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GENERALIZED SECOND ORDER SYMMETRIC DUALITY IN NONDIFFERENTIABLE MULTIOBJECTIVE PROGRAMMING

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Abstract. We introduce a pair of multiobjective generalized second order symmetric dual programs where the objective function contains a support function. Weak, strong and converse duality theorems for these second order problems are established under suitable generalized second order convexity assumptions. Also, we give some special cases of our second order symmetric duality results.

1. INTRODUCTION

Symmetric duality in nonlinear programming was first proposed by Dorn [6], who defined symmetric duality in mathematical programming if the dual of the dual is the primal problem. Subsequently, Dantzig et al. [5] and Mond [12] formulated a pair of symmetric dual programs and established duality under convexity-concavity assumptions. Mond and Weir [15] then weakened the hypothesis by assuming pseudoconvexity-pseudoconcavity conditions. Later, Mond and Schechter [14] constructed two new symmetric dual pairs where the objective function contains a support function.

In multiobjective optimization case, Weir and Mond [19] discussed symmetric duality in multiobjective programming by using the concept of efficiency. Mond and Weir [16] proved symmetric duality theorems for nonlinear multiobjective programming. Gulati et al. [7] also established duality results for multiobjective symmetric dual problems without non-negativity constraints.

On the other hand, Mangasarian [11] considered a nonlinear programming and introduced second order duality under certain inequality conditions. Mond [13] was the first one to present second order symmetric dual models and proved second

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order symmetric duality results under the assumptions of second order convexity on functions involved in the primal problem. Later, Jeyakumar [9] and Yang [20] also gave second order Mangasarian type duality results by using ρ-convexity and generalized representation conditions, respectively. Bector and Chandra [3] formulated Mond-Weir type second order dual programs and established second order symmetric duality results for these programs. Later on, Yang [21] generalized the results of Bector and Chandra [3] to nonlinear programs involving second order pseudoinvexity.

Recently, Hou and Yang [8] obtained symmetric duality results for nondifferentiable nonlinear programs under second order F-pseudoconvexity assumptions. More recently, Suneja et al. [18] presented a pair of Mond-Weir type multiobjective second order symmetric dual programs and gave their duality results. Yang et al. [22] introduced a pair of Wolfe type nondifferentiable second order symmetric dual programs and established weak and strong duality theorems under second order F-convexity conditions.

Very recently, Yang et al. [24] showed a pair of second order symmetric models for a class of nondifferentiable multiobjective programs, which is Mond-Weir type. And Kim et al. [10] and Yang et al. [23] introduced slightly different pairs of Wolfe type second order symmetric dual programs in differentiable multiobjective nonlinear programming and presented duality results for these programs. In the nondifferentiable case, Wolfe type second order symmetric dual programs is not yet introduced.

By modifying these two type models, in this paper, we give a pair of multiobjective generalized second order symmetric dual programs where the objective function contains a support function. Weak duality, strong duality and converse duality theorems are established under second order F-convexity and F-concavity assumptions. The symmetric dual results presented in this paper include the already known results in [1, 2, 4, 8, 18].

This paper is organized as follows. In section 2, we introduce mathematical notations and give definitions. In section 3, we formulate the pair of multiobjective generalized second order symmetric dual programs involving a support function and establish weak, strong and converse duality theorems under second order F-convexity and F-concavity assumptions. Finally, in section 4, we apply our generalized second order symmetric dual models and results to several dual models and results.

2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and let $\mathbb{R}^n_+$ be its non-negative orthant.

The following convention for inequalities will be used:

If $x, u \in \mathbb{R}^n$, then

$$x \leqq q\, u \iff u - x \in \mathbb{R}^n_+ \ ;$$

$$x \leq u \iff u - x \in \mathbb{R}^n_+ \setminus \{0\} \ ;$$

$$x < u \iff u - x \in int\mathbb{R}^n_+ \ ;$$

$$x \nleq u \text{ is the negation of } x \leq u \ .$$

For $x,\ u \in \mathbb{R}$, $x \leqq q\, u$ and $x < u$ have the usual meaning.

Let $f(x, y)$ be a real valued thrice continuously differentiable function defined on an open set in $\mathbb{R}^n \times \mathbb{R}^m$. Let $\nabla_x f(\overline{x}, \overline{y})$ denote the gradient vector of f with respect to x at $(\overline{x}, \overline{y})$. Also, let $\nabla_{xx} f(\overline{x}, \overline{y})$ denote the $n \times n$ symmetric Hessian matrix with respect to x evaluated at $(\overline{x}, \overline{y})$. $\nabla_y f(\overline{x}, \overline{y})$ and $\nabla_{yy} f(\overline{x}, \overline{y})$ are defined similarly. $(\frac{\partial}{\partial y_i})(\nabla_{yy} f(\overline{x}, \overline{y}))$ is the $m \times m$ matrix obtained by differentiating the elements of $\nabla_{yy} f(\overline{x}, \overline{y})$ with respect to y_i, $i = 1, 2, \cdots, m$.

Consider the following multiobjective programming problem:

$$\begin{array}{lll} (MP) & \text{Minimize} & f(x) = (f_1(x), f_2(x), \cdots, f_k(x)) \\ & \text{subject to} & g(x) \leqq q\, 0, x \in X, \end{array}$$

where X is an open set of $\mathbb{R}^n$, $f : X \to \mathbb{R}^k$ and $g : X \to \mathbb{R}^m$.

Definition 2.1. A feasible point $\overline{x}$ is said to be a weak efficient solution of (MP), if there exists no other $x \in X$ with $f(x) < f(\overline{x})$.

Definition 2.2. A feasible point $\overline{x}$ is said to be an efficient solution of (MP), if there exists no other $x \in X$ with $f(x) \leq f(\overline{x})$.

Definition 2.3. A feasible point $\overline{x}$ is said to be a properly efficient solution of (MP), if it is an efficient solution of (MP) and if there exists a scalar $M > 0$ such that for each i and $x \in X$ satisfying $f_i(x) < f_i(\overline{x})$, we have $\frac{f_i(\overline{x}) - f_i(x)}{f_j(x) - f_j(\overline{x})} \leq M$ for some j satisfying $f_j(x) > f_j(\overline{x})$.

