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ON THE PROJECTION DYNAMICAL SYSTEMS IN BANACH SPACES

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Abstract. We study dynamical systems of the projection gradient type for convex constrained minimization problems, clearance type dynamical system for fixed point problems with nonexpansive self-mappings and descent-like dynamical system for variational inequalities with maximal monotone operators in Banach spaces. We prove the weak convergence of dynamical trajectories and establish the estimates of the convergence rate with respect to functionals of the problems. We also produce some strong convergence theorem. The results presented in the paper are new even in Hilbert spaces.

1. Introduction and Preliminaries

Let B be a real uniformly convex and uniformly smooth Banach space and B^* be a dual space. Denote the norm in B by $\|.\|$ and the norm in B^* by $\|.\|_*$, and the dual product between vectors of B and B^* by $\langle \phi, x\rangle$, where $x \in B$ and $\phi \in B^*$. In a Hilbert space H, $\langle \phi, x\rangle$ is an inner (or scalar) product.

We recall that uniform convexity of B means that for any given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in B$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x-y\| = \epsilon$, the inequality

$$\|x+y\| \leq 2(1-\delta)$$

holds. The function

$$\delta_B(\epsilon) = \inf\{1 - 2^{-1}\|x+y\| : \|x\| = 1, \|y\| = 1, \|x-y\| = \epsilon\}$$

is called the modulus of convexity of B. The function $\delta_B(\epsilon)$ defined on the interval [0,2) is continuous, increasing and $\delta_B(0) = 0$.

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Uniform smoothness of B means that for any given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in B$ with $\|x\| = 1$ and $\|y\| \leq \delta$, the inequality

$$2^{-1}(\|x+y\| + \|x-y\|) - 1 \leq \epsilon\|y\|$$

holds. The function

$$\rho_B(\tau) = \sup\{2^{-1}(\|x+y\| + \|x-y\|) - 1 : \|x\| = 1, \|y\| = \tau\}$$

is called the modulus of smoothness of B. The function $\rho_B(\tau)$ defined on the interval $[0,\infty)$ is convex, continuous, increasing and $\rho_B(0) = 0$.

The space B is uniformly convex if and only if

$$\delta_B(\epsilon) > 0 \ \ \forall \epsilon > 0,$$

and it is uniformly smooth if and only if

$$\lim_{\tau \to 0} \frac{\rho_B(\tau)}{\tau} = 0.$$

It is well known that every uniformly convex and every uniformly smooth Banach space is reflexive. The spaces of number sequences l^p, Lebesgue spaces L^p and Sobolev spaces W_m^p with $p \in (1, \infty),\ m > 0$ are examples of uniformly convex and, at the same time, uniformly smooth Banach spaces.

In what follows, we denote

$$h_B(\tau) = \frac{\rho_B(\tau)}{\tau}.$$

The function $h_B(\tau)$ is non-decreasing and $h_B(0) = \lim_{\tau \to o^+} h_B(\tau) = 0$. Therefore, the estimate

$$h_B(C\tau) \leq h_B(\tau)$$

holds for all C such that $0 \leq C \leq 1$. As it was shown in [8],

$$h_B(C\tau) \leq LCh_B(\tau), \quad \forall\, C \geq 1,$$

where $1 < L < 1.7$ is the Figiel's constant. Thus,

$$h_B(C\tau) \leq max\{1, LC\} h_B(\tau), \quad \forall\, C \geq 0. \tag{1.1}$$

Let $J : B \to B^*$ and $J^* : B^* \to B$ be normalized duality mappings in B and B^*, respectively. Recall that J is normalized duality mappings in a Banach space B if the following equalities are true for all $x \in B$:

$$\langle Jx, x \rangle = \|Jx\|_*^2 = \|x\|^2.$$

A normalized duality mapping has a lot of remarkable properties which can be seen, for example, in [3]. In particular, J is a monotone operator, i.e.,

$$\langle Jx - Jy, x - y\rangle \geq 0, \quad \forall x, y \in B. \tag{1.2}$$

Moreover, if $\|x\| \leq R$ and $\|x\| \leq R$, then

$$\langle Jx - Jy, x - y\rangle \geq \frac{R^2}{2L}\delta_B\Big(\frac{\|x-y\|}{2R}\Big), \quad \forall x, y \in B, \tag{1.3}$$

and

$$\|Jx - Jy\| \leq 8Rh_B\Big(\frac{16L\|x-y\|}{R}\Big), \quad \forall x, y \in B. \tag{1.4}$$

Normalized duality mappings in B^* are defined by the same way. In uniformly convex and uniformly smooth Banach spaces $JJ^* = J^*J = I$, where I is the identity operator.

As examples, below we give analytical representations of the normalized duality mappings in the spaces l^p, L^p and W^p_m with $p \in (1, \infty),\ m > 0$:

(i) $l^p : Jx = \|x\|^{2-p}_{l^p} y \in l^q,\ \ x = \{x_1, x_2, ...\},\ \ y = \{x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, ...\}$,

(ii) (1.5) $\qquad L^p(\Omega) : Jx = \|x\|^{2-p}_{L^p}|x(s)|^{p-2}x(s) \in L^q(\Omega)$,

(iii) $W^p_m(\Omega) : Jx = \|x\|^{2-p}_{W^p_m} \sum\limits_{|\alpha|\leq m} (-1)^{|\alpha|}D^\alpha(|D^\alpha x(s)|^{p-2}D^\alpha x(s)) \in W^q_{-m}(\Omega)$,

where $p^{-1} + q^{-1} = 1$. Recall that the normalized duality mappings J and J^* in Hilbert spaces H are identity operators.

In this paper we study the special types of dynamical systems to find approximately minimal points of convex functional, solutions of variational inequalities with maximal monotone operators and fixed points of nonexpansive operators. R. Courant was the first to study, in [15], the behavior of trajectories of the gradient differential equation

$$\frac{dx(t)}{dt} = -\nabla f(x(t)),\ \ t_0 \leq t < \infty,\ \ x(t_0) = x_0,$$

to define stationary points of the classical variational functional $f : H \to R$ (i.e., unconstrained problems in a Hilbert space when gradient ∇f of the functional f acts from H to H). It is easy to see that if $\|\nabla f(x)\|^2 \geq c(f(x) - f^*)$ with any constant $c > 0$, then the following estimate holds:

$$f(x(t)) - f^* \leq (f(x_0) - f^*)e^{-c(t-t_0)},$$

where $f^* = min_{x\in H} f(x)$. It is clear that $f(x(t)) \to f^*$ as $t \to \infty$. After this, dynamical systems became the subject of many investigations (we refer readers to [1, 2, 11, 12], where there is a short survey of the literature). Along with minimization problems, the authors have considered operator equations, saddle point and fixed point problems, stochastic approximation and optimal control problems, in both well-posed and ill-posed statements. In [1], the differential descent equation

$$\frac{dx(t)}{dt} = -\frac{f(x(t)) - f^*}{\|\nabla(x(t))\|_*^2}\nabla(x(t)), \quad t \geq t_0, \quad x(t_0) = x_0, \tag{1.6}$$

was studied in a Hilbert space and in [7] the equation

$$\frac{dx(t)}{dt} = -\frac{f(x(t)) - f^*}{\|\nabla(x(t))\|_*^2}J^*\nabla(x(t)), \quad t \geq t_0, \quad x(t_0) = x_0, \tag{1.7}$$

in a Banach space. Only under the existence condition of the trajectories $x(t)$ generated by (1.6) or (1.7) on the interval $[t_0, \infty)$, the following functional equality is satisfied:

$$f(x(t)) - f^* = (f(x_0) - f^*)e^{t-t_0}.$$

Namely this fact allowed us to prove strong and weak convergence of $x(t)$ to minimal points x^* of $f(x)$ in degenerate and nondegenerate cases [1, 2, 7].

For solving operator equations $Ax = 0$ with operators $A : B \to B$, the convergence of trajectories of the steepest descent type dynamical system

$$\frac{dx(t)}{dt} = -A(x(t)), \;\; t_0 \leq t < \infty, \;\; x(t_0) = x_0,$$

and estimates of their convergence rate have been established in Hilbert and Banach spaces [2, 7]. In the case $A : B \to B^*$, the following dynamical system has been introduced and studied in [5]:

$$\frac{dJx(t)}{dt} = -A(x(t)), \;\; t_0 \leq t < \infty, \;\; x(t_0) = x_0.$$

Similar results were obtained for the continuous analogy of the Newton-Kantorovich method

$$\frac{dx(t)}{dt} = -D^{-1}A(x(t)), \;\; t_0 \leq t < \infty, \;\; x(t_0) = x_0,$$

where $D = A'$ is the Fréchet derivative of A.

