TAIWANESE JOURNAL OF MATHEMATICS
Vol. 11, No. 4, pp. 1019-1035, September 2007
This paper is available online at http://www.math.nthu.edu.tw/tjm/

# ANNIHILATORS OF PRINCIPAL IDEALS IN THE EXTERIOR ALGEBRA

Cemal Koç and Songül Esin

Dedicated to the memory of Professor Masatoshi Ikeda.

**Abstract.** In this paper we describe annihilators of principal ideals of exterior algebras. For odd elements we establish formulae for dimensions of their principal ideals and their annihilators. For even elements we exhibit (multiplicative) generators for annihilator ideals.

## 1. INTRODUCTION

Given a form $\mu$ and a form $\omega$ the determination of whether $\omega$ can be factorized as $\omega = \mu \wedge \tau$ by means of a number of exterior equations is proved to be a significant factorization problem in differential geometry (see Section 2.4 in [1]). Motivated by this factorization problem, in [1] I. Dibag determined annihilators of 2-vectors, by exhibiting its generators (as ideal). This is accomplished by setting up a duality between left and right ideals of the exterior algebra, that is to say by using Frobeniusean property of exterior algebras. Our motivation for this paper is purely algebraic, our objective is to extend this result to arbitrary elements of the exterior algebra. Throughout the text we fix a finite dimensional vector space $V$ over a base field $F$, its exterior algebra $E = E(V)$ in which the multiplication is denoted in the ordinary form $ab$ in place of the common notation $a \wedge b$, and we use the following notations:

$$\begin{aligned}
S \quad &:= \{1, ..., s\} \\
M_i \quad &:= \{x_{i1}, ..., x_{in_i}\} \ \ ; \ i \in S \\
M \quad &:= \bigcup_{i=1}^{s} M_i \\
\mu_i \quad &:= x_{i1}...x_{in_i} \ , \quad i \in S \\
\mu \quad &:= \mu_1 + ... + \mu_s \\
(\mu) &:= E\mu E, \text{ the ideal of } E \text{ generated by } \mu
\end{aligned}$$

Received June 14, 2005; accepted February 23, 2006.
Communicated by Shun-Jen Cheng.
2000 *Mathematics Subject Classification*: 15A75.
*Key words and phrases*: Exterior algebra, Annihilator, Frobenius algebra.

$Ann_l(\mu) = \{a \in E : a\mu = 0\}$, the left annihilator of $\mu$in $E$
$Ann_r(\mu) = \{a \in E : \mu a = 0\}$, the right annihilator of $\mu$ in $E$
$Ann(\mu) \ = Ann_l(\mu) \cap Ann_r(\mu)$, the annihilator of $\mu$ in $E$
$\mathcal{G}(\mu) :=$ the set of all the so-called standard generators of the form

$$g = (\mu_{i_1} - \mu_{j_1})...(\mu_{i_r} - \mu_{j_r})u_{k_1}...u_{k_{s-2r}}$$

where the $x_{ij}$ are linearly independent elements of $V$, equivalently $\mu_1...\mu_s \neq 0$ and the $u_{k_t}$ are elements of $M$ such that $\mu_{i_1}...\mu_{i_r}\mu_{j_1}...\mu_{j_r}u_{k_t} \neq 0.$

It is well known that a Frobenius algebra is a finite dimensional algebra $A$ over a field $F$ which has a *nondegenerate bilinear form* $B$ satisfying the associativity condition $B(xy, z) = B(x, yz)$ for all $x, y, z \in A.$ The existence of such a bilinear form on $A$ is equivalent to the existence of the duality map $L \to Ann_r(L)$ (resp. $R \to Ann_l(R))$ from left ideals to their right annihilators (resp. from right ideals to their left annihilators) in $A$ which are inclusion preserving bijections between lattices of left and right ideals of $A$ satisfying

(a) $Ann_r(L_1 + L_2) = Ann_r(L_1) \cap Ann_r(L_2)$ ,
$Ann_r(L_1 \cap L_2) = Ann_r(L_1) + Ann_r(L_2)$

(b) $Ann_l(R_1 + R_2) = Ann_l(R_1) \cap Ann_l(R_2)$ ,
$Ann_l(R_1 \cap R_2) = Ann_l(R_1) + Ann_l(R_2)$

(c) $Ann_l(Ann_r(L)) = L$ and $Ann_r(Ann_l(R)) = R.$

(d) $\dim L + \dim Ann_r(L) = \dim R + \dim Ann_l(R) = \dim A.$

(For example see [2] or [3])

The most natural examples of Frobenius algebras are the group algebra of a finite group and the exterior algebra $E(V)$, on a finite dimensional vector space $V$. Our main concern is this exterior algebra. If $\{e_1, \cdots, e_n\}$ is taken to be a basis for $V$, then products of the form

$$e_I = e_{i_1} \cdots e_{i_k} \quad \text{with} \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n$$

constitute a basis $\{e_I \mid I \subset \{1, 2, \cdots, n\}\}$ for the exterior algebra $E(V)$ (For example see [4] or [5]). The map $B : E \times E \longrightarrow F$ given by

$$B(a, b) := \text{the coefficient of } e_1e_2 \cdots e_n \text{ in the product } ab$$

becomes a bilinear form on $E(V)$. Since $B(e_I, e_J) = \pm 1$ when $J$ is the complement of $I$ in $\{1, 2, \cdots, n\}$ and $B(e_I, e_J) = 0$ otherwise, this bilinear form is nondegenerate. Clearly it satisfies the associativity $B(ab, c) = B(a, bc)$ and therefore $E(V)$ is a Frobenius algebra. Thus the duality constructed in [1] is an immediate consequence of this fact and it is crucial for our proofs.

In the first section we investigate the case where $\mu$ is an odd element of $E$ that is to say $n_1, n_2, \ldots, n_s$ are odd numbers and we determine the ideals $(\mu)$ and $Ann(\mu)$ in a complete manner. In Section 2, we handle the case where $n_1, n_2, \ldots, n_s$ are all even and determine the generators of the ideal $Ann(\mu)$ under the assumption that the base field $F$ is of characteristic $0$, thus we obtain a generalization of the results of I. Dibag in [1]. In concluding we indicate that the restriction $Char(F) = 0$ cannot be removed.

## 2. Annihilators of Principal Ideals: Odd Case

Throughout this section we assume that $n_1, n_2, \ldots, n_s$ are odd numbers. Annihilators of odd elements $\mu = \mu_1 + \cdots + \mu_s$ can be described easily, even further in this case dimensions of $(\mu)$ and $Ann(\mu)$ can be computed explicitly. In this direction we first establish the following lemma which applies to linearly independent odd elements of the exterior algebra.

**Lemma 1.** *If* $\nu_1, \nu_2, ..., \nu_k$ *are algebra generators of any algebra* $A$ *such that*

(i) $\nu_i \nu_j = -\nu_j \nu_i$ *for all* $i, j$;

(ii) $\nu_i^2 = 0$ *for all* $i$ ;

(iii) $\nu_1 \nu_2 ... \nu_k \neq 0$

*then* $A$ *is isomorphic the exterior algebra of any vector space of dimension* $k$.

