TAIWANESE JOURNAL OF MATHEMATICS
Vol. 11, No. 4, pp. 1057-1062, September 2007
This paper is available online at http://www.math.nthu.edu.tw/tjm/

ON THE FUZZY SHEAF OF THE FUNDAMENTAL GROUPS

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Abstract. Let X be a fuzzy path connected space and H_{a_λ} be the fundamental group of X based for any $a_\lambda \in X$, that is $\pi_1(X, a_\lambda)$. Constructing the fuzzy sheaf of the fundamental groups of X, it is shown that there is a covariant functor from the category of fuzzy path connected topological spaces and fuzzy continuous mappings to the category of fuzzy sheaves and fuzzy sheaf homomorphisms.

1. Introduction

Let X be a set and I the unit interval $[0, 1]$. A fuzzy set X is characterized by a membership function μ_A which associates with each point $x \in X$ its "grade of membership" $\mu_A(x) \in I$.

Definition 1. A fuzzy point in X is a fuzzy set with membership function μ_{a_λ} defined by

$$\mu_{a_\lambda}(x) = \begin{cases} \lambda, \ x = a \\ 0, \ \text{otherwise} \end{cases}$$

for all $x \in X$. Where $0 < \lambda \leq 1$.

We denote by k_λ the fuzzy set in X with the constant membership function $\mu_{k_\lambda}(x) = \lambda$ for all $x \in X$ [3].

Definition 2. A fuzzy topology on a set X is a family τ of fuzzy sets in X which satisfies the following conditions:

(i) $k_0, k_1 \in \tau$

(ii) If $A, B \in \tau$, then $A \cap B \in \tau$

Received July 19, 2005, accepted December 19, 2005.
Communicated by Bor-Luh Lin.
2000 *Mathematics Subject Classification*: 54A40.
Key words and phrases: Fuzzy path connectedness, Fuzzy sheaf, Fuzzy sheaf homomorphism.

(iii) If $A_j \in \tau$ for all $j \in J$, then $\bigcup_{j \in J} A_j \in \tau$ $[1]$.

The pair (X, τ) is called a fuzzy topological space. Every member of τ is called an open fuzzy set. The complement of an open fuzzy set is called a closed fuzzy set.

Definition 3. Let τ be a fuzzy topology on a set X. A subfamily $\mathcal{B}$ of τ is called a base for τ if each member of τ can be expressed as the union of members of $\mathcal{B}$.

Definition 4. Let (X, τ_1), (Y, τ_2) be two fuzzy topological spaces. A mapping f of (X, τ_1) into (Y, τ_2) is fuzzy continuous iff for each open fuzzy set V in τ_2 the inverse image $f^{-1}(V)$ is in τ_1. Conversely, f is fuzzy open iff for each open fuzzy set U in τ_1, the image $f(U)$ is in τ_2 $[2]$.

Definition 5. A bijective mapping f of fuzzy topological space (X, τ_1) onto a fuzzy topological space (Y, τ_2) is called a fuzzy homeomorphism if it is fuzzy continuous and fuzzy open.

Definition 6. A mapping f of fuzzy topological space (X, τ_1) into a fuzzy topological space (Y, τ_2) is called a fuzzy sheaf if it is a locally fuzzy homeomorphism.

Definition 7. Let (X, τ) be a fuzzy topological space. If $\alpha : (I, \widetilde{\varepsilon}) \to (X, \tau)$ is a fuzzy continuous function and the fuzzy set A is connected in $(I, \widetilde{\varepsilon}_I)$ with $A(0) > 0$ and $A(1) > 0$, then the fuzzy set $\alpha(A)$ in (X, τ) is called a fuzzy path in (X, τ).

The fuzzy point $(\alpha(0))_{A(0)} = \alpha(0_{A(0)})$ and $(\alpha(1))_{A(1)} = \alpha(1_{A(1)})$ are called the initial point and the terminal point of the fuzzy path $\alpha(A)$, respectively $[5]$.

Definition 8. Let a_λ be a fuzzy point in a fuzzy topological space (X, τ). A fuzzy path $\alpha(A)$ in (X, τ) is called a fuzzy loop in (X, τ) based at a_λ if $\alpha(0_{A(0)}) = \alpha(1_{A(1)}) = a_\lambda$.