Section 2 of this paper deals with projection gradient dynamical systems in Banach spaces with both metric and generalized projection operators. We cite two closed works focusing on this topic. In [11, 12] Antipin has studied the following system in a finite-dimensional space R^n :

$$\frac{dx(t)}{dt} + x(t) = P_G\Big(x(t) - \nabla f(x(t))\Big), \ \ t_0 \le t < \infty, \ \ x(t_0) = x_0, \tag{1.8}$$

where $P_G : R^n \to G$ is the metric projection operator, and proved convergence of $x(t)$ to a minimal point x^* of $f(x)$ on a closed convex subset $G \subset R^n$. He has also shown that if x^* exists and $\nabla f(x(t))$ satisfied the Lipschitz condition, then there exist positive constants c_1 and c_2 such that

$$f(x(t)) - f^* \le \frac{1}{c_1 t + c_2}. \tag{1.9}$$

Let us observe that differential equation (1.8) can be obtained from the gradient iterative process

$$x(t + \Delta t) = x(t) + \Delta t\Big(x(t) + P_G\Big(x(t) - \nabla f(x(t))\Big)\Big),$$

if we write down it in the equivalent form

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} = x(t) + P_G\Big(x(t) - \nabla f(x(t))\Big)$$

and take the limit of this equality as $\Delta t \to 0$.

Unfortunately, the proof schemes of [11, 12] can not be carried from R^n to infinite-dimensional Banach spaces. We therefore used in Section 2 another approach suitable for both Hilbert and Banach spaces. Moreover, we do not require for $\nabla f(x(t))$ to be smooth and bounded, instead, we only assume that a solution $x(t)$ of the dynamical system exists. In turn, the estimate of the convergence rate to zero of the functional $f(x(t)) - f^*$ in a Banach space depends on its geometric characteristics: the modulus of convexity $\delta_B(\varepsilon)$ and modulus of smoothness $\rho_B(\tau)$. In a Hilbert space this estimate coincides with (1.9).

In Section 3 we study the dynamical system

$$\frac{dx(t)}{dt} = -(x(t) - Tx(t)), \ \ t_0 \le t < \infty, \ \ x(t_0) = x_0 \in G,$$

for a fixed point problem with nonexpansive operator $T : G \to G$, namely, to approximately find $x^* \in G$ such that $Tx^* = x^*$. Similarly to the functional $\|\nabla f(x)\|_*$ in optimization theory, the functional $\|(I - T)x\|$ plays fundamental role in fixed point theory. Similarly to the limit relation $\|\nabla f(x_n)\|_* \to 0$ as $n \to \infty$ (or $f(x_n) - f^* \to 0$) in optimization theory, the limit relation $\|(I - T)x_n\| \to 0$ plays fundamental role in fixed point theory. This has to do not only with the sequence $\{x_n\}$ but also with the trajectory $x(t)$. In what follows we call $F = I - T$ *the clearance operator* and $\|(I - T)x(t)\|$ *the clearance functional* on $x(t)$.

We want to attract attention of readers to the paper [13] of Baillon and Bruck. Its Theorem 7.1 contains the following remarkable result: If G is bounded with $\operatorname{diam} G \leq 1$ and $x(t)$ exists, then in any real Banach space B

$$\|x(t) - Tx(t)\| \leq \frac{1}{\sqrt{\pi t}}. \tag{1.10}$$

In Theorem 3.1 we proved weak convergence $x(t)$ to some x^* in a uniformly convex Banach space and limit relation $\|x(t) - Tx(t)\| \to 0$ as $t \to \infty$, provided that G is not necessarily bounded. However, the result on the convergence rate of the clearance functional on $x(t)$ established by our method is much worse than (1.10).

In Section 4 we present some results for variational inequalities with maximal monotone operators $A : B \to 2^{B^*}$ on convex closed subsets $G \subseteq B$. In our approach, based on the dynamical system introduced in [5], a variational inequality is equivalently replaced by the operator equation in the form of sum of the original operator A and normality operator N_G associated with G. We prove weak convergence of $x(t)$ for variational inequalities with arbitrary solutions and strong convergence for variational inequalities with the map-type uniformly monotone operators and separately with ψ-uniformly degenerate solutions $\bar{x}$. We also establish the strong convergence rate estimates of $x(t)$ to $\bar{x}$.

2. Gradient Dynamical Systems for Minimization Problems

Assume that $G \subseteq B$ is a convex closed set and $f : G \to R$ is a real convex differentiable functional. Let us recall that $grad\ f = \nabla f$ acts from B to B^*. We solve the problem: To find

$$x^* = Argmin\,\{f(x) \mid x \in G\}. \tag{2.1}$$

It is supposed that the solution set $\mathcal{N}$ of (2.1) is nonempty and $f^* = f(x^*) > -\infty$. It is easy to see that $\mathcal{N}$ is convex and closed.

We study the dynamical system

$$\frac{dx(t)}{dt} + x(t) = P_G\Big(x(t) - \alpha(t)J^*\nabla f(x(t))\Big), \ \ t_0 \leq t < \infty, \ \ x(t_0) = x_0, \tag{2.2}$$

where $P_G : B \to G \subset B$ is the metric projection operator in B, $0 < \alpha_1 \leq \alpha(t) \leq \alpha_2$, and assume that a solution $x(t)$ of the equation (2.2) exists.

Let us present the definition of the metric projection operator in a Banach space.

Definition 2.1. Let G be a nonempty convex closed subset of the space B. The operator $P_G : B \to G$ is called metric projection operator if it assigns to each

$x \in B$ its nearest point $\bar{x} \in G$, i.e., the solution $\bar{x}$ for the minimization problem

$$\|x - \bar{x}\| = \inf_{\xi \in G} \|x - \xi\|.$$

The basic variational principle for the metric projection $\bar{x} = P_G x$ is (see [3, 16, 21])

$$\langle J(x - P_G x), P_G x - y \rangle \geq 0, \quad \forall y \in G, \tag{2.3}$$

that is, $\bar{x} = P_G x$ is a metric projection of $x \in B$ onto G if and only if (2.3) holds.

Since the equality

$$x^* = P_G(x^* - \alpha J^* \nabla f(x^*))$$

is a necessary and sufficient condition for x^* to be minimal point of $f(x)$ on G [3], we conclude that x^* is the point of trajectory $x(t)$ in which $\dfrac{dx(t)}{dt} = 0$. We see from the equation (2.2) that $\dfrac{dx}{dt} + x(t) \in G$ at any point $t \geq t_0$. It is very important to emphasize that the same equation guarantees the inclusions $x(t) \in G$. It easily follows from the contradiction.

Theorem 2.2. *Assume that*

(1) *The set*

$$M = \Big\{x \in G \mid f(x) \leq f(x_0)\Big\} \tag{2.4}$$

is bounded;

(2) ∇f *is a bounded operator* (*i.e., it carries any bounded set of* B *to bounded set of* B^*). *Then the following assertions hold:*

- (i) *the trajectory* $x(t)$ *generated by* (2.2) *is bounded;*
- (ii) $f(x(t)) \to f^*$ *as* $t \to \infty$ *with the estimate* (2.14)*;*
- (iii) *there exists a weak accumulation point of* $x(t)$;
- (iv) *all weak accumulation points of* $x(t)$ *belong to* $\mathcal{N}$;
- (iv) *if either* $\mathcal{N}$ *is singleton, or* J *is sequentially weakly continuous operator* (*on any bounded set containing* $x(t)$), *then* $x(t)$ *weakly converges to some point of* $\mathcal{N}$.