*Proof.* It is sufficient to note that the vector space generators $\nu_{i_1} \nu_{i_2} ... \nu_{i_t}$ of $A$ are linearly independent. To see this take a relation

$$a_{i_1 i_2 \cdots i_t} \nu_{i_1} \nu_{i_2} ... \nu_{i_t} = 0$$

where $\{i_1, i_2, ..., i_r\}$ runs over all subsets of $\{1, 2, ..., k\}$. Considering a nonzero term $a_{i_1 i_2 \cdots i_r} \nu_{i_1} \nu_{i_2} ... \nu_{i_r}$, and multiplying the above relation through $\nu_{j_1} \nu_{j_2} ... \nu_{j_{k-r}}$ where $\{j_1, j_2, ..., j_{k-r}\}$ is the complementary set of $\{i_1, i_2, ..., i_r\}$ in $\{1, 2, ..., k\}$ we obtain

$$a_{i_1 i_2 \cdots i_r} \nu_1 \nu_2 ... \nu_k = 0$$

Since $a_{i_1 i_2 \cdots i_r} \neq 0$, this implies $\nu_1 \nu_2 ... \nu_k = 0$, which contradict our last assumption. Now the result follows from the universal property of exterior algebras (see for example [5]) by constructing a homomorphism from the exterior algebra of any vector space of dimension $k$ onto $A$. ∎

**Lemma 2.** *The notation being as in the introduction, the ideal* $\mu A$ *of the subalgebra* $A$ *of* $E$ *generated by* $\mu_1, ..., \mu_s$ *is of dimension* $2^{s-1}$.

*Proof.* By using Lemma 1, we can identify the algebra generated by $\mu_1, ..., \mu_s$ with the exterior algebra of the vector space spanned by $\mu_1, ..., \mu_s$. It is well known (cf. 1.2.11. Lemma in [1]) and easy to prove that

$$\dim(\mu) = \dim Ann(\mu A) = \frac{1}{2} \dim A = 2^{s-1}. \qquad \blacksquare$$

**Lemma 3.** *Let $A$ be the subalgebra of the exterior algebra generated by $\mu_1, ..., \mu_s$, and let $N$ be the set of all monomials in the $x_{ij}$ which are not divisible by any $\mu_k$. Then*

$$E = \bigoplus_{\nu \in N} A\nu \quad \textit{and} \quad \mu E = \bigoplus_{\nu \in N} \mu A\nu.$$

*Proof.* By Lemma 1, $A$ is an algebra isomorphic to the exterior algebra of the space spanned by $\mu_1, ..., \mu_s$, that is to say $A$ is a vector space with a basis consisting of products

$$\mu_{j_1} ... \mu_{j_t} \quad \text{where} \quad 1 \leq j_1 < j_2 < ... < j_t \leq s.$$

Obviously, for any $\nu \in N$, the product $\mu_{j_1} ... \mu_{j_t}\nu$ is either zero or a nonzero monomial, and further any two such nonzero products $\mu_{j_1} ... \mu_{j_t}\nu$ and $\mu_{k_1} ... \mu_{k_u}\nu^{'}$ are different unless $t = u$, $\mu_{j_1} = \mu_{k_1}, ..., \mu_{j_t} = \mu_{k_t}$, and $\nu = \nu^{'}$. Since any monomial of $E$ can be written in this form (up to order of factors), such products form a basis for $E$. This proves the first direct sum decomposition. The second one is an immediate consequence of the first one since $\mu \in A$. $\blacksquare$

Now we are in a position to describe the principal ideal $(\mu)$ and its annihilator.

**Theorem 4.** *If all the $n_i$ are odd, then* the principal ideal $(\mu)$ of $E$ is of dimension

$$\dim(\mu) = \frac{\dim E}{2}(1 - (1 - 2^{1-n_1})(1 - 2^{1-n_2}) ... (1 - 2^{1-n_s})).$$

*Proof.* Completing $M$ to a basis $\mathcal{B}$ of the underlying vector space $V$, say $\mathcal{B} = M \cup M'$ with $M \cap M' = \phi$, and denoting by $E_M$ (resp. $E_{M'}$) the subalgebra of $E$ generated by $M$ (resp. $M'$) we can say that

$$(\mu) = E\mu E = \mu E = \mu E_M \otimes E_{M'}$$

and thus we may assume WLG that $\dim V = n = n_1 + n_2 + \cdots + n_s$ and $\dim E = 2^n$. By Lemma 3, it is enough to determine $\dim(\mu A\nu)$ for each $\nu \in N$. For a fixed $\nu$ with $K = \{k \mid \mu_k \nu = 0\} = \{k_1, k_2, ..., k_t\}$, the space $\mu A\nu$ is spanned by elements of the form

$$\begin{aligned} \mu\mu_{j_1} ... \mu_{j_u}\nu &= ((\mu_{i_1} + \cdots + \mu_{i_r}) + (\mu_{k_1} + \cdots + \mu_{k_t}) \\ &\quad + (\mu_{j_1} + \cdots + \mu_{j_u}))\mu_{j_1} ... \mu_{j_u}\nu \\ &= ((\mu_{i_1} + \cdots + \mu_{i_r}) + (\mu_{j_1} + \cdots + \mu_{j_u}))\mu_{j_1} ... \mu_{j_u}\nu \end{aligned}$$

where $\{i_1, i_2, ..., i_r, j_1, j_2, ..., j_u\} = S - K$. That is to say,

$$\mu A\nu = ((\mu_{i_1} + \cdots + \mu_{i_r}) + (\mu_{j_1} + \cdots + \mu_{j_u}))A_K\nu$$

where $A_K$ is the subalgebra of $E$ generated by $\{\mu_l | \ l \in S - K \}$. Since $\nu$ has no common factor $x_{ij}$ with any generator $\mu_l$ of $A_K$, it is clear that

$$\begin{aligned}\dim(\mu A\nu) &= \dim(((\mu_{i_1} + \cdots + \mu_{i_r}) + (\mu_{j_1} + \cdots + \mu_{j_u}))A_K\nu) \\ &= \dim(((\mu_{i_1} + \cdots + \mu_{i_r}) + (\mu_{j_1} + \cdots + \mu_{j_u}))A_K)\end{aligned}$$

and therefore by Lemma 2, we have

$$\dim(\mu A\nu) = \dim(((\mu_{i_1} + \cdots + \mu_{i_r}) + (\mu_{j_1} + \cdots + \mu_{j_u}))A_K) = 2^{r+u-1} = 2^{s-t-1}.$$