Definition 9. Let F be a fuzzy set in a fuzzy topological space (X, τ). If for any two fuzzy points a_λ and b_μ in F, there is a fuzzy path from a_λ to b_μ contained in F, then F is said to be fuzzy path connected in (X, τ).

If $F = X$ in the above definition, we call (X, τ) a fuzzy path connected.

2. The Fuzzy Sheaf of the Fundamental Group of Fuzzy Topological Space

Let X be a fuzzy path connected topological space and H_{a_λ} be the fundamental group of X based for any $a_\lambda \in X$, that is $H_{a_\lambda} = \pi_1(X, a_\lambda)$ $[4]$.

Let $X = (X, x_p)$ be a pointed fuzzy topological space, for an arbitrary fixed fuzzy point $x_p \in X$. Let us denote the disjoint union of all fundamental groups obtained for each $a_\lambda \in X$, by H, i.e., $H = \bigvee_{a_\lambda \in X} H_{a_\lambda}$.

H is a set over X and the mapping $\psi : H \rightarrow X$ defined by $\psi(\sigma_{a_\lambda}) = \psi([\alpha(A)]_{a_\lambda}) = a_\lambda$ for any $\sigma_{a_\lambda} = [\alpha(A)]_{a_\lambda} \in H_{a_\lambda} \subset H$ is onto.

Let $W \subset X$ be an open fuzzy set. Define a mapping $s : W \rightarrow H$ such that $s(a_\lambda) = [\gamma^{-1}(H) * \alpha(A) * \gamma(G)]_{a_\lambda}$ for each $a_\lambda \in W$. Where $[\alpha(A)]_{x_p} \in H_{x_p}$ is any element and $[\gamma(G)]$ is an arbitrary fixed fuzzy homotopy class defines an homomorphism between H_{a_λ} and H_{x_p}. Then, the change of s depends on only the change of $\sigma_{x_p} = [\alpha(A)]_{x_p}$. Furthermore $\psi \circ s = 1_W$. Let us denote the totality of the mappings s defined on W by $\Gamma(W, H)$.

If B is a fuzzy base for X, then $B^* = \{s(W) : W \in B,\ s \in \Gamma(W, H)\}$ is a fuzzy base for H. The mappings ψ and s are fuzzy continuous in this topology. Moreover ψ is a locally fuzzy topological mapping. Then (H, ψ) is a fuzzy sheaf over X. (H, ψ) or only H is called ”The fuzzy sheaf of the fundamental groups” over X.

For any open fuzzy set $W \subset X$, an element s of $\Gamma(W, H)$ is called a fuzzy section of the fuzzy sheaf H over W. Furthermore, the group $H_{a_\lambda} = \pi_1(X, a_\lambda)$ is called the stalk of the fuzzy sheaf H for any $a_\lambda \in X$.

3. Characterizations

Let X_1, X_2 be fuzzy path connected topological spaces and H_1, H_2 be the corresponding fuzzy sheaves, respectively.

We begin by giving the following definitions.

Definition 10. Let $f^* : H_1 \rightarrow H_2$ be a mapping. If f^* is fuzzy continuous, a homomorphism on each stalk of H_1 and maps every stalk of H_1 into a stalk of H_2, then it is called a fuzzy sheaf homomorphism.

Let $f : X_1 \rightarrow X_2$ be a fuzzy continuous mapping and $f^* : H_1 \rightarrow H_2$ be a fuzzy sheaf homomorphism. If $f^*((H_1)_{a_{\lambda_1}}) \subset (H_2)_{f(a_{\lambda_1})}$ for each $a_{\lambda_1} \in X_1$, then f^* is called a stalk preserving fuzzy sheaf homomorphism with respect to f.

Definition 11. Let $f^* : H_1 \rightarrow H_2$ be a fuzzy sheaf homomorphism. If f^* is also a bijection, then f^* is called a fuzzy sheaf isomorphism.

Theorem 1. *Let $f : X_1 \rightarrow X_2$ be a fuzzy continuous mapping. Then there is a stalk preserving fuzzy sheaf homomorphism $f^* : H_1 \rightarrow H_2$ with respect to f.*

Proof. Let $a_{\lambda_1} \in X_1$ be any point and $\alpha(A)$ be a fuzzy loop based at a_{λ_1}. Then $(f \circ \alpha)(A)$ is a fuzzy loop based at $f(a_{\lambda_1})$ and $[(f \circ \alpha)(A)] \in H_{f(a_{\lambda_1})}$.