Proof. Denote

$$y(t) = x(t) - \alpha(t) J^* \nabla f(x(t)).$$

Then

$$\nabla f(x(t)) = -\frac{1}{\alpha(t)} J(y(t) - x(t)).$$

We estimate the following derivative:

$$\begin{aligned} \frac{df(x(t))}{dt} &= \left\langle \nabla f(x), \frac{dx(t)}{dt} \right\rangle \\ &= -\frac{1}{\alpha(t)} \left\langle J(y(t) - x(t)), \frac{dx(t)}{dt} \right\rangle \\ &= -\frac{1}{\alpha(t)} \left\langle J(y(t) - x(t)), P_G y(t) - x(t) \right\rangle \\ &= -\frac{1}{\alpha(t)} \left\langle J(y(t) - P_G y(t)), P_G y(t) - x(t) \right\rangle \\ &\quad -\frac{1}{\alpha(t)} \left\langle J(y(t) - x(t)) - J(y(t) - P_G y(t)), P_G y(t) - x(t) \right\rangle. \end{aligned} \tag{2.5}$$

By (2.3), we have

$$\left\langle J(y(t) - P_G y(t)), P_G y(t) - x(t) \right\rangle \geq 0 \tag{2.6}$$

because $x(t) \in G$. In view of (1.2),

$$\left\langle J(y(t) - x(t)) - J(y(t) - P_G y(t)), P_G y(t) - x(t) \right\rangle \geq 0.$$

Consequently,

$$\frac{df(x(t))}{dt} \leq 0, \quad \forall t \in [t_0, \infty),$$

and the estimate $f(x(t)) \leq f(x_0)$ holds. Then by (2.4), $x(t)$ is bounded for all $t \geq t_0$, say $\|x(t)\| \leq C_0$. We proved the claim (i). It follows from this:

(a) Since ∇f is bounded operator, there exists $C_1 > 0$ such that $\|\nabla f(x(t))\|_* \leq C_1$;

(b) We obtain

$$\|y(t) - x(t)\| \leq \alpha(t) \|\nabla f(x(t))\|_* \leq \alpha_2 C_1$$

and

$$\|y(t)\| \leq \|x(t)\| + \|y(t) - x(t)\| \leq C_0 + \alpha_2 C_1 = C_2;$$

(c) There exists $C_3 > 0$ such that $\|P_G y(t)\| \leq C_3$ and $\|y(t) - P_G y(t)\| \leq C_2 + C_3$.

(d) By (1.3), one gets

$$\begin{aligned} &\Big\langle J(y(t)-x(t)) - J(y(t)-P_Gy(t)), P_Gy(t)-x(t)\Big\rangle \\ &\qquad \geq \frac{R^2}{2L}\delta_B\Big(\frac{\|P_Gy(t)-x(t)\|}{2R}\Big), \\ &\qquad = \frac{R^2}{2L}\delta_B\Big(\frac{1}{2R}\Big\|\frac{dx(t)}{dt}\Big\|\Big), \end{aligned} \tag{2.7}$$

where $R = max\ \{\alpha_2 C_1, C_2 + C_3\} = C_2 + C_3$. Combining (2.5), (2.6) and (2.7) we can write down

$$\frac{df(x(t))}{dt} \leq -\frac{R^2}{2L\alpha_2}\delta_B\Big(\frac{1}{2R}\Big\|\frac{dx(t)}{dt}\Big\|\Big). \tag{2.8}$$

Next we have to estimate $\Big\|\dfrac{dx(t)}{dt}\Big\|$ from below. Let $x^* \in \mathcal{N}$. By making use of the convexity property of $f(x)$, we derive for any $t \geq t_0$:

$$\begin{aligned} f(x(t)) - f^* &\leq \langle \nabla f(x(t)), x(t) - x^*\rangle \\ &= -\frac{1}{\alpha(t)}\Big\langle J(y(t)-x(t)), x(t)-x^*\Big\rangle \\ &= -\frac{1}{\alpha(t)}\Big\langle J(y(t)-P_Gy(t)), x(t)-P_Gy(t)\Big\rangle \\ &\quad - \frac{1}{\alpha(t)}\Big\langle J(y(t)-P_Gy(t)), P_Gy(t)-x^*\Big\rangle \\ &\quad - \frac{1}{\alpha(t)}\Big\langle J(y(t)-x(t)) - J(y(t)-P_Gy(t)), x(t)-x^*\Big\rangle. \end{aligned} \tag{2.9}$$

Due to the Cauchy-Schwarz inequality, definition of J and (2.6), we continue (2.9) as follows:

$$\begin{aligned} f(x(t)) - f^* &\leq -\frac{1}{\alpha(t)}\Big\langle J(y(t)-P_Gy(t)), x(t)-P_Gy(t)\Big\rangle \\ &\quad -\frac{1}{\alpha(t)}\Big\langle J(y(t)-x(t)) - J(y(t)-P_Gy(t)), x(t)-x^*\Big\rangle \\ &\leq \frac{1}{\alpha_1}\|(y(t)-P_Gy(t)\|\Big\|\frac{dx(t)}{dt}\Big\| \\ &\quad +\frac{1}{\alpha_1}\|J(y(t)-x(t)) - J(y(t)-P_Gy(t))\|_*\|x(t)-x^*\|. \end{aligned}$$

Finally, by (1.4) and by (1.1),

$$f(x(t))-f^* \le \frac{C_0+C_3}{\alpha_1}\Big\|\frac{dx(t)}{dt}\Big\|+\frac{8RC^*}{\alpha_1}h_B\Big(16LR^{-1}\Big\|\frac{dx(t)}{dt}\Big\|\Big)$$
$$\le \frac{C_0+C_3}{\alpha_1}\Big\|\frac{dx(t)}{dt}\Big\|+\frac{8C^*}{\alpha_1}max\{R,16L^2\}h_B\Big(\Big\|\frac{dx(t)}{dt}\Big\|\Big). \tag{2.10}$$

where $C^* = C_0 + \|x^*\|$.

Denote $v = \Big\|\frac{dx(t)}{dt}\Big\|$, $C_4 = \frac{C_0+C_3}{\alpha_1}$, $C_5 = \frac{8C^*}{\alpha_1}\max\{R, 16L^2\}$ and

$$\zeta(v) = C_4 v + C_5 h_B(v). \tag{2.11}$$

The function $\zeta(v)$ is increasing, it therefore has increasing inverse function $\zeta^{-1}(s)$, where $s = f(x(t)) - f^* \ge 0$. Hence, (2.10) leads to the inequality

$$\Big\|\frac{dx(t)}{dt}\Big\| \ge \zeta^{-1}\Big(f(x(t)) - f^*\Big). \tag{2.12}$$

Since $\delta_B(\varepsilon)$ is increasing function, the substitution of (2.12) into (2.13) yields

$$\frac{d(f(x(t)) - f^*)}{dt} \le -\frac{R^2}{2L\alpha_2}\delta_B\Big(\frac{1}{2R}\zeta^{-1}\Big(f(x(t)) - f^*\Big)\Big).$$

Let the continuous and increasing function $\phi(s)$ have the form

$$\phi(s) = \delta_B\Big(\frac{1}{2R}\zeta^{-1}(s)\Big).$$

Then

$$\frac{d(f(x(t)) - f^*)}{dt} \le -\frac{R^2}{2L\alpha_2}\phi\Big(f(x(t)) - f^*\Big)$$

and

$$\int_{t_0}^{t}\frac{ds}{\phi(s)} \le -\frac{R^2}{2L\alpha_2}\int_{t_0}^{t}dt'. \tag{2.13}$$

Note that if $\Phi(s) = \int \frac{1}{\phi(s)}ds$ is any antiderivative and Φ^{-1} is the inverse function to Φ, then we derive from (2.13) the following estimate:

$$f(x(t)) - f^* \le \Phi^{-1}\Big[\Phi(f(x_0) - f^*) - c(t - t_0)\Big], \tag{2.14}$$

where $c = \frac{R^2}{2L\alpha_2}$. If $t \to \infty$ then $f(x(t)) \to f^*$. Thus, we proved the claim (ii).

Choose an arbitrary sequence $\{t_k\} \to \infty$ and consider $\{x_k\}$ such that $x_k = x(t_k)$. Any bounded point set in a reflexive Banach space is weakly compact, therefore, there exists a subsequence $\{x_{k_j}\} \subset \{x_k\}$ which weakly converges to some point $\tilde{x} \in B$ as $k_j \to \infty$. Since G is closed and convex, it is weakly closed. This guarantees the inclusion $\tilde{x} \in G$. Using the assertion (i), one gets $f(x_{k_j}) \to f^*$. By weak lower semicontinuity of convex functionals [18], we have

$$f(\tilde{x}) \leq \liminf_{j\to\infty} f(x_{k_j}) = \lim_{k\to\infty} f(x_k) = f^*.$$

This means that $f(\tilde{x}) = f^*$, i.e., a weak accumulation point of $x(t_i)$ exists and the set of all weak accumulation points of $\{x(t_i)\}$ is contained in $\mathcal{N}$, i.e., the claims (iii) and (iv) hold.