Now, for each $K = \{k_1, k_2, ..., k_t\}$, letting

$$N_K = \{\nu \in N | \nu\mu_k = 0 \text{ for } k \in K, \ \nu\mu_l \neq 0 \text{ for } l \notin K\}$$

we see that it consists of products $n_{k_1} \cdots n_{k_t}$ where each $n_{k_p}$ $(p = 1, 2, \cdots, t)$ is a product of elements of $M_{k_p}$ different from $1$ and $\mu_{k_p}$. Therefore it contains $|N_K| = (2^{n_{k_1}} - 2)...(2^{n_{k_t}} - 2)$ elements, and that for each $\nu \in N_K$, as we have just seen $\dim(\mu A\nu) = 2^{s-t-1}$ provided $t < s$. Thus, the dimension of $\bigoplus_{\nu \in N_K} \mu A\nu$ is

$$\begin{aligned}d_K &= 2^{s-t-1}(2^{n_{k_1}} - 2)...(2^{n_{k_t}} - 2) \\ &= 2^{s-1}\underbrace{(2^{n_{k_1}-1} - 1)}_{a_{k_1}}...\underbrace{(2^{n_{k_t}-1} - 1)}_{a_{k_t}} = 2^{s-1}a_{k_1}...a_{k_t}\end{aligned}$$

and therefore $d = \dim(\mu) = \dim(\bigoplus_{\nu \in N} \mu A\nu) = \dim(\bigoplus_{K \subset S} (\bigoplus_{\nu \in N_K} \mu A\nu))$ is obtained as

$$\begin{aligned}d &= \sum_{K \subset S} d_K \\ &= 2^{s-1} \sum_{\{k_1,...,k_t\} \subset S} a_{k_1}...a_{k_t} \\ &= 2^{s-1} \sum_{\{k_1,...,k_t\} \subseteq S} a_{k_1}...a_{k_t} - 2^{s-1}a_1...a_s \\ &= 2^{s-1}(1 + a_1) \cdots (1 + a_s) - 2^{s-1}a_1...a_s \\ &= 2^{s-1}(1 + a_1) \cdots (1 + a_s) \left(1 - \tfrac{a_1}{1+a_1} \cdots \tfrac{a_s}{1+a_s}\right)\end{aligned}$$

By letting $a_{k_i} = 2^{n_{k_i}-1} - 1$ one obtains

$$\begin{aligned}d &= 2^{s-1+(n_1-1)+\cdots+(n_s-1)}(1 - (1 - 2^{1-n_1})(1 - 2^{1-n_2})...(1 - 2^{1-n_s})) \\ &= 2^{-1+n_1+\cdots+n_s}(1 - (1 - 2^{1-n_1})(1 - 2^{1-n_2})...(1 - 2^{1-n_s})) \\ &= \frac{\dim E}{2}(1 - (1 - 2^{1-n_1})(1 - 2^{1-n_2})...(1 - 2^{1-n_s}))\end{aligned}$$

and the proof is completed. ∎

**Theorem 5.** *If all the $n_i$ are odd, then $Ann(\mu)$ is generated by $\mu$ and the products $P_J = x_{1j_1}x_{2j_2}...x_{sj_s}$ where $x_{ij} \in M_i$ and is of dimension*

$$\dim(Ann(\mu)) = \frac{\dim E}{2}(1 + (1 - 2^{1-n_1})(1 - 2^{1-n_2})...(1 - 2^{1-n_s})).$$

*Proof.* Let $\mathcal{A}(\mu)$ be the ideal generated by $\mu$ and the $P_J$. Since $\mathcal{A}(\mu) \subset Ann(\mu)$ to prove the theorem it is enough to show that $\dim(\mathcal{A}(\mu)) = \dim Ann(\mu)$. As in the proof of Theorem 4, we assume WLG that $\dim V = n = n_1 + n_2 + \cdots + n_s$ and we compute $\dim(\mathcal{A}(\mu))$. Obviously the basis elements of the ideal generated by the $P_J$ are of the form $m_1m_2\cdots m_s$ where each $m_i$ is a product of elements of $M_i$ different from $1$. Of these elements, those in which some $m_i = \mu_i$, are of the form

$$\begin{aligned} m_1\cdots m_i\cdots m_s &= m_1\cdots\mu_i\cdots m_s \\ &= m_1\cdots m_{i-1}(\mu_1 + \cdots + \mu_i + \cdots + \mu_s)m_{i+1}\cdots m_s \end{aligned}$$

and hence they are in $(\mu)$. As for elements $m_1m_2\cdots m_s$ with $m_i \neq \mu_i$ for all $i$, they span a subspace $P$ such that $P \cap (\mu) = 0$. Since the number of such $m_1m_2\cdots m_s$'s is $(2^{n_1} - 2)...(2^{n_s} - 2)$ we have

$$\begin{aligned} \dim(\mathcal{A}(\mu)) &= \dim(\mu) + (2^{n_1} - 2)...(2^{n_s} - 2) \\ &= \dim(\mu) + 2^{n_1+\cdots+n_s}(1 - 2^{1-n_1})...(1 - 2^{1-n_s}) \\ &= \dim(\mu) + \dim E(1 - 2^{1-n_1})...(1 - 2^{1-n_s}) \\ &= \frac{\dim E}{2}\left(1 - (1 - 2^{1-n_1})(1 - 2^{1-n_2})...(1 - 2^{1-n_s})\right) \\ &\quad + \dim E(1 - 2^{1-n_1})...(1 - 2^{1-n_s}) \\ &= \frac{\dim E}{2}(1 + (1 - 2^{1-n_1})(1 - 2^{1-n_2})...(1 - 2^{1-n_s})) \end{aligned}$$

Now, we see that

$$\dim(\mathcal{A}(\mu)) + \dim(\mu) = \dim E,$$

and using the duality we mentioned in the introduction we obtain

$$\dim(\mathcal{A}(\mu)) = \dim E - \dim(\mu) = \dim(Ann(\mu)).$$

This completes the proof. ∎

As a corollary we give the following strengthened generalization of Lemma 1.2.11 in [1].

**Corollary 6.** The notation being as in the introduction, the following statements are equivalent:

(i) *One of the* $\mu_i$ in $\mu = \mu_1 + ... + \mu_s$ *is of degree one*

(ii) $(\mu) = Ann(\mu)$

(iii) $\dim(\mu) = \dim(Ann(\mu))$

*Proof.* Supposing $\mu_1 = x_{11}$, we observe that for any $J = \{j_1 = 1, j_2, ...., j_s\}$ we have

$$P_J = x_{1j_1}x_{2j_2}...x_{sj_s} = \mu_1 x_{2j_2}...x_{sj_s} = \mu x_{2j_2}...x_{sj_s} \in (\mu)$$

so that $Ann(\mu) \subseteq (\mu)$. The other inclusion is obvious and thus it follows that (i) implies (ii). Trivially (ii) implies (iii). Finally, by equating the two dimensions in Theorems 4 and 5 we obtain

$$(1 - 2^{1-n_1})(1 - 2^{1-n_2})...(1 - 2^{1-n_s}) = 0$$

which forces $n_i = 1$ for some $i$. ■

## 3. Annihilators of Principal Ideals: Even Case

The investigation of this even case is more subtle. We make use of Dibag's techniques basically with some generalizations. Throughout this section we assume that the base field $F$ is of characteristic zero as well as $n_1, n_2, \ldots, n_s$ are even numbers. We constantly use elements of the form $g = (\mu_{i_1} - \mu_{j_1})...(\mu_{i_r} - \mu_{j_r})u_{k_1}...u_{k_{s-2r}}$ which will be referred to as "standard generators". As we indicated in the introduction their set will be denoted by $\mathcal{G}(\mu)$ and the ideal generated by elements of $\mathcal{G}(\mu)$ will be denoted by $\mathcal{A}(\mu)$.