Definition 2.4. A functional $F : X \times X \times \mathbb{R}^n \to \mathbb{R}$ is sublinear in its third component, if for all $x,\ u \in X$,

(i) $F(x, u; a_1 + a_2) \leqq q\, F(x, u; a_1) + F(x, u; a_2)$ for all $a_1, a_2 \in \mathbb{R}^n$; and

(ii) $F(x, u; \alpha a) = \alpha F(x, u; a)$ for all $\alpha \in \mathbb{R}_+$, and for all $a \in \mathbb{R}^n$.

For notational convenience, we write $F_{x,u}(a) = F(x, u; a)$.

Definition 2.5. Let $f_i(x, y)(i = 1, 2, \cdots, k)$ be a twice differentiable function from $X(\subset \mathbb{R}^n) \times Y(\subset \mathbb{R}^m)$ to $\mathbb{R}$.

(i) $f_i(.,v)$ is said to be second order (strict) F-convex at $u \in \mathbb{R}^n$, if there exists a sublinear functional F: $X \times X \times \mathbb{R}^n \to \mathbb{R}$ such that for $v \in \mathbb{R}^m$, $r \in \mathbb{R}^n$, $x \in \mathbb{R}^n$,

$$f_i(x,v) - f_i(u,v) \geqq q\,(>) F_{x,u}[\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v) r] - \frac{1}{2} r^T \nabla_{xx} f_i(u,v) r.$$

(ii) $f_i(x,.)$ is said to be second order F-concave at $y \in \mathbb{R}^m$, if there exists a sublinear functional G: $Y \times Y \times \mathbb{R}^m \to \mathbb{R}$ such that for $x \in \mathbb{R}^n$, $p \in \mathbb{R}^m$, $v \in \mathbb{R}^m$,

$$f_i(x,v) - f_i(x,y) \leqq q\, G_{v,y}[\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y) p] - \frac{1}{2} p^T \nabla_{yy} f_i(x,y) p.$$

(iii) $f_i(.,v)$ is said to be second order F-pseudoconvex at $u \in \mathbb{R}^n$, if there exists a sublinear functional F: $X \times X \times \mathbb{R}^n \to \mathbb{R}$ such that for $v \in \mathbb{R}^m$, $r \in \mathbb{R}^n$, $x \in \mathbb{R}^n$,

$$F_{x,u}[\nabla_x f_i(u,v) + \nabla_{xx} f_i(u,v) r] \geqq q\, 0 \Rightarrow f_i(x,v) \geqq q\, f_i(u,v) - \frac{1}{2} r^T \nabla_{xx} f_i(u,v) r.$$

(iv) $f_i(x,.)$ is said to be second order F-pseudoconcave at $y \in \mathbb{R}^m$, if there exists a sublinear functional G: $Y \times Y \times \mathbb{R}^m \to \mathbb{R}$ such that for $x \in \mathbb{R}^n$, $p \in \mathbb{R}^m$, $v \in \mathbb{R}^m$,

$$G_{v,y}[\nabla_y f_i(x,y) + \nabla_{yy} f_i(x,y) p] \leqq q\, 0 \Rightarrow f_i(x,v) \leqq q\, f_i(x,y) - \frac{1}{2} p^T \nabla_{yy} f_i(x,y) p.$$

f_i is second order F-concave(F-pseudoconcave) at $u \in X$ with respect to $r \in \mathbb{R}^n$, if $-f_i$ is second order F-convex(F-pseudoconcave) at $u \in X$ with respect to $r \in \mathbb{R}^n$.

Definition 2.6. [14] Let C be a compact convex set in $\mathbb{R}^n$. The support function $s(x|C)$ of C is defined by

$$s(x|C) := max\{x^T y : y \in C\}.$$

The support function $s(x|C)$, being convex and everywhere finite, has a subdifferential, that is, there exists z such that

$$s(y|C) \geq s(x|C) + z^T(y-x) \text{ for all } y \in C.$$

Equivalently,

$$z^T x = s(x|C).$$

The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) := \{z \in C : z^T x = s(x|C)\}.$$

For any set $S \subset \mathbb{R}^n$, the normal cone to S at a point $x \in S$ is defined by

$$N_S(x) := \{y \in \mathbb{R}^n : y^T(z - x) \leq 0 \ \text{ for all } \ z \in S\}.$$

It is readily verified that for a compact convex set C, y is in $N_C(x)$ if and only if $s(y|C) = x^T y$, or equivalently, x is in the subdifferential of s at y.

3. Generalized Second Order Symmetric Duality

We now propose the following pair of generalized second order nondifferentiable multiobjective programming problems with k-objectives:

(GMP) Minimize

$$\begin{aligned} &K(x, y, \lambda, w, p) \\ &= f(x, y) + s(x|B) - (y_J^T w)e - (y_I^T \nabla_{y_I}(\lambda^T f)(x, y))e \\ &-(y_I^T \nabla_{yy_I}(\lambda^T f)(x, y)p)e - \frac{1}{2}(p^T \nabla_{yy}(\lambda^T f)(x, y)p)e \end{aligned}$$

subject to

$$\nabla_y(\lambda^T f)(x, y) - w + \nabla_{yy}(\lambda^T f)(x, y)p \leqq q\, 0, \tag{1}$$

$$y_J^T[\nabla_{y_J}(\lambda^T f)(x, y) - w + \nabla_{yy_J}(\lambda^T f)(x, y)p] \geqq q\, 0, \tag{2}$$

$$w \in C_i, \quad \lambda > 0, \quad \lambda^T e = 1,$$

(GMD) Maximize

$$\begin{aligned} &G(u, v, \lambda, z, r) \\ &= f(u, v) - s(v|C) + (u_B^T z)e - (u_A^T \nabla_{x_A}(\lambda^T f)(u, v))e \\ &-(u_A^T \nabla_{xx_A}(\lambda^T f)(u, v)r)e - \frac{1}{2}(r^T \nabla_{xx}(\lambda^T f)(u, v)r)e \end{aligned}$$

subject to

$$\nabla_x(\lambda^T f)(u, v) + z + \nabla_{xx}(\lambda^T f)(u, v)r \geqq q\, 0, \tag{3}$$

$$u_B^T[\nabla_{x_B}(\lambda^T f)(u, v) + z + \nabla_{xx_B}(\lambda^T f)(u, v)r] \leqq q\, 0, \tag{4}$$

$$z \in B_i, \quad \lambda > 0, \quad \lambda^T e = 1,$$

where

(i) f is a thrice differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^k$;

