth condition (1), Lemma 2.2 (2) and Lemma 2.3 (2) yield

$$\begin{aligned}
&\mu(4Q)^{\frac{1}{s}-\frac{1}{u}} \left(\int_Q \left(\int_{\mathbf{R}^d \setminus 2Q} \frac{|(a(x) - m_{Q^*}(a)) \, f(y)|}{|z_Q - y|^{n-\alpha}} \, d\mu(y) \right)^u \, d\mu(x) \right)^{\frac{1}{u}} \\
&= \mu(4Q)^{\frac{1}{s}-\frac{1}{u}} \left(\int_Q |(a(x) - m_{Q^*}(a))|^u \, d\mu(x) \right)^{\frac{1}{u}} \cdot \int_{\mathbf{R}^d \setminus 2Q} \frac{|f(y)|}{|z_Q - y|^{n-\alpha}} \, d\mu(y) \\
&\le C \, \|a\|_* \| f \mid \mathcal{M}^p_q(\mu) \|.
\end{aligned}$$

The growth condition and Lemma 2.2 (1) yield

$$\mu(4Q)^{\frac{1}{s}-\frac{1}{u}} \left(\int_Q \left(\int_{\mathbf{R}^d \setminus 2Q} \frac{|(m_{Q^*}(a) - a(y))\, f(y)|}{|z_Q - y|^{n-\alpha}} \, d\mu(y) \right)^u d\mu(x) \right)^{\frac{1}{u}}$$
$$\leq C\, \|a\|_* \|f \mid \mathcal{M}^p_q(\mu)\|.$$

Thus, the estimate of the second term is finished. Putting these estimates all together, we have the desired. ■

Proof of Theorem 4.5. We keep the same notation as in the previous proof. By using Proposition 4.3 and

$$|[a,T]f_2(x)| = \left| \int_{\mathbf{R}^d \setminus 2Q} (a(x) - a(y)) K(x,y) f(y)\, d\mu(y) \right|$$
$$\leq C \int_{\mathbf{R}^d \setminus 2Q} \frac{|(a(x) - a(y)) f(y)|}{|x-y|^n} \, d\mu(y),$$

which follows from (15) and (17), the proof is the same as Theorem 4.5. ■

5. Appendix

Self-improvement of Theorem 1.3. Theorem 1.3 has a self-improvement by using Theorem 1.4, if $\mu(\mathbf{R}^d) < \infty$.

Theorem 5.1. *Suppose that* $\mu(\mathbf{R}^d) < \infty$ *and that the parameters are the same as in Theorem* 1.3. *Then we have*

$$\|f \mid \mathcal{M}^p_q(\mu)\| \sim \|f \mid L^1(\mu)\| + \|M^\sharp f \mid \mathcal{M}^p_q(\mu)\|.$$

Remark 5.2. Since $\mu(\mathbf{R}^d)$ is finite, we have $L^1(\mu) \supset \mathcal{M}^p_1(\mu)$. Thus this theorem is somehow stronger than Theorem 1.3, if μ is finite measure.

Proof. All we have to prove is that

$$\|f \mid \mathcal{M}^p_q(\mu)\| \leq C \left(\|f \mid L^1(\mu)\| + \|M^\sharp f \mid \mathcal{M}^p_q(\mu)\| \right),$$

the converse inequality being trivial. So that we may assume that the right-hand side is finite. In particular we may assume that $f \in L^1$. In this case the function $f - m_{\mathbf{R}^d}(f)$ satisfies the assumption of Theorem 1.4. So that we have

$$\|(f - m_{\mathbf{R}^d}(f)) \mid \mathcal{M}^p_q(\mu)\| \leq C \|M^\sharp (f - m_{\mathbf{R}^d}(f)) \mid \mathcal{M}^p_q(\mu)\| = \|M^\sharp f \mid \mathcal{M}^p_q(\mu)\|.$$

This estimate readily yields $\|f \mid \mathcal{M}^p_q(\mu)\| \le C\left(\|f \mid L^1(\mu)\| + \|M^\sharp f \mid \mathcal{M}^p_q(\mu)\|\right)$. ■

Boundedness of Different Commutators on Morrey Space.

Finally we consider another commutator with Lipschitz function and singular integral operator T or with Lipschitz function and fractional integral operator. Shirai [12] considered a commutator with $b \in \Lambda_\gamma$ and T and proved the boundedness of $[b, T]$ with Lebesgue measure. The same proof also holds in our nonhomogeneous space. For completeness we supply the proof.

Theorem 5.3. *Assume that the parameters satisfy that*

$$1 < q \le p,\ 1 < t \le s,\ \frac{p}{q} = \frac{s}{t},\ \frac{1}{s} = \frac{1}{p} - \frac{\alpha+\gamma}{n},\ 0 < \alpha < n,\ 0 < \gamma \le 1$$

Suppose that a continuous function b satisfies

$$|b(x) - b(y)| \le C|x-y|^\gamma \tag{19}$$

for $C > 0$. Then we have $[b, I_\alpha]$ is bounded from $\mathcal{M}^p_q(\mu)$ to $\mathcal{M}^s_t(\mu)$.

Proof. In fact by the definition of the commutator $[b, I_\alpha]$ together with (19), we have

$$|[b, I_\alpha]f(x)| \le C I_{\alpha+\gamma} f(x). \tag{20}$$

Thus we have, using the boundedness of $I_{\alpha+\gamma}$, we have the desired result. ■

Theorem 5.4. *Assume that the parameters satisfy that*

$$1 < q \le p,\ 1 < t \le s,\ \frac{p}{q} = \frac{s}{t},\ \frac{1}{s} = \frac{1}{p} - \frac{\gamma}{n},\ 0 < \gamma \le 1.$$

Suppose that b is the same function as in the previous theorem. Then $[b, T]$ is bounded from $\mathcal{M}^p_q(\mu)$ to $\mathcal{M}^s_t(\mu)$.

Proof. Similar to the previous theorem by virtue of the boundedness of I_γ. ■

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