**Definition 7.**

$(a)$ The product $u_{k_1}...u_{k_t}$ in the generator

$$g = (\mu_{i_1} - \mu_{j_1})...(\mu_{i_r} - \mu_{j_r})u_{k_1}...u_{k_t} \text{ where } 2r + t = s,$$

is called the tail of $g$. Two generators $g_1$ and $g_2$ in $\mathcal{G}(\mu)$ which have the same tail are said to be equivalent.

$(b)$ For each $x_{kl} \in M_k$, the product $x_{k1}x_{k2}\cdots x_{k(l-1)}x_{k(l+1)}\cdots x_{kn_k}$ is called the complement of $x_{kl}$ and it is denoted by $x^*_{kl}$.

$(c)$ For a generator $g=(\mu_{i_1}-\mu_{j_1})...(\mu_{i_r}-\mu_{j_r})u_{k_1}...u_{k_t}$ in $\mathcal{G}(\mu)$, its companion $g^*$ is defined by

$$g^*=(\mu_{i_1}-\mu_{j_1})...(\mu_{i_t}-\mu_{j_t})u^*_{k_1}...u^*_{k_t}.$$

$(d)$ For an element $u$ $=x_{kl}$ in $M_k$ the product $\mu_k=(-1)^{l-1}uu^*$ is denoted by $\mu_u$.

**Lemma 8.**

$(a)$ *If* $k_1,k_2,\cdots,k_m,\cdots,k_t$ *are distinct, the annihilator of the product* $u_{k_1}u_{k_2}\cdots u_{k_m}u^*_{k_{m+1}}\cdots u^*_{k_t}$ *in the subalgebra generated by* $\mu_1,...,\mu_s$ *is equal to the annihilator of the product* $\mu_{k_1}\mu_{k_2}\cdots\mu_{k_m}\mu_{k_{m+1}}\cdots\mu_{k_t}$ *in this subalgebra.*

$(b)$ *For any two generators* $g_1$ *and* $g_2$ *in* $\mathcal{G}(\mu)$ *there is an integer* $n$ *such that* $g_1g_2^*=n\mu_1\mu_2\cdots\mu_s$. *Further,* $n=0$ *unless* $g_1$ *and* $g_2$ *are equivalent.*

*Proof.*

$(a)$ Let $A$ be the subalgebra generated by $\mu_1,...,\mu_s$ of our exterior algebra $E$ and let $a$ be an element in $A$. Obviously

$$au_{k_1}u_{k_2}\cdots u_{k_m}u^*_{k_{m+1}}\cdots u^*_{k_t}=0$$

implies that $a\mu_{k_1}\mu_{k_2}\cdots\mu_{k_m}\mu_{k_{m+1}}\cdots\mu_{k_t}=0$. Conversely, writing $a$ as a linear combination of linearly independent products $\mu_{i_1}\mu_{i_2}\cdots\mu_{i_p}$, say $a=\sum a_{i_1\cdots i_p}\mu_{i_1}\cdots\mu_{i_p}$ and supposing $a\mu_{k_1}\mu_{k_2}\cdots\mu_{k_m}\mu_{k_{m+1}}\cdots\mu_{k_t}=0$, we obtain from

$$a\mu_{k_1}\cdots\mu_{k_m}\mu_{k_{m+1}}\cdots\mu_{k_t}=a_{i_1\cdots i_p}\mu_{i_1}\cdots\mu_{i_p}(\mu_{k_1}\cdots\mu_{k_m}\mu_{k_{m+1}}\cdots\mu_{k_t})=0$$

that each term must be equal to zero. This amounts to saying that in each term of $a$ one of the $\mu_{k_j}$ occurs. It follows from this that

$$\mu_{i_1}\mu_{i_2}\cdots\mu_{i_p}u_{k_1}u_{k_2}\cdots u_{k_m}u^*_{k_{m+1}}\cdots u^*_{k_t}=0$$

since each $u_{k_j}$ and $u^*_{k_j}$ contains at least one factor $x_{k_jl}$ of $M_{k_j}$. Hence

$$au_{k_1}u_{k_2}\cdots u_{k_m}u^*_{k_{m+1}}\cdots u^*_{k_t}=0.$$

Thus we see that

$$a\mu_{k_1}\mu_{k_2}\cdots\mu_{k_m}\mu_{k_{m+1}}\cdots\mu_{k_t}=0 \text{ implies } au_{k_1}u_{k_2}\cdots u_{k_m}u^*_{k_{m+1}}\cdots u^*_{k_t}=0.$$

$(b)$ Let

$$g_1 = (\mu_{i_1} - \mu_{j_1})...(\mu_{i_r} - \mu_{j_r})u_{k_1}...u_{k_t}$$

and

$$g_2 = (\mu_{i'_1} - \mu_{j'_1})...(\mu_{i'_{r'}} - \mu_{j'_{r'}})v_{k'_1}...v_{k'_{t'}}$$

with $2r + t = 2r^{'} + t^{'} = s$. There are two cases to consider:

**Case 1.** $t = t^{'}$ and $u_{k_1}...u_{k_t}v^*_{k'_1}...v^*_{k'_{t'}} = \pm\mu_{k_1}...\mu_{k_t}$ which means that $g_1$ and $g_2$ are equivalent. Then the expansion of the product $g_1g_2^*$ is a linear combination of products of the $\mu_k$ with integral coefficients. Each product involves $2r + t$ factors. Since $2r + t = s$ is the total number of the $\mu_k$ each product is either $0$ or $\pm\mu_1...\mu_s$, that is to say we have

$$g_1g_2^* = n\mu_1...\mu_s$$

for some integer $n$, as asserted.

**Case 2.** $u_{k_1}...u_{k_t}v^*_{k'_1}...v^*_{k'_{t'}} \neq \pm\mu_{k_1}...\mu_{k_t}$ that is $g_1$ and $g_2$ are not equivalent. Then combining the $u_{k_l}$ and their complements when they appear in the tails of $g_1$ and $g_2^*$ respectively, we can write

$$u_{k_1}...u_{k_t}v^*_{k'_1}...v^*_{k'_{t'}} = \mu_{l_1}...\mu_{l_m}u_{p_1}...u_{p_q}v^*_{p'_1}...v^*_{p'_{q'}} \text{ with } q + q^{'} \neq 0$$

$$g_1g_2^* = (\mu_{i_1} - \mu_{j_1})...(\mu_{i_r} - \mu_{j_r})(\mu_{i'_1} - \mu_{j'_1})...(\mu_{i'_{r'}} - \mu_{j'_{r'}})\mu_{l_1}...\mu_{l_m}u_{p_1}...u_{p_q}v^*_{p'_1}...v^*_{p'_{q'}}$$

with $q + q^{'} \neq 0$. Now, if $u_{p_1}...u_{p_q}v^*_{p'_1}...v^*_{p'_{q'}} = 0$ we are done, so we may assume it is non zero and therefore by (a)

$$g_1g_2^* = (\mu_{i_1} - \mu_{j_1})...(\mu_{i_r} - \mu_{j_r})(\mu_{i'_1} - \mu_{j'_1})...(\mu_{i'_{r'}} - \mu_{j'_{r'}})\mu_{l_1}...\mu_{l_m}u_{p_1}...u_{p_q}v^*_{p'_1}...v^*_{p'_{q'}}$$

is zero when

$$(\mu_{i_1} - \mu_{j_1})...(\mu_{i_r} - \mu_{j_r})(\mu_{i'_1} - \mu_{j'_1})...(\mu_{i'_{r'}} - \mu_{j'_{r'}})\mu_{l_1}...\mu_{l_m}\mu_{p_1}...\mu_{p_q}\mu_{p'_1}...\mu_{p'_{q'}} = 0.$$