(ii) r, z are vectors in $\mathbb{R}^n$, p, w are vectors in $\mathbb{R}^m$;

(iii) λ and $e = (1, \cdots, 1)^T$ are vectors in $\mathbb{R}^k$;

(iv) B_i and $C_i (i = 1, 2, \cdots, k)$ are compact convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, note that $B = (B_1, B_2, \cdots, B_k)^T$, $C = (C_1, C_2, \cdots, C_k)^T$; and

(v) $N = \{1, 2, \cdots, n\}$, $M = \{1, 2, \cdots, m\}$, $A \subset N$, $I \subset M$,$N \setminus A = B$ and $M \setminus I = J$. Note that A, B, I or J can be empty.

We prove the following duality results for the pairs (GMP) and (GMD).

Theorem 3.1. (Weak Duality) *Let* (x, y, λ, w, p) *be feasible for* (GMP) *and* (u, v, λ, z, r) *be feasible for* (GMD)*. Assume that*

(i) $f(., v) + ((.)^T z)e$ *is second order F-convex in the first variable,*

(ii) $f(x, .) - ((.)^T w)e$ *is second order F-concave in the second variable,*

(iii) $F_{x,u}(a) + a^T u \geqq q\, 0$ *for all* $a \in \mathbb{R}^n_+$*, and*

(iv) $G_{v,y}(b) + b^T y \geqq q\, 0$ *for all* $b \in \mathbb{R}^m_+$*.*

Then $K(x, y, \lambda, w, p) \nleq G(u, v, \lambda, z, r)$.

Proof. Assume to the contrary that $K(x, y, \lambda, w, p) \leq G(u, v, \lambda, z, r)$. Then since $\lambda > 0$ and $\lambda^T e = 1$, we have

$$(5)\quad (\lambda^T f)(x, y) + \lambda^T s(x|B) - y_J^T w - y_I^T \nabla_{y_I}(\lambda^T f)(x, y) - y_I^T \nabla_{yy_I}(\lambda^T f)(x, y)p$$
$$- \frac{1}{2} p^T \nabla_{yy}(\lambda^T f)(x, y)p$$
$$< (\lambda^T f)(u, v) - \lambda^T s(v|C) + u_B^T z - u_A^T \nabla_{x_A}(\lambda^T f)(u, v)$$
$$- u_A^T \nabla_{xx_A}(\lambda^T f)(u, v)r - \frac{1}{2} r^T \nabla_{xx}(\lambda^T f)(u, v)r.$$

By the hypothesis (i), for any $i = 1, 2, \cdots, k$,

$$[f_i(x, v) + x^T z] - [f_i(u, v) + u^T z]$$
$$\geqq q\, F_{x,u}[\nabla_x f_i(u, v) + z + \nabla_{xx} f_i(u, v)r] - \frac{1}{2} r^T \nabla_{xx} f_i(u, v)r,$$

which premultiplying by λ^T and using $\lambda > 0,\ \lambda^T e = 1$ and F is sublinear, we get

$$[(\lambda^T f)(x, v) + x^T z] - [(\lambda^T f)(u, v) + u^T z]$$
$$\geqq q\, F_{x,u}[\nabla_x(\lambda^T f)(u, v) + z + \nabla_{xx}(\lambda^T f)(u, v)r] - \frac{1}{2} r^T \nabla_{xx}(\lambda^T f)(u, v)r.$$

Using the hypothesis (iii) for $\overline{a} := \nabla_x(\lambda^T f)(u, v) + z + \nabla_{xx}(\lambda^T f)(u, v)r \geqq q\, 0$(by the constraint (3)) and the constraint (4),

$$
\begin{aligned}
&[(\lambda^T f)(x,v)+x^Tz]-[(\lambda^T f)(u,v)+u^Tz]\\
&\geqq_q -u^T[\nabla_x(\lambda^T f)(u,v)+z+\nabla_{xx}(\lambda^T f)(u,v)r]-\frac{1}{2}r^T\nabla_{xx}(\lambda^T f)(u,v)r\\
&=-u_A^T[\nabla_{x_A}(\lambda^T f)(u,v)+z+\nabla_{xx_A}(\lambda^T f)(u,v)r]\\
&\quad -u_B^T[\nabla_{x_B}(\lambda^T f)(u,v)+z+\nabla_{xx_B}(\lambda^T f)(u,v)r]-\frac{1}{2}r^T\nabla_{xx}(\lambda^T f)(u,v)r\\
&\geqq_q -u_A^T[\nabla_{x_A}(\lambda^T f)(u,v)+z+\nabla_{xx_A}(\lambda^T f)(u,v)r]-\frac{1}{2}r^T\nabla_{xx}(\lambda^T f)(u,v)r.
\end{aligned}
$$

Then, we obtain

$$
\begin{aligned}
&(\lambda^T f)(x,v)+x^Tz-(\lambda^T f)(u,v)\\
&\geqq_q -u_A^T\nabla_{x_A}(\lambda^T f)(u,v)+u_B^Tz-u_A^T\nabla_{xx_A}(\lambda^T f)(u,v)r-\frac{1}{2}r^T\nabla_{xx}(\lambda^T f)(u,v)r.
\end{aligned}
$$