If $\mathcal{N}$ is the singleton, i.e. $\mathcal{N} = \{x^*\}$, then the trajectory $x(t)$ weakly converges to x^*. The proof of (v), Part 2, is the same as in [7], where the dynamical system (1.7) was studied for unconstrained minimization problems in a Banach space. Take any $x^* \in M$ and introduce the Lyapunov functional

$$W(x, x^*) = \|x\|^2 - 2\langle Jx, x^*\rangle + \|x^*\|^2.$$

Consider it on the solution $x(t)$ of the differential equation (2.2):

$$W(x(t), x^*) = \|x(t)\|^2 - 2\langle Jx(t), x^*\rangle + \|x^*\|^2. \tag{2.15}$$

Suppose that $x(t)$ has two weak limit points $\hat{x} \in M$ and $\tilde{x} \in M$ as $t \to \infty$. Then there exist two sequences $\{x_n\}$ and $\{x_m\}$, where $x_n = x(t_n)$ and $x_m = x(t_m)$, which weakly converge to $\hat{x}$ and $\tilde{x}$, respectively. Subsequently, we have

$$W(x(t), \hat{x}) = \|x(t)\|^2 - 2\langle Jx(t), \hat{x}\rangle + \|\hat{x}\|^2.$$

and

$$W(x(t), \tilde{x}) = \|x(t)\|^2 - 2\langle Jx(t), \tilde{x}\rangle + \|\tilde{x}\|^2.$$

From this we obtain

$$\begin{aligned} v &= \lim_{t\to\infty} [W(x(t), \hat{x}) - W(x(t), \tilde{x})] \\ &= 2\lim_{t\to\infty} \langle Jx(t), \tilde{x} - \hat{x}\rangle + \|\hat{x}\|^2 - \|\tilde{x}\|^2. \end{aligned}$$

Therefore,

$$v = 2\lim_{n\to\infty} \langle Jx_n, \tilde{x} - \hat{x}\rangle + \|\hat{x}\|^2 - \|\tilde{x}\|^2$$

and

$$v = 2\lim_{m\to\infty} \langle Jx_m, \tilde{x} - \hat{x}\rangle + \|\hat{x}\|^2 - \|\tilde{x}\|^2.$$

Thus,

$$v = 2\langle J\hat{x}, \tilde{x} - \hat{x}\rangle + \|\hat{x}\|^2 - \|\tilde{x}\|^2$$

and

$$v = 2\langle J\tilde{x}, \tilde{x} - \hat{x}\rangle + \|\hat{x}\|^2 - \|\tilde{x}\|^2.$$

Finally, we derive

$$0 = \langle J\tilde{x} - J\hat{x}, \tilde{x} - \hat{x}\rangle$$

which implies $\tilde{x} = \hat{x}$. This means that $x(t)$ weakly converges to some point $\bar{x}$. The proof is accomplished. ■

Theorem 2.2 is new for the problem (2.1) even in a Hilbert space H. Consider dynamical system (2.2) with the metric projection operator $P_G : H \to G \subset H$:

$$\frac{dx(t)}{dt} + x(t) = P_G\Big(x(t) - \alpha(t)\nabla f(x(t))\Big), \;\; t_0 \le t < \infty, \;\; x(t_0) = x_0, \tag{2.16}$$

Recall that [17]

$$\delta_H(\varepsilon) = 1 - \Big(1 - \frac{\varepsilon^2}{4}\Big)^{\frac{1}{2}}, \quad 0 \le \varepsilon \le 2.$$

and

$$\rho_H(\tau) = (1 + \tau^2)^{\frac{1}{2}} - 1, \quad \tau \ge 0.$$

It is easy to see that

$$\delta_H(\varepsilon) \ge \frac{\varepsilon^2}{8} \quad \text{and} \quad \rho_H(\tau) \le \tau^2. \tag{2.17}$$

Then $\zeta(v) \le C_6 v$ with $C_6 = C_4 + C_5$, where C_4 and C_5 are given as in (2.11). The inverse function for $s = f(x(t)) - f^*$ gives $\zeta^{-1}(s) \ge \dfrac{s}{C_6}$, and then $\phi(s) \ge \dfrac{s^2}{32(C_6 R)^2}$. Thus

$$\frac{d(f(x(t)) - f^*)}{dt} \le -C_7(f(x(t)) - f^*)^2, \quad C_7 = \frac{1}{64\alpha_2 L C_6^2}.$$

The formula (2.14) takes the form

$$f(x(t)) - f^* \le \frac{1}{(f(x_0) - f^*)^{-1} + C_7(t - t_0)}. \tag{2.18}$$

Since duality mapping J is the identity operator in a Hilbert space H, it is always sequentially weakly continuous. Furthermore, the Lyapunov functional (2.15)

$$W(x(t), x^*) = \|x(t) - x^*\|^2$$

has a limit for all $x^* \in \mathcal{N}$. All these arguments enable us to state the following theorem.

Theorem 2.3. *Assume that the conditions* (1) *and* (2) *of Theorem* 2.2 *are fulfilled in* H. *Then the following assertions hold:*

(i) *the trajectory* $x(t)$ *generated by* (2.16) *is bounded;*

(ii) $f(x(t)) \to f^*$ *as* $t \to \infty$ *with the estimate* (2.18)*;*

(iii) $x(t)$ *weakly converges to some point* $\tilde{x}^* \in \mathcal{N}$.

Remark 1.1. (2.18) leads to the asymptotic result

$$f(x(t)) - f^* \le O\Big(\frac{1}{t}\Big). \tag{2.19}$$

Let us present more general estimates for the Banach spaces l^p, L^p and W^p_m with $p \in (1, \infty)$, $m > 0$. It is known that [3, 10]

$$\rho_B(\tau) \le (p-1)\tau^2, \quad \delta_B(\varepsilon) \ge p^{-1}\Big(\frac{\varepsilon}{2}\Big)^p \quad \text{for} \quad p \ge 2, \tag{2.20}$$

and

$$\rho_B(\tau) \le \frac{\tau^p}{p}, \quad \delta_B(\varepsilon) \ge \frac{(p-1)\varepsilon^2}{16} \quad \text{for} \quad 1 < p \le 2, \tag{2.21}$$

In the case $p \ge 2$, by (2.20), we have: $\zeta(v) \le C_6 v$, where $C_6 = C_4 + C_5(p-1)$. The inverse function $\zeta^{-1}(s) \ge \dfrac{s}{C_6}$. Consequently, $\phi(s) \ge \dfrac{s^p}{p(4RC_6)^p}$. Then

$$\frac{d(f(x(t)) - f^*)}{dt} \le -C_7(f(x(t)) - f^*)^p, \quad C_7 = \frac{1}{2Lp\alpha_2(4C_6)^p R^{p-2}}.$$

This differential inequality implies the integral inequality

$$\int_{t_0}^{t} \frac{ds}{s^p} \le -C_7 \int_{t_0}^{t} dt',$$

where $s = f(x(t)) - f^*$. Since $\displaystyle\int \frac{ds}{s^p} = \frac{s^{1-p}}{1-p} + const.$, we have

$$f(x(t)) - f^* \le \frac{1}{[(f(x_0) - f^*)^{1-p} + C_7(p-1)(t-t_0)]^{\frac{1}{p-1}}}, \quad p \ge 2. \tag{2.22}$$

The asymptotic estimate is

$$f(x(t)) - f^* \le O\Big(t^{\frac{1}{1-p}}\Big), \quad p \ge 2. \tag{2.23}$$

Consider the case $1 < p \le 2$. We proved in Theorem 2.2 that the operator ∇f is uniformly bounded on the trajectory $x(t)$, i.e., $\|\nabla f(x(t))\|_* \le C_1$. The following estimate has been established in [4]:

$$\|P_G x - P_G y\| \le r_1 \delta_B^{-1}\Big(2Lr_2\rho_B\Big(\|x-y\|\Big)\Big),$$

where

$$r_1 = 2\max\{1,\ \|x - P_G y\|,\ \|y - P_G x\|\}$$

and

$$r_2 = 16 + 24\max\{L,\ \|x - P_G y\|,\ \|y - P_G x\|\}.$$

Then there exist positive constants R_1 and R_2 such that

$$\begin{aligned} \Big\|\frac{dx(t)}{dt}\Big\| &= \|P_G\Big(x(t) - \alpha(t)J^*\nabla f(x(t))\Big) - P_G x(t)\| \\ &\le R_1\delta_B^{-1}\Big(R_2\rho_B\Big(\alpha_2\|\nabla f(x(t))\|_*\Big)\Big) \\ &\le R_1\delta_B^{-1}\Big(R_2\rho_B(\alpha_2 C_1)\Big) = R_3. \end{aligned} \tag{2.24}$$