However, when we expand the expression

$$(\mu_{i_1} - \mu_{j_1})...(\mu_{i_r} - \mu_{j_r})(\mu_{i'_1} - \mu_{j'_1})...(\mu_{i'_{r'}} - \mu_{j'_{r'}})\mu_{l_1}...\mu_{l_m}\mu_{p_1}...\mu_{p_q}\mu_{p'_1}...\mu_{p'_{q'}} = 0$$

we see that each term contains $r + r^{'} + m + q + q^{'} = s + \frac{q+q^{'}}{2} > s$ factors, since $s = 2r+m+q = 2r^{'}+m+q^{'}$. Hence in each term at least one $\mu_i$ will be repeating. It follows that each of its terms is 0 and thus $g_1g_2^* = 0$. ∎

**Lemma 9.** $(\mu) \cap (\mu_s) = [Ann(\mu_1 + ... + \mu_{s-1}) + (\mu_1 + ... + \mu_{s-1})]\,\mu_s$.

*Proof.* Suppose that $\omega \in [Ann(\mu_1 + ... + \mu_{s-1}) + (\mu_1 + ... + \mu_{s-1})]\,\mu_s$, say $\omega = (\alpha + \beta)\mu_s$ such that $\alpha(\mu_1 + ... + \mu_{s-1}) = 0$ and $\beta = (\mu_1 + ... + \mu_{s-1})\tau$. Also $\alpha\mu_s = \alpha(\mu_1 + ... + \mu_{s-1} + \mu_s) = \alpha\mu$ and $\beta\mu_s = (\mu_1 + ... + \mu_{s-1})\tau\mu_s = \mu\tau\mu_s$, consequently $\omega \in (\mu)$. Since $\omega \in (\mu_s)$ is obvious, $\omega \in (\mu) \cap (\mu_s)$.

For the reversed inclusion, let $\omega = \mu\tau = \mu_s\rho$; then

$$\omega = \mu\tau = \mu\left(\sum_{\substack{i_1<...<i_p\\ p<n_s}} \tau_{i_1...i_p} x_{si_1}...x_{si_p} + \tau_1\mu_s\right)$$

where the $\tau_{i_1...i_p}$ and $\tau_1$ are in the subalgebra of $E$ generated by the set $M - M_s$. For each $\tau_{i_1...i_p} x_{si_1}...x_{si_p}$ with $p < n_s$ there exists $x_{si}$ such that

$$\tau_{i_1...i_p} x_{si_1}...x_{si_p} x_{si} \neq 0.$$

This yields

$$\omega x_{si} = \mu\left(\sum_{\substack{i_1<...<i_p\\ p<n_s}} \tau_{i_1...i_p} x_{si_1}...x_{si_p}\right) x_{si} = 0$$

and therefore,

$$(\mu_1 + ... + \mu_{s-1})\left(\sum_{\substack{i_1<...<i_p\\ p<n_s}} \tau_{i_1...i_p} x_{si_1}...x_{si_p}\right) x_{si} = 0$$

$$\left(\sum_{\substack{i_1<...<i_p\\ p<n_s}} (\mu_1 + ... + \mu_{s-1})\tau_{i_1...i_p} x_{si_1}...x_{si_p} x_{si}\right) = 0$$

implying $(\mu_1 + ... + \mu_{s-1})\tau_{i_1...i_p} = 0$ and finally implying $\tau_{i_1...i_p} \in Ann(\mu_1 + ... + \mu_{s-1})$. Then letting

$$\psi = \sum_{\substack{i_1<...<i_p\\ p<n_s}} \tau_{i_1...i_p} x_{si_1}...x_{si_p} \in Ann(\mu_1 + ... + \mu_{s-1})$$

we see that

$$\begin{aligned}\omega &= (\mu_1 + ... + \mu_{s-1} + \mu_s)\psi + \mu\tau_1\mu_s = \psi\mu_s + (\mu_1 + ... + \mu_{s-1})\tau_1\mu_s \\ &= [\psi + (\mu_1 + ... + \mu_{s-1})\tau_1]\mu_s\end{aligned}$$

and hence $\omega \in [Ann(\mu_1 + ... + \mu_{s-1}) + (\mu_1 + ... + \mu_{s-1})]\, \mu_s$. ■

**Lemma 10.** $Ann(\mathcal{A}(\mu)) \cap (\mu_s) = Ann\{\mathcal{A}(\mu_1 + ... + \mu_{s-2})\mu_{s-1} + ... + \mathcal{A}(\mu_2 + ... + \mu_{s-1})\mu_1\}\mu_s$.

*Proof.* Let $\omega \in Ann(\mathcal{A}(\mu)) \cap (\mu_s)$. Since $\omega \in (\mu_s)$ we can write $\omega = \tau\mu_s$ and we may assume that $\tau$ is in the subalgebra generated by the set $M - M_s$.

On the other hand $\omega \in Ann(\mathcal{A}(\mu))$ yields $g_1\omega = 0$ for any generator $g_1 = (\mu_{i_1} - \mu_{j_1})...(\mu_{i_r} - \mu_{j_r})u_{k_1}...u_{k_{s-2r}}$ of $\mathcal{A}(\mu)$, in particular for those in which $s \notin \{k_1, ..., k_{s-2r}\}$. Suppose $g_1$ is such a generator and assume WLG that $j_r = s$. Then

$$\begin{aligned} 0 &= g_1\omega = (\mu_{i_1} - \mu_{j_1})...(\mu_{i_r} - \mu_{j_r})\mu_s u_{k_1}...u_{k_{s-2r}}\tau \\ &= (\mu_{i_1} - \mu_{j_1})...(\mu_{i_{r-1}} - \mu_{j_{r-1}})\mu_{i_r}\mu_s u_{k_1}...u_{k_{s-2r}}\tau \\ &= [(\mu_{i_1} - \mu_{j_1})...(\mu_{i_{r-1}} - \mu_{j_{r-1}})u_{k_1}...u_{k_{s-2r}}]\mu_{i_r}\tau\mu_s \\ &= g_1^{'}\mu_{i_r}\tau\mu_s \end{aligned}$$

where $g_1 \in \mathcal{A}(\mu_1 + ... + \widehat{\mu_{i_r}} + ... + \mu_{s-1})$ , $(1 \le i_r \le s-1)$. Since $g_1\mu_{i_r}\tau$ is in the subalgebra generated by the set $M - M_s$, the equality $g_1\mu_{i_r}\tau\mu_s = 0$ implies $g_1\mu_{i_r}\tau = 0$. Thus

$$\tau \in Ann\{\mathcal{A}(\mu_1 + ... + \widehat{\mu_{i_r}} + ... + \mu_{s-1})\mu_{i_r}\} \text{ for all } i_r.$$

i.e.