On the other hand, if $\alpha_1(A_1)$ and $\alpha_2(A_2)$ are fuzzy loops based at a_{λ_1} in X_1 such that $\alpha_1(A_1) \simeq_{a_{\lambda_1}} \alpha_2(A_2)$ then $(f \circ \alpha_1)(A_1) \simeq_{f(a_{\lambda_1})} (f \circ \alpha_2)(A_2)$. Thus we can define the mapping $f^* : H_1 \to H_2$ such that

$$f^*(\sigma_{a_{\lambda_1}}) = f^*([\alpha(A)]_{a_{\lambda_1}}) = [(f \circ \alpha)(A)]_{f(a_{\lambda_1})}$$

for any $[\alpha(A)]_{a_{\lambda_1}} = \sigma_{a_{\lambda_1}} \in (H_1)_{a_{\lambda_1}} \subset H_1$.

(1) f^* is fuzzy continuous.

Let $U_2 \subset f^*(H_1) \subset H_2$, be an open fuzzy set. Without loss of generality, we assume that $U_2 = s^2(W_2)$ where $W_2 \subset X_2$ is an open fuzzy set and $s^2 \in \Gamma(W_2, H_2)$. Thus $\psi_2(U_2) = \psi_2(s^2(W_2)) = W_2$. Since f is fuzzy continuous, $f^{-1}(W_2) = W_1 \subset X_1$ is an open fuzzy set. Now let $\sigma_{f(a_{\lambda_1})} \in U_2$ be an element. Then there exists at least one element $\sigma_{a_{\lambda_1}} \in U_1 = (f^*)^{-1}(U_2)$ such that $f^*(\sigma_{a_{\lambda_1}}) = f^*([\alpha(A)]_{a_{\lambda_1}}) = [(f \circ \alpha)(A)]_{f(a_{\lambda_1})}$.

Since $\psi_1(\sigma_{a_{\lambda_1}}) = a_{\lambda_1} \in W_1$, there is a fuzzy section $s^1 \in \Gamma(W_1, H_1)$ such that $s^1(a_{\lambda_1}) = \sigma_{a_{\lambda_1}}$ and $s^1(W_1) \subset H_1$ is an open fuzzy set. Also $s^1(W_1) \subset U_1$. It is easily seen that $U_1 = \bigcup_{i \in I} s_i^1(W_1)$.

Therefore, $U_1 \subset H_1$ is an open fuzzy set, that is f^* is a fuzzy continuous mapping.

(2) f^* preserves the stalk with respect to f.

In fact, for any $\sigma_{a_{\lambda_1}} = [\alpha(A)]_{a_{\lambda_1}} \in H_1$

$$\begin{aligned}
(f \circ \psi_1)([\alpha(A)]_{a_{\lambda_1}}) &= f(\psi_1([\alpha(A)]_{a_{\lambda_1}})) = f(a_{\lambda_1}) \\
(\psi_2 \circ f^*)([\alpha(A)]_{a_{\lambda_1}}) &= \psi_2(f^*([\alpha(A)]_{a_{\lambda_1}})) \\
&= \psi_2([(f \circ \alpha)(A)]_{f(a_{\lambda_1})}) = f(a_{\lambda_1})
\end{aligned}$$

(3) For every $a_{\lambda_1} \in X_1$, the map $f^*|_{(H_1)_{a_{\lambda_1}}} : (H_1)_{a_{\lambda_1}} \to (H_2)_{f(a_{\lambda_1})}$ is homomorphism.

In fact, if $\alpha_1(A_1), \beta_1(B_1)$ are fuzzy loops based at $a_{\lambda_1} \in X_1$ and $(f \circ \alpha_1)(A_1)$, $(f \circ \beta_1)$ (B_1) are the corresponding fuzzy loops based at $f(a_{\lambda_1}) \in X_2$,

$$\begin{aligned}
\gamma(G) &= (\alpha_1 * \beta_1)(G) = \alpha_1(A_1) * \beta_1(B_1) \\
&= \begin{cases} \alpha_1((2t)_{A_1(2t)}) & , 0 \le t \le \frac{1}{2} \\ \beta_1((2t-1)_{B_1(2t-1)}) & , \frac{1}{2} \le t \le 1 \end{cases}
\end{aligned}$$

and

$$(f\circ\gamma)(G)=\begin{cases} f(\alpha_1((2t)_{A_1(2t)})) & ,0\le t\le \frac{1}{2} \\ f(\beta_1((2t-1)_{B_1(2t-1)})) & ,\frac{1}{2}\le t\le 1 \end{cases}.$$