Now we obtain from (2.21) and (2.24)

$$\zeta(v) \le C_4 v + C_5 p^{-1} v^{p-1} \le C_8 v^{p-1}, \quad C_8 = C_4 R_3^{2-p} + C_5 p^{-1}.$$

The inverse function $\zeta^{-1}(s) \ge C_8^{-\frac{1}{p-1}} s^{\frac{1}{p-1}}$ and function $\phi(s) \ge \dfrac{p-1}{64R^2} C_8^{-\frac{2}{p-1}} s^{\frac{2}{p-1}}$, where $s = f(x(t)) - f^*$. Consequently,

$$\frac{d(f(x(t)) - f^*)}{dt} \le -C_9 (f(x(t)) - f^*)^{\frac{2}{p-1}}, \quad C_9 = \frac{p-1}{128L\alpha_2 C_8^{\frac{2}{p-1}}}.$$

Integrating this differential inequality we can write down

$$\int_{t_0}^{t} \frac{ds}{s^{\frac{2}{p-1}}} \le -C_9 \int_{t_0}^{t} dt'.$$

Since

$$\int \frac{ds}{s^{\frac{2}{p-1}}} = \frac{(p-1)s^{\frac{p-3}{p-1}}}{p-3} + const.,$$

one gets

$$f(x(t)) - f^* \le \frac{1}{\left[(f(x_0)-f^*)^{\frac{p-3}{p-1}} + \frac{3-p}{p-1}C_9(t-t_0)\right]^{\frac{p-1}{3-p}}}, \quad 1 < p \le 2. \tag{2.25}$$

We see that the asymptotic estimate is

$$f(x(t)) - f^* \le O\Big(t^{\frac{1-p}{3-p}}\Big), \quad 1 < p \le 2. \tag{2.26}$$

Remark 2.5. The spaces l^p, L^p and W^p_m with $p = 2$ are Hilbert ones. In these cases, (2.22) and (2.25) coincide with (2.18) (up to constant), in turn, (2.23) and (2.26) coincide with (2.19).

Emphasize that the differential system (2.5) use the analytical representations of the duality mapping J^*. For instance, in the spaces l^q and L^q with $q \in (1, \infty)$, which are dual spaces to l^p and L^p with $p^{-1} + q^{-1} = 1$, are written down by analogy with (1.5) as

(i) $l^q : J^*x = \|x\|^{2-q}_{l^q} y \in l^p, \quad x = \{x_1, x_2, ...\}, \quad y = \{x_1|x_1|^{q-2}, x_2|x_2|^{q-2}, ...\},$

(ii) $L^q : J^*x = \|x\|^{2-q}_{L^q}|x|^{q-2}x \in L^p.$

Unfortunately, we are not able to present J^* in the analytical form for Sobolev spaces W^q_{-m} because it needs to solve some boundary value problem for PDE in W^p_m. This is the first reason we propose below another version of (2.2) which exploits J only in the original space B. The second reason is that we use different version of the projection operator which has of much better properties in a Banach space than the metric projection operator there.

Next we show that Theorem 2.2 is valid for the dynamical system:

$$\frac{dx(t)}{dt} + x(t) = \pi_G\Big(Jx(t) - \alpha(t)\nabla f(x(t))\Big), \;\; t_0 \le t < \infty, \;\; x(t_0) = x_0, \tag{2.27}$$

where $\pi_G : B^* \to G \subset B$ is the generalized projection operator, $0 < \alpha_1 \le \alpha(t) \le \alpha_2$. We assume again that a solution $x(t)$ of the equation (2.27) exists.

Recall definition of the generalized projection operator $\pi_G : B^* \to G$ [3]. Introduce the Lyapunov functional

$$W(\phi, \xi) = \|\phi\|^2_* - 2\langle \phi, \xi\rangle + \|\xi\|^2,$$

where $\phi \in B^*$ and $\xi \in B$.

Definition 2.6. Operator $\pi_G : B^* \to G$ is called the generalized projection operator onto G if it assigns to each $\phi \in B^*$ a minimum point $\tilde{\phi} \in G$ of the functional $W(\phi, \xi)$, i.e., a solution of the following minimization problem:

$$\pi_G \phi = \tilde{\phi}; \quad \tilde{\phi} : W(\phi, \tilde{\phi}) = \inf_{\xi \in G} W(\phi, \xi).$$

The basic variational principle for the generalized projection $\tilde{\phi}$ is

$$\langle \phi - J\tilde{\phi}, \tilde{\phi} - \xi \rangle \geq 0, \quad \forall \xi \in G, \tag{2.28}$$

that is, $\tilde{\phi} = \pi_G \phi$ is a generalized projection of $\phi \in B^*$ onto G if and only if (4.4) holds (see [3]).

Theorem 2.7. *Under the conditions of Theorem* 2.2*, all its assertions hold for the dynamical system* (2.27).

Proof. Denote

$$\psi(t) = Jx(t) - \alpha(t) \nabla f(x(t)).$$

We have

$$\nabla f(x(t)) = -\frac{1}{\alpha(t)} (\psi(t) - Jx(t)).$$

The following calculations are similar to Theorem 2.2:

$$\begin{aligned} \frac{df(x(t))}{dt} &= \left\langle \nabla f(x), \frac{dx(t)}{dt} \right\rangle \\ &= -\frac{1}{\alpha(t)} \left\langle \psi(t) - Jx(t), \frac{dx(t)}{dt} \right\rangle \\ &= -\frac{1}{\alpha(t)} \left\langle \psi(t) - Jx(t), \pi_G \psi(t) - x(t) \right\rangle \\ &= -\frac{1}{\alpha(t)} \left\langle \psi(t) - J\pi_G y(t), \pi_G \psi(t) - x(t) \right\rangle \\ &\quad -\frac{1}{\alpha(t)} \left\langle J\pi_G \psi(t) - Jx(t), \pi_G \psi(t) - x(t) \right\rangle. \end{aligned} \tag{2.29}$$

Since $x(t) \in G$, we conclude from (4.4) that

$$\left\langle \psi(t) - J\pi_G y(t), \pi_G \psi(t) - x(t) \right\rangle \geq 0. \tag{2.30}$$

In view of (1.2)

$$\left\langle J\pi_G \psi(t) - Jx(t), \pi_G \psi(t) - x(t) \right\rangle \geq 0.$$

Consequently,

$$\frac{df(x(t))}{dt} \leq 0, \quad \forall t \in [t_0, \infty),$$

and boundedness of $x(t)$ follows from the condition (2.4).

Using the same constants as in Theorem 2.2 we derive

$$\begin{aligned} &\Big\langle J\pi_G\psi(t) \, - \, Jx(t), \pi_G\psi(t) - x(t)\Big\rangle \\ &\qquad \geq \frac{R^2}{2L}\delta_B\Big(\frac{\|\pi_G\psi(t) - x(t)\|}{2R}\Big), \\ &\qquad = \frac{R^2}{2L}\delta_B\Big(\frac{1}{2R}\Big\|\frac{dx(t)}{dt}\Big\|\Big). \end{aligned} \tag{2.31}$$

Combining (2.29), (2.30) and (2.31), one gets (2.8).

Next we estimate $f(x(t)) - f^*$ from above:

$$\begin{aligned} f(x(t)) - f^* &\leq \langle \nabla f(x(t)), x(t) - x^*\rangle \\ &= -\frac{1}{\alpha(t)}\Big\langle \psi(t) - Jx(t), x(t) - x^*\Big\rangle \\ &= -\frac{1}{\alpha(t)}\Big\langle \psi(t) - J\pi_G\psi(t), x(t) - \pi_G\psi(t)\Big\rangle \\ &- \frac{1}{\alpha(t)}\Big\langle \psi(t) - J\pi_G\psi(t), \pi_G\psi(t) - x^*\Big\rangle \\ &- \frac{1}{\alpha(t)}\Big\langle J\pi_G\psi(t)) - Jx(t), x(t) - x^*\Big\rangle \\ &\leq \frac{1}{\alpha_1}\|\psi(t) - J\pi_G\psi(t)\|_*\Big\|\frac{dx(t)}{dt}\Big\| \\ &+ \frac{1}{\alpha_1}\|J\pi_G\psi(t)) - Jx(t)\|_*\|x(t) - x^*\|. \end{aligned}$$

The rest of the proof follows the pattern of Theorem 2.2. ■

3. Clearance Dynamical System for Fixed Point Problems

Assume that B is a uniformly convex Banach space, $G \subseteq B$ is a convex closed subset and T is a nonexpansive self-mapping, that is, $T : G \to G$ and

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in G.$$

It is supposed that the fixed point set of T

$$\mathcal{F} = \{x^* : x^* = Tx^*\} \neq \emptyset.$$

Then $\mathcal{F}$ is convex and closed.