$$\tau \in Ann\{\mathcal{A}(\mu_1 + ... + \mu_{s-2})\mu_{s-1}\} \cap ... \cap Ann\{\mathcal{A}(\mu_2 + ... + \mu_{s-1})\mu_1\}$$

Since, $E$ is a Frobenius algebra, we have

$$\begin{aligned} &Ann\{\mathcal{A}(\mu_1 + ... + \mu_{s-2})\mu_{s-1}\} \cap ... \cap Ann\{\mathcal{A}(\mu_2 + ... + \mu_{s-1})\mu_1\} \\ = \; &Ann\{\mathcal{A}(\mu_1 + ... + \mu_{s-2})\mu_{s-1} + ... + \mathcal{A}(\mu_2 + ... + \mu_{s-1})\mu_1\}, \end{aligned}$$

and it follows that

$$\tau \in Ann\{\mathcal{A}(\mu_1 + ... + \mu_{s-2})\mu_{s-1} + ... + \mathcal{A}(\mu_2 + ... + \mu_{s-1})\mu_1\}$$

and hence

$$\omega \in Ann\{\mathcal{A}(\mu_1 + ... + \mu_{s-2})\mu_{s-1} + ... + \mathcal{A}(\mu_2 + ... + \mu_{s-1})\mu_1\}\mu_s.$$

For the reversed inclusion take

$$\omega \in Ann\{\mathcal{A}(\mu_1 + ... + \mu_{s-2})\mu_{s-1} + ... + \mathcal{A}(\mu_2 + ... + \mu_{s-1})\mu_1\}\mu_s$$

$$= [\bigcap_i Ann\{\mathcal{A}(\mu_1 + \cdots + \widehat{\mu_i} + \cdots + \mu_{s-1})\mu_i\}]\mu_s$$

say $\omega = \tau\mu_s$ where $\tau \in \bigcap_i Ann\{\mathcal{A}(\mu_1 + \cdots + \widehat{\mu_i} + \cdots + \mu_{s-1})\mu_i\}$, and take any generator

$$g_1 = (\mu_{i_1} - \mu_{j_1})...(\mu_{i_r} - \mu_{j_r})u_{k_1}...u_{k_{s-2r}}$$

of $\mathcal{A}(\mu)$.

If $s \in \{k_1, ..., k_{s-2r}\}$, the equality $g_1\omega = (\mu_{i_1}-\mu_{j_1})...(\mu_{i_r}-\mu_{j_r})u_{k_1}...u_{k_{s-2r}}\tau\mu_s = 0$ is obvious. So it is enough to consider the case $s \notin \{k_1, ..., k_{s-2r}\}$. Again we may assume WLG that $j_r = s$. Then

$$\begin{aligned} g_1\omega &= (\mu_{i_1} - \mu_{j_1})...(\mu_{i_r} - \mu_{j_r})u_{k_1}...u_{k_{s-2r}}\tau\mu_s \\ &= [(\mu_{i_1} - \mu_{j_1})...(\mu_{i_{r-1}} - \mu_{j_{r-1}})u_{k_1}...u_{k_{s-2r}}\mu_{i_r}]\mu_s\tau = 0 \end{aligned}$$

since $\tau \in \bigcap_i Ann\{\mathcal{A}(\mu_1 + \cdots + \widehat{\mu_i} + \cdots + \mu_{s-1})\mu_i\}$. Thus in any case $g_1\omega = 0$, showing that $\omega = \tau\mu_s \in Ann(\mathcal{A}(\mu)) \cap (\mu_s)$. ■

**Proposition 11.** *Let $\{\mu_{m_1}, ..., \mu_{m_{2r}}\}$ be a subset of $\{\mu_1, ..., \mu_s\}$ and let U be the subspace of $E$ spanned by the products $(\mu_{i_1} - \mu_{j_1})...(\mu_{i_r} - \mu_{j_r})$ where $\{i_1, ..., i_r, j_1, ..., j_r\} = \{m_1, ..., m_{2r}\}$. Then the bilinear form $\Psi$ on $U$ defined by $uv = (-1)^r\Psi(u, v)\mu_{m_1}...\mu_{m_{2r}}$ is positive definite and hence it is nondegenerate provided that the base field $F$ is formally real field.*

*Proof.* Let $u = \sum a_{k_1...k_r}\mu_{k_1}...\mu_{k_r} \in U$, then form $u' = \sum a_{k_1...k_r}\mu_{k'_1}...\mu_{k'_r}$ with $\{k'_1, ..., k'_r\} = \{m_1, ..., m_{2r}\} - \{k_1, ..., k_r\}$. Then we note that $u' = (-1)^r u$ and hence

$$uu' = (-1)^r u^2 = \left(\sum a^2_{k_1...k_r}\right)\mu_{m_1}...\mu_{m_{2r}}$$

which yields $u^2 = (-1)^r \sum a^2_{k_1...k_r}\mu_{m_1}...\mu_{m_{2r}}$. Thus $\Psi(u, u) = \sum a^2_{k_1...k_r}$ which is positive for each nonzero u in U. ■

**Corollary 12.** *Let G= $\{g_\alpha\}$ be a linearly independent subset of $\mathcal{G}(\mu)$ and let $n_{\alpha\beta}$ be the integer such that $g_\alpha g^*_\beta = n_{\alpha\beta}\mu_1...\mu_s$ for each pair $g_\alpha, g_\beta \in$G. If the base field $F$ is of characteristic zero, then the $n_{\alpha\beta}$ form an invertible matrix $N = [n_{\alpha\beta}]$.*

*Proof.* Let *K* be the prime subfield of $F$. The coefficients of the $g_\alpha$ are in *K* and *K* can be embedded into a formally real field *R* since $Char(F) = 0$. Thus we may assume WLG that $F$ is a formally real field. By Lemma 8(b), $n_{\alpha\beta} = 0$ unless

$g_\alpha$ and $g_\beta$ are equivalent. Now, we select an ordering of $G$ such that the matrix of the $n_{\alpha\beta}$ is of the form

$$N = \begin{bmatrix} N_1 & 0 & \cdots & 0 \\ 0 & N_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_t \end{bmatrix}$$

where each block $N_i$ corresponds to an equivalence class of $G$. As for an equivalent pair $g_\alpha = \widetilde{g_\alpha}\tau$, $g_\beta = \widetilde{g_\beta}\tau$ with the same tail $\tau = u_{k_1} \cdots u_{k_q}$ we have

$$g_\alpha g_\beta^* = (-1)^t \widetilde{g_\alpha}\widetilde{g_\beta}\mu_{k_1}...\mu_{k_q} = (-1)^t \Psi(\widetilde{g_\alpha}, \widetilde{g_\beta})\mu_1...\mu_s$$

where $t$ depends on the tail $\tau$ only, and $\Psi$ is the bilinear form in Proposition 11. Therefore each block $N_i$ is of the form $N_i = [(-1)^t \Psi(\widetilde{g_\alpha}, \widetilde{g_\beta})]$ and hence it is invertible by Proposition 11. ■