Similarly, using the hypotheses (ii), (iv) for $\overline{b} := -[\nabla_y(\lambda^T f)(x,y)-w+\nabla_{yy}(\lambda^T f)(x,y)p] \geqq_q 0$(by the constraint (1)) and the constraint (2), we also obtain

$$
\begin{aligned}
&(\lambda^T f)(x,v)-v^Tw-(\lambda^T f)(x,y)\\
&\leqq_q -y_I^T\nabla_{y_I}(\lambda^T f)(x,y)-y_J^Tw-y_I^T\nabla_{yy_I}(\lambda^T f)(x,y)p-\frac{1}{2}p^T\nabla_{yy}(\lambda^T f)(x,y)p.
\end{aligned}
$$

Combining these two inequalities, we get

$$
\begin{aligned}
&(\lambda^T f)(x,y)+x^Tz-y_J^Tw-y_I^T\nabla_{y_I}(\lambda^T f)(x,y)-y_I^T\nabla_{yy_I}(\lambda^T f)(x,y)p\\
&\quad -\frac{1}{2}p^T\nabla_{yy}(\lambda^T f)(x,y)p\\
&\geqq_q (\lambda^T f)(u,v)-v^Tw+u_B^Tz-u_A^T\nabla_{x_A}(\lambda^T f)(u,v)-u_A^T\nabla_{xx_A}(\lambda^T f)(u,v)r\\
&\quad -\frac{1}{2}r^T\nabla_{xx}(\lambda^T f)(u,v)r.
\end{aligned}
$$

Since $x^Tz \leqq_q s(x|B_i)$ and $v^Tw \leqq_q s(v|C_i)$ for any $i=1,2,\cdots,k$, we get, by $\lambda > 0$ and $\lambda^T e = 1$, $x^Tz \leqq_q \lambda^T s(x|B)$ and $v^Tw \leqq_q \lambda^T s(v|C)$. Finally, using these, we obtain

$$
\begin{aligned}
&(\lambda^T f)(x,y)+\lambda^T s(x|B)-y_J^Tw-y_I^T\nabla_{y_I}(\lambda^T f)(x,y)-y_I^T\nabla_{yy_I}(\lambda^T f)(x,y)p\\
&\quad -\frac{1}{2}p^T\nabla_{yy}(\lambda^T f)(x,y)p
\end{aligned}
$$

$$\geqq q\,(\lambda^T f)(u,v) - \lambda^T s(v|C) + u_B^T z - u_A^T \nabla_{x_A}(\lambda^T f)(u,v) - u_A^T \nabla_{xx_A}(\lambda^T f)(u,v)r$$
$$-\frac{1}{2} r^T \nabla_{xx}(\lambda^T f)(u,v)r,$$

which contradicts (5). Thus $K(x,y,\lambda,w,p) \nleq G(u,v,\lambda,z,r)$. ■

Theorem 3.2. (Strong Duality) *Let f be a thrice differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^k$. Let $(\overline{x},\overline{y},\overline{\lambda},\overline{w},\overline{p})$ be an efficient solution for (GMP); fix $\lambda=\overline{\lambda}$ in (GMD) and suppose that*

(i) $\nabla_{yy}(\overline{\lambda}^T f)$ *is non-singular,*

(ii) $\nabla_{y_J}(\overline{\lambda}^T f) - \overline{w} + \nabla_{yy_J}(\overline{\lambda}^T f)\overline{p} \neq 0$,

(iii) *the set* $\{\nabla_{y_J} f_1, \nabla_{y_J} f_2, \cdots, \nabla_{y_J} f_k, \overline{w}\}$ *is linearly independent, and*

(iv) *the matrix* $\frac{\partial}{\partial y_i}(\nabla_{yy}(\overline{\lambda}^T f))$ *is positive or negative definite, for some* $i \in I$,

where $f = f(\overline{x},\overline{y})$.
Then there exists $\overline{z} \in B_i (i=1,2,\cdots,k)$ *such that* $(\overline{x},\overline{y},\overline{\lambda},\overline{z},\overline{r}=0)$ *is a feasible solution for* (GMD) *and* $K(\overline{x},\overline{y},\overline{\lambda},\overline{w},\overline{p}) = G(\overline{x},\overline{y},\overline{\lambda},\overline{z},\overline{r})$.

Moreover, if the hypotheses of Theorem 3.1.(Weak Duality) are satisfied for all feasible solutions of (GMP) *and* (GMD), *then* $(\overline{x},\overline{y},\overline{\lambda},\overline{z},\overline{r})$ *is a properly efficient solution for* (GMD).

Proof. Since $(\overline{x},\overline{y},\overline{\lambda},\overline{w},\overline{p})$ is an efficient solution of (GMP), by Fritz-John neccessary optimality conditions [17], there exist $\alpha \in R_+^k$, $\beta \in R_+^m$, $\mu \in R_+$ and $\delta \in R_+^k$ such that

$$(6)\quad \nabla_x(\alpha^T f) + \gamma(\alpha^T e) - (\nabla_{y_I x}(\overline{\lambda}^T f)\ \ \nabla_{y_J x}(\overline{\lambda}^T f))\left(\ \mu\overline{y}_J - \beta_J\ \right)$$
$$-\nabla_x\left\{(\nabla_{yy_I}(\overline{\lambda}^T f)\overline{p}\ \ \nabla_{yy_J}(\overline{\lambda}^T f)\overline{p})\begin{pmatrix} (\alpha^T e)\overline{y}_I + \frac{1}{2}(\alpha^T e)\overline{p}_I - \beta_I \\ \mu\overline{y}_J + \frac{1}{2}(\alpha^T e)\overline{p}_J - \beta_J \end{pmatrix}\right\} = 0,$$