In order to find $x^* \in \mathcal{F}$, we investigate on the interval $[t_0, \infty)$ the following dynamical system:

$$\frac{dx(t)}{dt} = -\lambda(t)\Big(x(t) - Tx(t)\Big), \quad x(t_0) = x_0 \in G, \tag{3.1}$$

where $0 < \lambda_1 \le \lambda(t) \le 1$. Assume that a solution $x(t)$ of the equation (3.1) exists.

Consider differential equation:

$$\frac{dx(t)}{dt} + x(t) = P_G\Big(x(t) - \lambda(t)(x(t) - Tx(t))\Big), \quad x(t_0) = x_0 \in G, \quad t \ge t_0, \tag{3.2}$$

where $P_G : B \to G$ is the metric projection operator in B. Since the right-hand side of (3.2) always belongs to G, we conclude that $x(t)$ and $\dfrac{dx(t)}{dt}$ also belong to G. Since G is convex and T is nonexpansive, one gets

$$P_G\Big(x(t) - \lambda(t)(x(t) - Tx(t))\Big) = x(t) - \lambda(t)\Big(x(t) - Tx(t)\Big).$$

This means that the equations (3.1) and (3.2) are equivalent and we do not need to use the metric projection operator P_G in the right-hand side of (3.1) to guarantee the inclusion $x(t) \subset G$. Let us emphasize again that the limit behaviour of the clearance functional $\|x(t) - Tx(t)\|$ plays very important role in our research.

Theorem 3.1. *Assume that the conditions of this section are fulfilled. Then the following assertions hold:*

(i) *the trajectory $x(t)$ generated by (3.1) is bounded and $\|x(t) - x^*\|$ has a limit for any $x^* \in \mathcal{F}$;*

(ii) *the functional $\|x(t) - Tx(t)\| \to 0$ and derivative $\dfrac{dx(t)}{dt} \to 0$ as $t \to \infty$;*

(iii) *there exists a weak accumulation point of $x(t)$;*

(iv) *all weak accumulation points of $x(t)$ belong to $\mathcal{F}$;*

(v) *if $\mathcal{F}$ is a singleton, then $x(t)$ weakly converges to $\{\tilde{x}^*\} = \mathcal{F}$.*

(vi) *if J is sequentially weakly continuous on a bounded set containing $x(t)$, then it weakly converges to a point $\tilde{x}^* \in \mathcal{F}$.*

Proof. It is obvious that for any $x^* \in \mathcal{F}$, (3.1) implies

$$\Big\langle \frac{d(x(t) - x^*)}{dt}, J(x(t) - x^*)\Big\rangle = -\lambda(t)\Big\langle x(t) - Tx(t), J(x(t) - x^*)\Big\rangle.$$

Then

$$\frac{d\|x(t)-x^*\|^2}{dt} \leq -2\lambda_1 \Big\langle x(t) - Tx(t), J(x(t)-x^*)\Big\rangle, \quad x(t_0) = x_0 \in G. \tag{3.3}$$

The right-hand side of (3.3) is negative because T is nonexpansive. This implies

$$\frac{d\|x(t)-x^*\|^2}{dt} \leq 0. \tag{3.4}$$

It follows from (3.4) that $\|x(t)-x^*\|$ is monotonically decreasing, $\|x(t)-x^*\| \leq \|x_0 - x^*\|$, $x(t)$ is bounded by $R = \|x_0\| + 2\|x^*\|$ (let us recall we did not assume a priori that G is bounded). Besides, $\|x(t)-x^*\|$ has a limit as $t \to \infty$ for all $x^* \in \mathcal{F}$.

We have proved in [6] (see also [9]) the following result:

Lemma 3.2. *If $F = I - T$ with a nonexpansive mapping T, then for all $x, y \in D(T)$,*

$$\langle Fx - Fy, J(x-y)\rangle \geq R_1^2 \delta_B\Big(\frac{\|Fx-Fy\|}{2R_1}\Big),$$

where

$$R_1 = \sqrt{\frac{\|x-y\|^2 + \|Ax-Ay\|^2}{2}} \leq \|x-y\|.$$

If $\|x\| \leq R$ and $\|y\| \leq R$ with $x, y \in D(A)$, then $R_1 \leq 2R$ and

$$\langle Fx - Fy, J(x-y)\rangle \geq L^{-1}R^2 \delta_B\Big(\frac{\|Fx-Fy\|}{4R}\Big).$$

This lemma for $x = x(t)$ and $y = x^*$ gives

$$\langle x(t) - Tx(t), J(x(t)-x^*)\rangle \geq L^{-1}R^2 \delta_B\Big(\frac{\|x(t)-Tx(t)\|}{4R}\Big).$$

Now the estimate (3.3) can be continued by means of the previous inequality as follows:

$$\frac{d\|x(t)-x^*\|^2}{dt} \leq -2\lambda_1 L^{-1}R^2 \delta_B\Big(\frac{\|x(t)-Tx(t)\|}{4R}\Big).$$

Since $\|x(t)-x^*\|$ has a limit, we conclude on the base of the properties of $\delta_B(\varepsilon)$ that the clearance functional

$$\|x(t) - Tx(t)\| \to 0 \ \text{ as } \ t \to \infty. \tag{3.5}$$

The equation (3.1) implies $\left\|\frac{dx(t)}{dt}\right\| \to 0$ as $t \to \infty$ because $\lambda(t) \le 1$. Thus, the claim (ii) is true.

The existence of a weak accumulation point $\tilde{x}$ of $x(t)$ is shown as in Theorem 2.2. The inclusion $\tilde{x} \in \mathcal{F}$ follows from the fact that T is demiclosed. The proof of (v) and (vi) can be found in [9]. As regards to finite-dimensional spaces, see Remark 3.3. ■

We are not able to obtain by our method explicit estimates of the convergence rate of the clearance functional (3.5) for the whole trajectory $x(t)$, however, it is possible to do it for some totality

$$\mathcal{T} = \cup_{k=1}^{\infty}[t_k, t_k'], \quad t_0 \le t_k \le t_k' < t_{k+1}. \tag{3.6}$$

Indeed, let any continuous and decreasing function $g(t) : [t_0, \infty) \to [0, \infty)$ satisfies the conditions: $g(t) \to 0$ as $t \to \infty$ and

$$\int_{t_0}^{\infty} g(t)dt = \infty. \tag{3.7}$$

For every $t \in [t_0, \infty)$, consider the following alternative:

$$(H_1): \quad \delta_B\Big(\frac{\|x(t) - Tx(t)\|}{4R}\Big) \le g(t)$$

or

$$(H_2): \quad \delta_B\Big(\frac{\|x(t) - Tx(t)\|}{4R}\Big) > g(t).$$

Denote

$$\mathcal{T}_1 : \{t \in [t_0, \infty) \mid (H_1) \text{ is true}\}$$

and

$$\mathcal{T}_2 : \{t \in [t_0, \infty) \mid (H_2) \text{ is true.}\}$$

It is clear $\mathcal{T}_1 \cup \mathcal{T}_2 = [t_0, \infty)$. It is easy to prove that $\mathcal{T}_1$ is a unbounded set. In fact, suppose that $\mathcal{T}_1$ is bounded. Then there is $\bar{t} > t_0$ such that for all $t \ge \bar{t}$ the hypothesis (H_2) is true. Consequently,

$$\frac{d\|x(t) - x^*\|^2}{dt} \le -2\lambda_1 L^{-1} R^2 g(t), \quad \forall t \ge \bar{t}.$$

This implies the inequality

$$\|x(t) - x^*\|^2 \le \|x(\bar{t}) - x^*\|^2 - 2\lambda_1 L^{-1} R^2 \int_{\bar{t}}^{t} g(t)dt, \quad \forall t \ge \bar{t}, \tag{3.8}$$

which, in view of (3.7), contradicts to the fact $\|x(t) - x^*\|^2 \geq 0$ for all $t \geq t_0$. Thus, the nonempty set $\mathcal{T}$ defined by (3.6) coincides with $\mathcal{T}_1$ and for all $t \in \mathcal{T}$

$$\|x(t) - Tx(t)\| \leq 4R\delta_B^{-1}(g(t)).$$

The result (3.8) enable us to estimate a lower bound of any interval $[t'_k, t_{k+1}] \subset \mathcal{T}_2$:

$$t_{k+1} \leq \bar{\bar{t}} : \left\{ \bar{\bar{t}} = max\ t \mid \int_{t'_k}^{t} g(t)dt \leq \frac{L(R + \|x^*\|)^2}{2\lambda_1 R^2} \right\}.$$

As the examples we present: $g(t) = \dfrac{c_1}{t}$, $g(t) = \dfrac{c_2}{t \ln t}$, $g(t) = \dfrac{c_3}{t \ln t \ln \ln t}$, where c_1, c_2, c_3 are arbitrary positive constants, etc.