**Lemma 13.** $\mathcal{A}(\mu) \cap Ann(\mathcal{A}(\mu)) = \bigoplus_i \mathcal{A}(\mu_1 + ... + \widehat{\mu_i} + ... + \mu_s)\mu_i.$

*Proof.* In order to prove the inclusion $\supseteq$ we take a generator $g \in \mathcal{A}(\mu_1 + ... + \widehat{\mu_i} + ... + \mu_s)$ for any $i$ and show that $g\mu_i \in \mathcal{A}(\mu) \cap Ann(\mathcal{A}(\mu))$. This generator is of the form $g = (\mu_{i_1} - \mu_{j_1})...(\mu_{i_r} - \mu_{j_r})u_{k_1}...u_{k_{s-1-2r}}$ where all $i_m, j_m, k_m \neq i$. Then obviously

$$g\mu_i = (\mu_{i_1} - \mu_{j_1})...(\mu_{i_r} - \mu_{j_r})u_{k_1}...u_{k_{s-1-2r}}x_{i1}x_{i2}...x_{in_i} \in \mathcal{A}(\mu).$$

We claim that $g\mu_i \in Ann(\mathcal{A}(\mu))$ that is to say, for any generator

$$g_1 = (\mu_{l_1} - \mu_{t_1})...(\mu_{l_p} - \mu_{t_p})v_{q_1}...v_{q_{s-2p}}$$

of $\mathcal{A}(\mu)$ we have $g_1 g\mu_i = 0$. Because otherwise we would have

$$g_1 g\mu_i = g\mu_i(\mu_{l_1} - \mu_{t_1})...(\mu_{l_p} - \mu_{t_p})v_{q_1}...v_{q_{s-2p}} \neq 0$$

for some $g_1$. Now, among such $g_1$'s pick up the one with smallest $p$. Then we must have $v_{q_1}...v_{q_{s-2p}}\mu_i \neq 0$ implying $\mu_i = \mu_t$ (or $\mu_l$) for some $t$ (or $l$). We may assume WLG that $\mu_i = \mu_{t_p}$ and hence

$$g_1 g\mu_i = \pm g\mu_i(\mu_{l_1} - \mu_{t_1})...(\mu_{l_{p-1}} - \mu_{t_{p-1}})\mu_{l_p}v_{q_1}...v_{q_{s-2p}} \neq 0$$

that is to say $g_1 g\mu_i \neq 0$ for $g_2 = (\mu_{l_1} - \mu_{t_1})...(\mu_{l_{p-1}} - \mu_{t_{p-1}})v_{q_1}...v_{q_{s-2p}}\nu$ with $\nu = x_{l_p 1}$ which contradicts the minimality of $p$.

As for the inclusion $\subseteq$, take $\omega \in \mathcal{A}(\mu) \cap Ann(\mathcal{A}(\mu))$ and write $\omega = \sum_{\alpha} a_\alpha g_\alpha$ as an element in $\mathcal{A}(\mu)$. Since we also have $\omega \in Ann(\mathcal{A}(\mu))$, for each generator $g_\gamma$ of $\mathcal{A}(\mu)$ we have $\omega g_\gamma = 0$ and hence $\omega g_\beta^* = 0$ for all $\beta$. By Lemma 8(b), we obtain,

$$0 = \omega g_\beta^* = \sum_\alpha a_\alpha g_\alpha g_\beta^* = \sum_\alpha a_\alpha n_{\alpha\beta} \mu_1 ... \mu_s = \sum_\alpha a_\alpha \mu_1 ... \mu_s n_{\alpha\beta}$$

for all $g_\beta$ . Since the matrix $[n_{\alpha\beta}]$ is invertible by Corollary 12, it follows that

$$a_\alpha \mu_1 ... \mu_s = 0$$

for all $\alpha$. So $a_\alpha$ is in the ideal generated by the $x_{ij}$. Since

$$x_{ij} g_\alpha = x_{ij}(\mu_{i_1} - \mu_{j_1})...(\mu_{i_r} - \mu_{j_r}) u_{k_1} ... u_{k_{s-2r}} = \begin{cases} 0 & \text{if} \quad i \in \{k_1, ..., k_{s-2r}\} \\ \mu_l g_{\alpha'} x_{ij} & \text{if} \quad i \notin \{k_1, ..., k_{s-2r}\} \end{cases}$$

where $g_{\alpha'} x_{ij}$ is a generator of $\mathcal{A}(\mu_1 + ... + \widehat{\mu_{\alpha'}} + ... + \mu_s)$. Hence we see that $a_\alpha g_\alpha \in \bigoplus_i \mathcal{A}(\mu_1 + ... + \widehat{\mu}_i + ... + \mu_s)\mu_i$ and the required inclusion follows. ■

**Theorem 14.** *If* $Char(F) = 0$, *then* $Ann(\mathcal{A}(\mu)) = (\mu)$, *equivalently* $Ann(\mu) = \mathcal{A}(\mu)$.

*Proof.* The inclusion $(\mu) \subseteq Ann(\mathcal{A}(\mu))$ is obvious. Now, for the reversed inclusion we use induction on $s$. For $s = 1$, $\mu = x_{11} x_{12} ... x_{1n_1}$, generators of $\mathcal{A}(\mu)$ are $x_{11}, x_{12}, ..., x_{1n_1}$ and therefore $(\mu) = Ann(\mathcal{A}(\mu))$. Suppose that the assumption is true for $s - 1$. Let $\mu = \mu_1 + ... + \mu_s$. Since $\mathcal{A}(\mu_1 + ... + \mu_{s-1}) \cap (x_{s1}, ..., x_{sn_s}) \subseteq \mathcal{A}(\mu)$ and $E$ is a Frobenius algebra, we have

$$\begin{aligned} Ann(\mathcal{A}(\mu)) &\subseteq Ann\{\mathcal{A}(\mu_1 + ... + \mu_{s-1}) \cap (x_{s1}, ..., x_{sn_s})\} \\ &= Ann\{\mathcal{A}(\mu_1 + ... + \mu_{s-1})\} + Ann\{(x_{s1} ..., x_{sn_s})\} \\ &= (\mu_1 + ... + \mu_{s-1}) + (\mu_s) \end{aligned}$$

by the induction hypothesis. Thus any $\omega \in Ann(\mathcal{A}(\mu))$ is of the form $\omega = \tau(\mu_1 + ... + \mu_{s-1}) + \tau' \mu_s$ or $\omega = \tau\mu + \alpha\mu_s$, where $\alpha = \tau' - \tau$. Now,

$$\alpha \mu_s = \omega - \tau\mu \in Ann(\mathcal{A}(\mu)) \cap (\mu_s).$$

On the other hand

$$\begin{aligned}
Ann(\mathcal{A}(\mu))\cap(\mu_s) &= Ann\{\mathcal{A}(\mu_1+...+\mu_{s-2})\mu_{s-1}+...+\mathcal{A}(\mu_2+...+\mu_{s-1})\mu_1\}\mu_s \\
&\quad by\ Lemma\ 10 \qquad (1)\\
&= Ann\{\mathcal{A}(\mu_1+...+\mu_{s-1})\cap Ann(\mathcal{A}(\mu_1+...+\mu_{s-1}))\}\mu_s \\
&\quad by\ Lemma\ 13 \qquad (2)\\
&= \{Ann(\mathcal{A}(\mu_1+...+\mu_{s-1}))+\mathcal{A}(\mu_1+...+\mu_{s-1})\}\mu_s \\
&\quad by\ Frobenius\, property \qquad (3)\\
&= \{(\mu_1+...+\mu_{s-1})+\mathcal{A}(\mu_1+...+\mu_{s-1})\}\mu_s \\
&\quad by\ induction\ hypothesis \qquad (4)\\
&= (\mu)\cap(\mu_s) by\ Lemma\ 9. \qquad (5)
\end{aligned}$$

Therefore $\alpha\mu_s \in (\mu)\cap(\mu_s)$ . Thus $\omega = \tau\mu+\alpha\mu_s \in (\mu)$ which yields $Ann(\mathcal{A}(\mu)) \subseteq (\mu)$. ■

Now we give a proposition which allows us to discuss the nonzero characteristic cases.