$$(7)\quad -(\nabla_{y_I y_I}(\overline{\lambda}^T f)\ \ \nabla_{y_J y_I}(\overline{\lambda}^T f))\begin{pmatrix} (\alpha^T e)\overline{y}_I - \beta_I + (\alpha^T e)\overline{p}_I \\ \mu\overline{y}_J - \beta_J + (\alpha^T e)\overline{p}_J \end{pmatrix}$$
$$-\nabla_{y_I}\left\{(\nabla_{yy_I}(\overline{\lambda}^T f)\overline{p}\ \ \nabla_{yy_J}(\overline{\lambda}^T f)\overline{p})\begin{pmatrix} (\alpha^T e)\overline{y}_I + \frac{1}{2}(\alpha^T e)\overline{p}_I - \beta_I \\ \mu\overline{y}_J + \frac{1}{2}(\alpha^T e)\overline{p}_J - \beta_J \end{pmatrix}\right\} = 0,$$

$$(8)\quad (\alpha-\mu\overline{\lambda})^T\nabla_{y_J}f-(\alpha^Te-\mu)\overline{w}$$
$$-(\nabla_{y_Iy_J}(\overline{\lambda}^Tf)\ \ \nabla_{y_Jy_J}(\overline{\lambda}^Tf))\begin{pmatrix}(\alpha^Te)\overline{y}_I-\beta_I+\mu\overline{p}_I\\ \mu\overline{y}_J-\beta_J+\mu\overline{p}_J\end{pmatrix}$$
$$-\nabla_{y_J}\left\{(\nabla_{yy_I}(\overline{\lambda}^Tf)\overline{p}\ \ \nabla_{yy_J}(\overline{\lambda}^Tf)\overline{p})\begin{pmatrix}(\alpha^Te)\overline{y}_I+\frac{1}{2}(\alpha^Te)\overline{p}_I-\beta_I\\ \mu\overline{y}_J+\frac{1}{2}(\alpha^Te)\overline{p}_J-\beta_J\end{pmatrix}\right\}=0,$$

$$(9)\quad (\nabla_{yy_I}(\overline{\lambda}^Tf)\ \ \nabla_{yy_J}(\overline{\lambda}^Tf))\begin{pmatrix}\beta_I-(\alpha^Te)\overline{p}_I-(\alpha^Te)\overline{y}_I\\ \beta_J-(\alpha^Te)\overline{p}_J-\mu\overline{y}_J\end{pmatrix}=0,$$

$$(10)\quad (\nabla_{y_I}f\ \ \nabla_{y_J}f)\begin{pmatrix}\beta_I-(\alpha^Te)\overline{y}_I\\ \beta_J-\mu\overline{y}_J\end{pmatrix}$$
$$-(\nabla_{yy_I}f\overline{p}\ \ \nabla_{yy_J}f\overline{p})\begin{pmatrix}(\alpha^Te)\overline{y}_I+\frac{1}{2}(\alpha^Te)\overline{p}_I-\beta_I\\ \mu\overline{y}_J+\frac{1}{2}(\alpha^Te)\overline{p}_J-\beta_J\end{pmatrix}-\delta=0,$$

$$(11)\quad \beta^T[\nabla_y(\overline{\lambda}^Tf)-\overline{w}+\nabla_{yy}(\overline{\lambda}^Tf)\overline{p}]=0,$$

$$(12)\quad \mu\overline{y}_J^T[\nabla_{y_J}(\overline{\lambda}^Tf)-\overline{w}+\nabla_{yy_J}(\overline{\lambda}^Tf)\overline{p}]=0,$$

$$(13)\quad \delta^T\overline{\lambda}=0,$$

$$(14)\quad -(\alpha^Te)\overline{y}_J-\beta+\mu\overline{y}_J\in N_{C_i}(\overline{w}),\ \ i=1,2,\cdots,k,$$

$$(15)\quad \gamma\in B_i,\ \ \gamma^T\overline{x}=s(\overline{x}|B_i),\ \ i=1,2,\cdots,k,$$

$$(16)\quad (\alpha,\beta,\mu,\delta)\neq 0.$$

By the hypothesis (i), (9) yields

$$(17)\qquad \beta_I=(\alpha^Te)(\overline{p}_I+\overline{y}_I)\ \text{ and }\ \beta_J=(\alpha^Te)\overline{p}_J+\mu\overline{y}_J.$$

We claim that $\alpha\neq 0$. Indeed, if $\alpha=0$, then (17) gives $\beta_I=0$ and $\beta_J=\mu\overline{y}_J$. This together with (8) yields $\mu(\nabla_{y_J}(\overline{\lambda}^Tf)-\overline{w}+\nabla_{yy_J}(\overline{\lambda}^Tf)\overline{p})=0$. By the hypothesis (ii), this implies $\mu=0$ and hence $\beta=0$. Using $\alpha=0$ and (17) in (10), we get $\delta=0$, contradicting (16). Therefore

$$\alpha\geq 0.$$

Using (17) in (7), we get $\frac{1}{2}(\alpha^Te)\nabla_{y_I}(\overline{p}^T\nabla_{yy}(\overline{\lambda}^Tf)\overline{p})=0$, which using the hypothesis (iv) and $\alpha\geq 0$ implies

$$\overline{p}=0.$$

Then (17) by this implies

$$\beta_I = (\alpha^T e)\overline{y}_I \text{ and } \beta_J = \mu \overline{y}_J. \tag{18}$$

Using $\overline{p} = 0$ and (18) in (8), we get

$$(\alpha - \mu\overline{\lambda})^T \nabla_{y_J} f - (\alpha^T e - \mu)\overline{w} = 0, \tag{19}$$

since the hypothesis (iii), (19) implies

$$\alpha = \mu\overline{\lambda} \text{ and } \alpha^T e = \mu,$$

and, since $\alpha \geq 0$,

$$\mu > 0.$$

Using $\alpha \geq 0$, $\overline{p} = 0$ and (18) in (6), we obtain

$$\nabla_x(\alpha^T f) + \gamma(\alpha^T e) = 0.$$

Since $\alpha = \mu\overline{\lambda}$, $\lambda^T e = 1$ and $\mu > 0$, we get

$$\nabla_x(\overline{\lambda}^T f) + \gamma = 0, \tag{20}$$

and

$$\overline{x}_B^T \nabla_{x_B}(\overline{\lambda}^T f) + \overline{x}_B^T \gamma = 0. \tag{21}$$

Now, taking $\overline{z} := \gamma \in B_i$ for $i = 1, 2, \cdots, k$, we find that $(u, v, \overline{\lambda}, z, r) = (\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ satisfies constraints of (GMD) and is therefore a feasible solution for (GMD).