Remark 3.3. It is well known that all finite-dimensional Banach spaces including finite-dimensional Hilbert space are isomorphic. In a finite-dimensional Hilbert space strong convergence and weak convergence coincide. Theorems 2.2, 2.3 and 2.7 therefore give the strong convergence of trajectories $x(t)$ in finite-dimensional spaces B and H, respectively. However, finite-dimensional Banach spaces are not isometric between themselves and so the estimates of convergence rate for the difference $f(x(t)) - f^*$ are different there: as in the general case, they are defined by (2.14) and (2.18).

4. Descent-like Dynamical System for Variational Inequalities

In this section we study the problem of approximately solving a variational inequality

$$\langle Ax - f, v - x \rangle \geq 0, \quad \forall v \in G, \tag{4.1}$$

with a maximal monotone operator $A : D(A) \subset B \rightarrow 2^{B^*}$, "right-hand side" $f \in B^*$ and convex closed subset $G \subset D(A)$. Assume that

$$int\, G \neq \emptyset \quad \text{or} \quad int\, D(A) \cap G \neq \emptyset. \tag{4.2}$$

We present two definitions of a solution of the variational inequality and assume that the solution set $\mathcal{N}$ of (4.1) is nonempty. Then it is convex and closed.

Definition 4.1. An element $\bar{x} \in G$ is called a solution of (4.1) if there exists an element $z \in A\bar{x}$ such that

$$\langle z - f, v - \bar{x} \rangle \geq 0, \quad \forall v \in G.$$

Definition 4.2. An element $\bar{x} \in G$ is called a solution of (4.1) if

$$\langle z - f, v - \bar{x} \rangle \geq 0, \quad \forall v \in G, \ \forall z \in Av.$$

The following two lemmas can be found, for example, in [10], p. 73-74:

Lemma 4.3. (Minty-Browder). *If the condition* (4.2) *holds, then Definitions* 4.1 *and* 4.2 *are equivalent.*

Recall that the indicator function I_G associated with G is defined as

$$I_G(x) = \begin{cases} 0, & x \in G, \\ +\infty, & x \notin G. \end{cases}$$

Its subdifferential $\partial I_G : B \to 2^{B^*}$, represented by the formula

$$\partial I_G(x) = \begin{cases} \theta_{B^*}, & x \in int\ G; \\ \emptyset, & x \notin G; \\ \lambda J x, & x \in \partial G,\ \lambda \geq 0, \end{cases} \tag{4.3}$$

is a maximal monotone operator with $D(\partial I_G) = G$ and due to (4.2), the sum $A + \partial I_G$ are also maximal monotone operators with $D(A + \partial I_G) = G$.

Lemma 4.4. (Minty-Browder). *Let* ∂I_G *be a subdifferential of the indicator function* I_G *associated with* G*. If the condition (4.2) holds, then a solution* $\bar{x} \in G$ *of the variational inequality (4.1) is a solution in the sense of the following inclusion:*

$$f \in A\bar{x} + \partial I_G(\bar{x}). \tag{4.4}$$

The inverse conclusion is also true.

It is well known that if G is a nonempty convex and closed subset of B, then the subdifferential ∂I_G is the normality operator N_G given as follows:

$$N_G(x) = \begin{cases} \{\psi \in X^* \mid \langle \psi, x - y \rangle \geq 0 \quad \forall y \in G\} & \text{if } \ x \in G; \\ \emptyset & \text{if } \ x \notin G. \end{cases} \tag{4.5}$$

By making use of the metric projection operator $P_G : B \to G$ and generalized projection operator $\pi_G : B^* \to G$, two equivalence theorems were proved in [3]:

Theorem 4.5. *Let be an arbitrary operator from Banach space* B *to* B^*, α *an arbitrary fixed positive number,* $f \in B^*$*. Then the point* $\bar{x} \in G$ *is a solution of* *variational inequality* (4.1) *if and only if* $\bar{x}$ *is a solution of the following operator equation in* B :

$$x = P_G(x - \alpha J^*(Ax - f)).$$

Theorem 4.6. *Let* A *be an arbitrary operator from Banach space* B *to* B^*, α *an arbitrary fixed positive number,* $f \in B^*$. *Then the point* $\bar{x} \in G$ *is a solution of variational inequality* (4.1) *if and only if* $\bar{x}$ *is a solution of the following operator equation in* B :

$$x = \pi_G(Jx - \alpha(Ax - f)).$$

On the basis of Theorems 4.5 and 4.6 we are able to construct the following dynamical systems variational inequality (4.1):

$$\frac{dx(t)}{dt} + x(t) = P_G\Big(x(t) - \alpha(t)J^*Ax(t)\Big), \;\; t_0 \le t < \infty, \;\; x(t_0) = x_0,$$

and

$$\frac{dx(t)}{dt} + x(t) = \pi_G\Big(Jx(t) - \alpha(t)Ax(t)\Big), \;\; t_0 \le t < \infty, \;\; x(t_0) = x_0.$$

However, we are unable to prove the convergence of their trajectories $x(t)$ to a solution $\bar{x}$ of (4.1) even if $x(t) \in G$ exists. Therefore, we use here the idea of [5] produced for operator equations and suitable for operator inclusions. Consider the dynamical system:

$$\frac{dJx(t)}{dt} + Ax(t) + \partial I_G(x(t)) = f, \;\; t_0 \le t < \infty, \;\; x(t_0) = x_0. \tag{4.6}$$

and assume that a solution $x(t)$ of (4.6) exists on the interval $[t_0, \infty)$.

We further suppose that operator $F = A + \partial I_G$ satisfies the following condition **(P)**: For any sequence $\{x_n\} \subset G$ and any $\bar{v} \in G$ such that $0 \in F\bar{v}$, the limit relation

$$\langle Fx_n, x_n - \bar{v}\rangle \to 0 \quad \text{implies} \quad Fx_n \to 0 \quad \text{as} \quad n \to \infty;$$

Theorem 4.7. *Assume that all the conditions of this section are fulfilled. Then the following assertions hold:*

(i) *the trajectory* $x(t)$ *generated by the dynamical system* (4.6) *is bounded;*

(ii) *there exists a weak accumulation point of* $x(t)$;

(iii) *all weak accumulation points of* $x(t)$ *belong to* $\mathcal{N}$;

(iv) *if either $\mathcal{N}$ is singleton, or J is sequentially weakly continuous* $($*on any bounded set containing* $x(t))$, *then* $x(t)$ *weakly converges to some point of* $\mathcal{N}$.

Proof. Let $\bar{x} \in \mathcal{N}$. Since $x(t)$ exists on $[t_0, \infty)$ and $D(I_G) = G$, there is no necessity to prove the inclusion $x(t) \in G$. In fact, the later is assumed. From the definition of J we have: $\|x\|^2 = \|Jx\|_*^2$ and $(\|Jx\|_*^2)'_{Jx} = 2x$. Therefore, for the functional (2.15) one gets

$$\frac{dW(x(t), \bar{x})}{dJx(t)} = 2(x(t) - \bar{x}),$$

and then

$$\begin{aligned} \frac{dW(x(t), \bar{x})}{dt} &= \left\langle \frac{dJx(t)}{dt}, \frac{dW(x(t), \bar{x})}{dJx(t)} \right\rangle \\ &= -2\langle Ax(t) + \partial I_G(x(t)) - f, x(t) - \bar{x} \rangle. \end{aligned} \tag{4.7}$$

Since the operator $A + \partial I_G$ is monotone, we conclude by (4.4) that

$$\frac{dW(x(t), \bar{x})}{dt} \leq 0.$$

This implies that the functional $W(x(t), \bar{x})$ monotonically decreases, it is bounded by $W(x_0, \bar{x})$ and has a limit as $t \to \infty$. Then we can assert that $x(t)$ is also bounded because [5]

$$(\|x(t)\| - \|\bar{x}\|)^2 \leq W(x(t), \bar{x}).$$

i.e.,

$$\begin{aligned} \|x(t)\| &\leq \sqrt{\|\bar{x}\|^2 + W(x_0, \bar{x})} \\ &\leq \sqrt{\|x_0\|^2 + 2\|\bar{x}\|(\|x_0\| + \|\bar{x}\|)} = r. \end{aligned} \tag{4.8}$$