**Proposition 15.** *Let $Char(F) = p$. For $\omega = \mu_1 \cdots \mu_m u_{2m} u_{2m+1} \cdots u_s$ with $u_i \in M_i$, we have*

$(a)$ $\omega \in Ann(\mathcal{A}(\mu))$

$(b)$ $\omega \in Ann(\mu)$ *if and only if* $m < p$

*Proof.*

$(a)$ $\omega \in Ann(\mathcal{A}(\mu))$ means that $\omega g = 0$ for all $g \in \mathcal{G}(\mu)$. In fact, for any $g = (\mu_{i_1} - \mu_{j_1})\cdots(\mu_{i_r} - \mu_{j_r})v_{k_1}\cdots v_{k_{s-2r}}$ in $\mathcal{G}(\mu)$ we have

$$\omega g = \mu_1 \cdots \mu_m u_{2m} u_{2m+1} \cdots u_s (\mu_{i_1} - \mu_{j_1}) \cdots (\mu_{i_r} - \mu_{j_r}) v_{k_1} \cdots v_{k_{s-2r}}$$

which is zero unless $\mu_1, \cdots, \mu_m$ appear in distinct factors of the product $(\mu_{i_1} - \mu_{j_1})\cdots(\mu_{i_r} - \mu_{j_r})$. Therefore $\omega g \neq 0$ implies that $m \leq r$ and the above expression for $\omega g$ is a linear combination of nonzero terms containing factors taken from at least $t$ distinct $M_i'$s where

$$t = m + r + (s - 2m + 1) = s + (r - m) + 1$$

since $s - 2m + 1 > s - 2r$. However, the number of all the distinct $M_i$ is $s$ and therefore $t \geq s + 1$ is impossible. Hence $\omega g = 0$ for all $g \in \mathcal{G}(\mu)$ as asserted.

$(b)$ Suppose that $m < p$ and let

$$\alpha = \mu_1 + \cdots + \mu_m, \quad \beta = \mu_{m+1} + \cdots + \mu_{2m-1}.$$

We obtain $\alpha^m = m!\mu_1 \cdots \mu_m$, and $\beta^m = 0$ since $m$ is greater than the number of terms of $\beta$. Therefore it follows that

$$\begin{aligned} m!\mu_1 \cdots \mu_m &= \alpha^m = \alpha^m - (-\beta)^m \\ &= (\alpha + \beta)(\alpha^{m-1} - \alpha^{m-2}\beta + \cdots + (-1)^{m-1}\beta^{m-1}) \end{aligned}$$

since $Char(F) = p > m$, it follows that $m! \neq 0$ and hence by letting

$$\gamma = \frac{1}{m!}(\alpha^{m-1} - \alpha^{m-2}\beta + \cdots + (-1)^{m-1}\beta^{m-1})$$

we obtain

$$\mu_1 \cdots \mu_m = (\alpha + \beta)\gamma,$$

that is

$$\begin{aligned} \mu_1 \cdots \mu_m u_{2m} \cdots u_s &= u_{2m} \cdots u_s(\mu_1 + \cdots + \mu_m + \mu_{m+1} + \cdots + \mu_{2m-1})\gamma \\ &= u_{2m} \cdots u_s(\mu_1 + \cdots + \mu_m + \mu_{m+1} + \cdots + \mu_{2m-1} \\ &\quad + \mu_{2m} + \cdots + \mu_s)\gamma \\ &= u_{2m} \cdots u_s \mu\gamma. \end{aligned}$$

To complete the proof it remains to show that $\omega \notin (\mu)$ when $p \leq m$. Suppose on the contrary that $\omega \in (\mu)$. Then $\omega = \mu\gamma$ implies that $\omega\mu^{p-1} = \mu^p\gamma$. Since $\mu^p = 0$ and

$$\begin{aligned} \omega\mu^{p-1} &= \mu_1 \cdots \mu_m u_{2m} \cdots u_s(\mu_1 + \cdots + \mu_s)^{p-1} \\ &= \mu_1 \cdots \mu_m u_{2m} \cdots u_s(\mu_{m+1} + \cdots + \mu_{2m-1})^{p-1} \neq 0 \end{aligned}$$

we obtain a contradiction. ∎

**Corollary 16.** *If $Char(F) > \frac{s+1}{2}$ then any element of the form $\omega = \mu_1 \cdots \mu_m u_{2m} \cdots u_s$ is in $(\mu)$.*

*Proof.* Since the number of $u_k$ which occur in $\omega$ is $s - 2m + 1 \geq 0$, we obtain $m \leq \frac{s+1}{2} < p = Char(F)$ and the result follows from the proposition. ∎

This proposition shows that the hypothesis $Char(F) = 0$ of Theorem 14 can not be removed, for counter examples in the case $Char(F) = p \leq \frac{s+1}{2}$ are abundant. However if $Char(F) = p > \frac{s+1}{2}$ we conjecture that the assertion of Theorem 14 is true. In an earlier version of this paper whose abstract appeared in [6] we stated this result but we noticed a gap which still awaits proof. However, the proof of Corollary 12 can also be adopted to assert that the result is true for sufficiently large prime characteristics.

## Acknowledgment

The authors would like to thank the referee for several very helpful suggestions.

## References

1. I. Dibag, Duality for Ideals in the Grassmann Algebra, *J. Algebra*, **183** (1996), 24-37.
2. C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, John Wiley, New York, 1962.
3. G. Karpilovsky, *Symmetric and G-algebras*, 1990.
4. T. Y. Lam, *The Algebraic Theory of Quadratic Forms*, W. A. Benjamin, Inc., 1973.
5. T. Yokonuma, *Tensor Spaces and Exterior Algebra*, Translations of Mathematical Monographs, AMS, Providence, Rhode Island, 1992, Vol. 108.
6. C. Koç and S. Esin, *Annihilators of Principal Ideals in the Grassman Algebra*, in Applications of Geometric Algebra in Computer Science and Engineering, (Leo Dorst, Chris Doran and Joan Lasenby, eds.), Birkhauser, 2002, pp. 193-194.

Cemal Koç and Songül Esin
Department of Mathematics,
Dogus University,
Istanbul,
Turkey
E-mail: ckoc@dogus.edu.tr
E-mail: sesin@dogus.edu.tr