Moreover, using (18) in (11), we get

$$(\alpha^T e)\overline{y}_I^T[\nabla_{y_I}(\overline{\lambda}^T f) - \overline{w}] + \mu\overline{y}_J^T[\nabla_{y_J}(\overline{\lambda}^T f) - \overline{w}] = 0,$$

by (12), $(\alpha^T e)\overline{y}_I^T[\nabla_{y_I}(\overline{\lambda}^T f) - \overline{w}] = 0$, and since $\alpha^T e > 0$,

$$\overline{y}_I^T \nabla_{y_I}(\overline{\lambda}^T f) = \overline{y}_I^T \overline{w}. \tag{22}$$

Using $\alpha \geq 0$, $\overline{p} = 0$, (18) in (14), we get $\overline{y} \in N_{C_i}(\overline{w})$ for $i = 1, 2, \cdots, k$, and so $\overline{y}^T \overline{w} = s(\overline{y}|C_i)$ for $i = 1, 2, \cdots, k$, i.e.,

$$(\overline{y}^T \overline{w})e = s(\overline{y}|C). \tag{23}$$

Premultiplying (20) by $\overline{x}_A^T$, we get

$$\overline{x}_A^T \nabla_{x_A}(\overline{\lambda}^T f) + \overline{x}_A^T \overline{z} = 0. \tag{24}$$

Therefore, using (15), (22)-(24), we obtain

$$\begin{aligned}
&K(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p}) \\
&= f + s(\overline{x}|B) - (\overline{y}_J^T \overline{w})e - (\overline{y}_I^T \nabla_{y_I}(\overline{\lambda}^T f))e \\
&\quad -(\overline{y}_I^T \nabla_{yy_I}(\overline{\lambda}^T f)\overline{p})e - \frac{1}{2}(\overline{p}^T \nabla_{yy}(\overline{\lambda}^T f)\overline{p})e \\
&= f + (\overline{z}^T \overline{x})e - (\overline{y}_J^T \overline{w})e - (\overline{y}_I^T \overline{w})e \\
&= f + (\overline{z}^T \overline{x}_A)e + (\overline{z}^T \overline{x}_B)e - s(\overline{y}|C) \\
&= f - s(\overline{y}|C) - (\overline{x}_A^T \nabla_{x_A}(\overline{\lambda}^T f))e + (\overline{x}_B^T \overline{z})e \\
&\quad -(\overline{u}_A^T \nabla_{xx_A}(\overline{\lambda}^T f)\overline{r})e - \frac{1}{2}(\overline{r}^T \nabla_{xx}(\overline{\lambda}^T f)\overline{r})e \\
&= G(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r}).
\end{aligned}$$

That is, the objective values of (GMP) at $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p})$ and the objective values of (GMD) at $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r})$ are equal, i.e.,

$$K(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p}) = G(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r}). \tag{25}$$

Now, we claim that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r})$ is a properly efficient solution for (GMD). If $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r})$ is not efficient for (GMD) then there exists a feasible solution $(u, v, \overline{\lambda}, z, r)$ of (GMD) such that which by (25) gives

$$K(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p}) \leq G(u, v, \overline{\lambda}, z, r),$$

which is a contradiction to Theorem 3.1(Weak Duality).
If $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p})$ is not properly efficient for (GMD), then for some feasible $(u, v, \overline{\lambda}, z, r)$ of (GMD) and some i,

$$\begin{aligned}
&f_i(u, v) - s(v|C_i) + u_B^T z - u_A^T \nabla_{x_A}(\overline{\lambda}^T f)(u, v) \\
&-u_A^T \nabla_{xx_A}(\overline{\lambda}^T f)(u, v)r - \frac{1}{2} r^T \nabla_{xx}(\overline{\lambda}^T f)(u, v)r \\
&> f_i(\overline{x}, \overline{y}) + s(\overline{x}|B_i) - \overline{y}_J^T \overline{w} - \overline{y}_I^T \nabla_{y_I}(\overline{\lambda}^T f)(\overline{x}, \overline{y}),
\end{aligned}$$

we have

$$f_i(u,v) - s(v|C_i) + u_B^T z - u_A^T \nabla_{x_A}(\overline{\lambda}^T f)(u,v) - u_A^T \nabla_{xx_A}(\overline{\lambda}^T f)(u,v)r$$
$$-\frac{1}{2} r^T \nabla_{xx}(\overline{\lambda}^T f)(u,v)r$$
$$-f_i(\overline{x},\overline{y}) - s(\overline{x}|B_i) + \overline{y}_J^T \overline{w} + \overline{y}_I^T \nabla_{y_I}(\overline{\lambda}^T f)(\overline{x},\overline{y})$$
$$> M \ [\ f_j(\overline{x},\overline{y}) + s(\overline{x}|B_j) - \overline{y}_J^T \overline{w} - \overline{y}_I^T \nabla_{y_I}(\overline{\lambda}^T f)(\overline{x},\overline{y})$$
$$-f_j(u,v) + s(v|C_j) - u_B^T z + u_A^T \nabla_{x_A}(\overline{\lambda}^T f)(u,v) + u_A^T \nabla_{xx_A}(\overline{\lambda}^T f)(u,v)r$$
$$+\frac{1}{2} r^T \nabla_{xx}(\overline{\lambda}^T f)(u,v)r\ ],$$

for all $M > 0$ and all j satisfying

$$f_j(\overline{x},\overline{y}) + s(\overline{x}|B_j) - \overline{y}_J^T \overline{w} - \overline{y}_I^T \nabla_{y_I}(\overline{\lambda}^T f)(\overline{x},\overline{y})$$
$$> f_j(u,v) - s(v|C_j) + u_B^T z - u_A^T \nabla_{x_A}(\overline{\lambda}^T f)(u,v) - u_A^T \nabla_{xx_A}(\overline{\lambda}^T f)(u,v)r$$
$$-\frac{1}{2} r^T \nabla_{xx}(\overline{\lambda}^T f)(u,v)r.$$