Thus, in reflexive space B there exists the subsequence $\{x(t_k)\}$ weakly convergent to some point of G, i.e., $\{x_k\} \rightharpoonup u \in G$. Since $W(x(t), \bar{x})$ has a limit as $t \to \infty$, we conclude from (4.7) that

$$\lim_{t \to \infty} \frac{dW(x(t), \bar{x})}{dt} = -2 \lim_{t \to \infty} \langle Ax(t) + \partial I_G(x(t)) - f, x(t) - \bar{x} \rangle = 0. \tag{4.9}$$

Now the condition (**P**) gives

$$\langle Ax_k + \partial I_G(x_k) - f, x_k - \bar{x} \rangle \to 0 \text{ as } k \to \infty. \tag{4.10}$$

By Lemma 4.4,

$$f \in A\bar{x} + \partial I_G(\bar{x}),$$

and the limit relation

$$Ax_k + \partial I_G(x_k) \to f \tag{4.11}$$

follows now from the condition (**P**). Since any maximal monotone operator is demi-closed, we conclude from (4.11) and $\{x_k\} \rightharpoonup u \in G$ that

$$Au + \partial I_G(u) = f.$$

Due to Lemma 4.4 again, u is a solution of (4.1). The rest of the proof follows the pattern of Theorem 2.2. ∎

Observe that the inverse-strongly-monotone operators F (see, for instance, [14, 22]) satisfy the condition (**P**). Indeed, let $\langle Fx_n, x_n - \bar{v}\rangle \to 0$ as $n \to \infty$. We have

$$\begin{aligned} &\langle Ax_k + \partial I_G(x_k) - f, x_k - \bar{x}\rangle \\ &= \langle (Ax_k + \partial I_G(x_k) - f) - (A\bar{x} + \partial I_G(\bar{x}) - f), x_k - \bar{x}\rangle \\ &\geq \|Ax_k + \partial I_G(x_k) - f\|^2. \end{aligned}$$

Thus, $\lim\limits_{n\to\infty} (Ax_k + \partial I_G(x_k)) = f$.

Next we consider the strong convergence of trajectories of dynamical system (4.6) to solutions of the variational inequalities with ψ-uniformly degenerate solutions and with map-type uniformly monotone operators.

Definition 4.8. A (unique) solution $\bar{x}$ of (4.1) is called ψ-uniformly degenerate, if there exists an increasing and positive on $(0,\infty)$ function $\psi(s)$ with $\psi(0) = 0$ such that

$$\langle z - f, x - \bar{x}\rangle \geq \psi(\|x - \bar{x}\|), \quad \forall x \in G, \quad \forall z \in Ax.$$

Theorem 4.9. *If $\bar{x}$ is ψ-uniformly degenerate solution of variational inequality* (4.1)*, then the trajectories $x(t)$ of dynamical system* (4.6) *are bounded, i.e., $\|x(t)\| \leq r$, and strongly converge to $\bar{x}$ with the following estimate of the convergence rate:*

$$\|x(t) - \bar{x}\| \leq 4R\delta_B^{-1}\Big(2^{-1}LR^{-2}\Phi^{-1}\Big[\Phi(W(x_0,\bar{x})) - c(t-t_0)\Big]\Big), \tag{4.12}$$

where $\Phi(s) = \int \frac{1}{\phi(s)} ds$, $\phi(s) = \psi(\rho_B^{-1}(D^{-1}s))$, $D = 16L\max\{R, 4L\}$, $R = max\{r, \|\bar{x}\|\}$ *and* Φ^{-1} *is the inverse function to* Φ.

Proof. If the solution $\bar{x}$ of (4.1) is ψ-uniformly degenerate, then

$$\frac{dW(x(t), \bar{x})}{dt} \leq -\psi(\|x - \bar{x}\|)$$

because, in view of (4.5), for all $x \in G$

$$\langle \partial I_G(x), x - \bar{x} \rangle \geq 0.$$

In [3] it was shown that if $R = max\{r, \|\bar{x}\|\}$ then

$$2L^{-1}R^2\delta_B\Big(\frac{\|x - \bar{x}\|}{4R}\Big) \leq W(x, \bar{x}) \leq 4LR^2\rho_B\Big(\frac{4\|x - \bar{x}\|}{R}\Big). \tag{4.13}$$

By making use of (1.1), we find that $\rho_B(C\tau) \leq D\rho_B(\tau)$, where $D = Cmax\{1, LC\}$. Then the right inequality of (4.13) yields

$$W(x, \bar{x}) \leq D\rho_B(\|x - \bar{x}\|)$$

with $D = 16Lmax\{R, 4L\}$. Since $\rho_B(\tau)$ is an increasing function, one gets

$$\|x - \bar{x}\| \geq \rho_B^{-1}(D^{-1}W(x, \bar{x})). \tag{4.14}$$

Consequently,

$$\frac{dW(x(t), \bar{x})}{dt} \leq -\phi(W(x(t), \bar{x})),$$

where the function $\phi(s) = \psi(\rho_B^{-1}(D^{-1}s))$ is increasing and positive for all $s > 0$ and $\phi(0) = 0$. Similarly to (2.14),

$$W(x(t), \bar{x}) \leq \Phi^{-1}\Big[\Phi(W(x_0, \bar{x})) - c(t - t_0)\Big].$$

If $t \to \infty$ then $W(x(t), \bar{x}) \to 0$. As before, (4.8) is true for $x(t)$ in our case. Therefore, the left inequality of (4.13) implies strong convergence of $x(t)$ to $\bar{x}$ with the estimate (4.12). The proof follows. ∎

Theorem 4.10. *Assume that an operator* $F = A + \partial I_G - f$ *is map-type uniformly monotone, that is, there exists an increasing and positive on* $(0, \infty)$ *function* $\zeta(s)$ *with* $\zeta(0) = 0$ *such that* $\lim_{s\to\infty} \frac{\zeta(s)}{s} = \infty$ *and*

$$\langle Fx_1 - Fx_2, x_1 - x_2 \rangle \geq \zeta(\|Fx_1 - Fx_2\|_*), \quad \forall x_1, x_2 \in G, \tag{4.15}$$

the inverse operator F^{-1} *is uniformly continuous, i.e., there exists an increasing and positive on* $(0,\infty)$ *function* $\omega(s)$ *with* $\omega(0)=0$ *such that*

$$\|F^{-1}z_1 - F^{-1}z_2\| \le \omega(\|z_1 - z_2\|_*), \quad \forall z_1, z_2 \in B^*.$$

If $\bar{x}$ *is a* $($*unique*$)$ *solution of* (4.1), *then all the trajectories* $x(t)$ *of dynamical system* (4.6) *strongly converge to* $\bar{x}$ *with the estimate of the convergence rate* (4.12), *where* $\phi(s) = 2\zeta\Big(\omega^{-1}(\rho^{-1}(D^{-1}s))\Big)$.

Proof. From (4.7) and (4.15) we obtain

$$\frac{dW(x(t),\bar{x})}{dt} \le -2\zeta(\|Fx(t) - F\bar{x}\|_*).$$

Since F^{-1} is uniformly continuous, one gets

$$\|Fx(t) - F\bar{x}\|_* \ge \omega^{-1}(x(t) - \bar{x}).$$

Consequently,

$$\frac{dW(x(t),\bar{x})}{dt} \le -2\zeta\Big(\omega^{-1}(x(t) - \bar{x})\Big).$$

By (4.14),

$$\frac{dW(x(t),\bar{x})}{dt} \le -2\zeta\Big(\omega^{-1}(\rho_B^{-1}(D^{-1}W(x(t),\bar{x}))\Big),$$

or

$$\frac{dW(x(t),\bar{x})}{dt} \le -\phi(W(x(t),\bar{x})),$$

where $\phi(s) = 2\zeta\Big(\omega^{-1}(\rho_B^{-1}(D^{-1}s))\Big)$ is the increasing and positive for all $s > 0$ function with $\phi(0) = 0$. The rest of the proof follows the pattern of the previous theorem. ∎

Remark 4.11. Note that in [19] and [20], the variational inequalities

$$\langle Ax, v - x\rangle \ge 0, \;\; \forall v \in G,$$

with operators $A : G \to B^*$ are considered on convex subsets of B. By making use of the metric and generalized projection operators onto tangent cones to G, the authors have constructed more general dynamical systems and solved the equivalence problem between critical points of variational inequalities and stationary points of these systems. However, the question on convergence of trajectories of such dynamical systems remains still open.

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