That is $(f\circ\gamma)(G)=(f\circ\alpha_1)(A_1)*(f\circ\beta_1)(B_1)$. Hence

$$\begin{aligned} f^*([\alpha_1(A_1)]_{a_{\lambda_1}}).f^*([\beta_1(B_1)]_{a_{\lambda_1}}) &= [(f\circ\alpha_1)(A_1)]_{f(a_{\lambda_1})}.[(f\circ\beta_1)(B_1)]_{f(a_{\lambda_1})} \\ &= [(f\circ\alpha_1)(A_1)*(f\circ\beta_1)(B_1)]_{f(a_{\lambda_1})} \\ &= [f\circ(\alpha_1(A_1)*\beta_1(B_1))]_{f(a_{\lambda_1})} \\ &= f^*([\alpha_1(A_1)*\beta_1(B_1)]_{a_{\lambda_1}}). \end{aligned}$$ ■

Now, let C be the category of fuzzy path connected topological spaces and fuzzy continuous mappings and D be the category of fuzzy sheaves and fuzzy sheaf homomorphisms. Let us define a mapping $F:C\to D$ as follows:

$$F(f)=f^*:H_1\to H_2$$

for any fuzzy continuous mapping $f:X_1\to X_2$. Then

(1) If $f=1_X$, then $F(1_X)=1_{F(X)}$, since

$$(1_X)^*([\alpha(A)]_{a_\lambda})=[(1_X\circ\alpha)(A)]_{a_\lambda}=[\alpha(A)]_{a_\lambda}$$

for any $\sigma_{a_\lambda}=[\alpha(A)]_{a_\lambda}\in H_{a_\lambda}$.

(2) If $f_1:X_1\to X_2$ and $f_2:X_2\to X_3$ are any two fuzzy continuous mappings, then $f_2\circ f_1=f_2f_1:X_1\to X_3$ is also a fuzzy continuous mapping and

$$F(f_2f_1)=(f_2f_1)^*:H_1\to H_3.$$

However, $(f_2f_1)^*([\alpha(A)]_{a_\lambda})=[(f_2f_1\circ\alpha)(A)]_{(f_2f_1)(a_\lambda)}$ for any $[\alpha(A)]_{a_\lambda}\in H_1$. Since

$$(f_2f_1\circ\alpha)(A)\simeq_{f(a_\lambda)} f_2((f_1\circ\alpha)(A)),$$

it can be written that

$$\begin{aligned} [(f_2f_1\circ\alpha)(A)]_{(f_2f_1)(a_\lambda)} &= [f_2((f_1\circ\alpha)(A))]_{(f_2f_1)(a_\lambda)} \\ &= f_2^*[(f_1\circ\alpha)(A)]_{f_1(a_\lambda)} \\ &= (f_2^*\circ f_1^*)([\alpha(A)]_{a_\lambda}). \end{aligned}$$

We then have:

Theorem 2. *There is a covariant functor from the category of fuzzy path connected topological spaces and fuzzy continuous mappings to the category of fuzzy sheaves and fuzzy sheaf homomorphisms.*

Let $f : X_1 \to X_2$ be a fuzzy topological mapping, then there exists the fuzzy continuous mapping $f^{-1} : X_2 \to X_1$ such that $ff^{-1} = 1_{X_2}$, $f^{-1}f = 1_{X_1}$.

From Theorem 1, there are the mappings $(f^{-1})^* : H_2 \to H_1$, $(ff^{-1})^* = (1_{X_2})^* : H_2 \to H_2$, $(f^{-1}f)^* = (1_{X_1})^* : H_1 \to H_1$. From Theorem 2, $(ff^{-1})^* = f^*(f^{-1})^* = 1_{F(X_2)}$, $(f^{-1}f)^* = (f^{-1})^* f^* = 1_{F(X_1)}$. Hence $(f^{-1})^* = (f^*)^{-1}$. Thus f^* is a fuzzy sheaf isomorphism.

Therefore, We can give the following corollary.

Corollary 3. *Let $f : X_1 \to X_2$ be a fuzzy topological mapping. Then the corresponding fuzzy sheaves H_1 and H_2 are fuzzy isomorphic.*

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