This means that $f_i(u,v) - s(v|C_i) + u_B^T z - u_A^T \nabla_{x_A}(\overline{\lambda}^T f)(u,v) - u_A^T \nabla_{xx_A}(\overline{\lambda}^T f)(u,v)r$ $-\frac{1}{2} r^T \nabla_{xx}(\overline{\lambda}^T f)(u,v)r - f_i(\overline{x},\overline{y}) - s(\overline{x}|B_i) + \overline{y}_J^T \overline{w} + \overline{y}_I^T \nabla_{y_I}(\overline{\lambda}^T f)(\overline{x},\overline{y})$ can be made arbitrarily large. Thus since $\overline{\lambda} > 0$ and $\overline{\lambda}^T e = 1$, we obtain

$$(\overline{\lambda}^T f)(u,v) - \overline{\lambda}^T s(v|C) + u_B^T z - u_A^T \nabla_{x_A}(\overline{\lambda}^T f)(u,v) - u_A^T \nabla_{xx_A}(\overline{\lambda}^T f)(u,v)r$$
$$-\frac{1}{2} r^T \nabla_{xx}(\overline{\lambda}^T f)(u,v)r$$
$$> (\overline{\lambda}^T f)(\overline{x},\overline{y}) + \overline{\lambda}^T s(\overline{x}|B) - \overline{y}_J^T \overline{w} - \overline{y}_I^T \nabla_{y_I}(\overline{\lambda}^T f)(\overline{x},\overline{y}),$$

which again contradicts Theorem 3.1(Weak Duality). ∎

Theorem 3.3. (Converse Duality) *Let f be a thrice differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^k$. Let $(\overline{u},\overline{v},\overline{\lambda},\overline{z},\overline{r})$ be an efficient solution for (GMD);fix $\lambda = \overline{\lambda}$ in (GMP) and suppose that*

(i) $\nabla_{xx}(\overline{\lambda}^T f)$ *is non-singular,*
(ii) $\nabla_{x_B}(\overline{\lambda}^T f) + \overline{z} + \nabla_{xx_B}(\overline{\lambda}^T f)\overline{r} \neq 0$,
(iii) *the set* $\{\nabla_{x_B} f_1, \nabla_{x_B} f_2, \cdots, \nabla_{x_B} f_k, \overline{z}\}$ *is linearly independent, and*
(iv) *the matrix* $\frac{\partial}{\partial x_i}(\nabla_{xx}(\overline{\lambda}^T f))$ *is positive or negative definite, for some* $i \in A$.

where $f = f(\overline{u}, \overline{v})$.
Then there exists $\overline{w} \in C_i (i = 1, 2, \cdots, k)$ *such that* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p} = 0)$ *is a feasible solution for* (GMP) *and* $G(\overline{u}, \overline{v}, \overline{\lambda}, \overline{z}, \overline{r}) = K(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p})$.

Moreover, if the hypotheses of Theorem 3.1(Weak Duality) are satisfied for all feasible solutions of (GMD) *and* (GMP)*, then* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p})$ *is a properly efficient solution for* (GMP).

Proof. It follows on the lines of Theorem 3.2. ∎

Remark 3.1. In case of $\lambda \geq 0$, if we replace the second order F-convexity(or F-concavity) by the second order strict F-convexity(or strict F-concavity), then the same duality results also hold.

4. Special Cases

In this section, we consider some special cases of the programs (GMP) and (GMD) by choosing particular forms.

4.1. Mond-Weir Type Symmetric Duality

If $I = \emptyset$ and $A = \emptyset$, then our pair of programs (GMP) and (GMD) is reduced to the following multiobjective second order symmetric dual problems (MMP) and (MMD), which are Mond-Weir type ones and are different from the multiobjective second order symmatric dual problems considered in [24].

$$
\begin{aligned}
(MMP)\quad \text{Minimize}\quad & K_M(x, y, \lambda, w, p) \\
& = f(x, y) + s(x|B) - (y^T w)e - \tfrac{1}{2}(p^T \nabla_{yy}(\lambda^T f)(x, y)p)e \\
\text{subject to}\quad & \nabla_y(\lambda^T f)(x, y) - w + \nabla_{yy}(\lambda^T f)(x, y)p \leqq_q 0, \\
& y^T[\nabla_y(\lambda^T f)(x, y) - w + \nabla_{yy}(\lambda^T f)(x, y)p] \geqq_q 0, \\
& w \in C_i, \quad \lambda > 0, \quad \lambda^T e = 1, \\
(MMD)\quad \text{Maximize}\quad & G_M(u, v, \lambda, z, r) \\
& = f(u, v) - s(v|C) + (u^T z)e - \tfrac{1}{2}(r^T \nabla_{xx}(\lambda^T f)(u, v)r)e \\
\text{subject to}\quad & \nabla_x(\lambda^T f)(u, v) + z + \nabla_{xx}(\lambda^T f)(u, v)r \geqq_q 0, \\
& u^T[\nabla_x(\lambda^T f)(u, v) + z + \nabla_{xx}(\lambda^T f)(u, v)r] \leqq_q 0, \\
& z \in B_i, \quad \lambda > 0, \quad \lambda^T e = 1,
\end{aligned}
$$

Theorem 4.1. (Weak Duality) *Let* (x, y, λ, w, p) *be feasible for* (MMP) *and* (u, v, λ, z, r) *be feasible for* (MMD)*. Assume that*

(i) $(\lambda^T f)(.,v) + (.)^T z$ *is second order F-pseudoconvex in the first variable,*

(ii) $(\lambda^T f)(x,.) - (.)^T w$ *is second order F-pseudoconcave in the second variable,*

(iii) $F_{x,u}(a) + a^T u \geqq q\, 0$ *for all* $a \in \mathbb{R}^n_+$*, and*

(iv) $G_{v,y}(b) + b^T y \geqq q\, 0$ *for all* $b \in \mathbb{R}^m_+$.

Then $K_M(x, y, \lambda, w, p) \nleq G_M(u, v, \lambda, z, r)$.

Theorem 4.2. (Strong Duality) *Let* f *be a thrice differentiable function from* $\mathbb{R}^n \times \mathbb{R}^m$ *to* $\mathbb{R}^k$ *and* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p})$ *be an efficient solution for* (MMP)*; fix* $\lambda = \overline{\lambda}$ *in* (MMD). *Suppose that*

(i) $\nabla_{yy}(\overline{\lambda}^T f)$ *is non-singular,*

(ii) $\nabla_y(\overline{\lambda}^T f) - \overline{w} + \nabla_{yy}(\overline{\lambda}^T f)\overline{p} \neq 0$,

(iii) *the set* $\{\nabla_y f_1, \nabla_y f_2, \cdots, \nabla_y f_k, \overline{w}\}$ *is linearly independent, and*

(iv) *the matrix* $\frac{\partial}{\partial y_i}(\nabla_{yy}(\overline{\lambda}^T f))$ *is positive or negative definite,*

for some $i \in \{1, 2, \cdots, m\}$*, where* $f = f(\overline{x}, \overline{y})$.
Then there exists $\overline{z} \in B_i (i = 1, 2, \cdots, k)$ *such that* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ *is a feasible solution for* (MMD) *and* $K_M(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p}) = G_M(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r})$.

Moreover, if the hypotheses of Theorem 4.1.(Weak Duality) are satisfied for all feasible solutions of (MMP) *and* (MMD)*, then* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r})$ *is a properly efficient solution for* (MMD).

Theorem 4.3. (Converse Duality) *Let* f *be a thrice differentiable function from* $\mathbb{R}^n \times \mathbb{R}^m$ *to* $\mathbb{R}^k$ *and* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{z}, \overline{r})$ *be an efficient solution for* (MMD)*; fix* $\lambda = \overline{\lambda}$ *in* (MMP). *Suppose that*

(i) $\nabla_{xx}(\overline{\lambda}^T f)$ *is non-singular,*

(ii) $\nabla_x(\overline{\lambda}^T f) + \overline{z} + \nabla_{xx}(\overline{\lambda}^T f)\overline{r} \neq 0$,

(iii) *the set* $\{\nabla_x f_1, \nabla_x f_2, \cdots, \nabla_x f_k, \overline{z}\}$ *is linearly independent, and*

(iv) *the matrix* $\frac{\partial}{\partial x_i}(\nabla_{xx}(\overline{\lambda}^T f))$ *is positive or negative definite, for some* $i \in \{1, 2, \cdots, n\}$,

where $f = f(\overline{u}, \overline{v})$.
Then there exists $\overline{w} \in C_i (i = 1, 2, \cdots, k)$ *such that* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p} = 0)$ *is a feasible solution for* (MMP) *and* $G_M(\overline{u}, \overline{v}, \overline{\lambda}, \overline{z}, \overline{r}) = K_M(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p})$.

Moreover, if the hypotheses of Theorem 4.1.(Weak Duality) are satisfied for all feasible solutions of (MMD) *and* (MMP)*, then* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p})$ *is a properly efficient solution for* (MMP).

The proof of above duality theorems for Mond-Weir type models can be connected with Theorems 3.1, 3.2 and 3.3.

4.2. Wolfe Type Symmetric Duality

If $J = \emptyset$ and $B = \emptyset$, then our pair of programs (GMP) and (GMD) is reduced to the following multiobjective second order symmetric dual problems (WMP) and(WMD), which are Wolfe type ones.

$$
\begin{array}{lll}
(WMP) & \text{Minimize} & K_W(x, y, \lambda, w, p) \\
& & = f(x, y) + s(x|B) - (y^T \nabla_y (\lambda^T f)(x, y))e \\
& & -(y^T \nabla_{yy} (\lambda^T f)(x, y)p)e - \frac{1}{2}(p^T \nabla_{yy} (\lambda^T f)(x, y)p)e \\
& \text{subject to} & \nabla_y (\lambda^T f)(x, y) - w + \nabla_{yy} (\lambda^T f)(x, y)p \leqq q\, 0, \\
& & w \in C_i, \quad \lambda > 0, \quad \lambda^T e = 1, \\
(WMD) & \text{Maximize} & G_W(u, v, \lambda, z, r) \\
& & = f(u, v) - s(v|C) - (u^T \nabla_x (\lambda^T f)(u, v))e \\
& & -(u^T \nabla_{xx} (\lambda^T f)(u, v)r)e - \frac{1}{2}(r^T \nabla_{xx} (\lambda^T f)(u, v)r)e \\
& \text{subject to} & \nabla_x (\lambda^T f)(u, v) + z + \nabla_{xx} (\lambda^T f)(u, v)r \geqq q\, 0, \\
& & z \in B_i, \quad \lambda > 0, \quad \lambda^T e = 1,
\end{array}
$$

Theorem 4.4. (Weak Duality) *Let (x, y, λ, w, p) be feasible for (WMP) and (u, v, λ, z, r) be feasible for (WMD). Assume that*

- (i) *$f(., v) + ((.)^T z)e$ is second order F-convex in the first variable,*
- (ii) *$f(x, .) - ((.)^T w)e$ is second order F-concave in the second variable,*
- (iii) *$F_{x,u}(a) + a^T u \geqq q\, 0$ for all $a \in \mathbb{R}^n_+$, and*
- (iv) *$G_{v,y}(b) + b^T y \geqq q\, 0$ for all $b \in \mathbb{R}^m_+$.*

Then $K_W(x, y, \lambda, w, p) \nleq G_W(u, v, \lambda, z, r)$.

Theorem 4.5. (Strong Duality) *Let f be a thrice differentiable function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^k$ and $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p})$ be an efficient solution for (WMP); fix $\lambda = \overline{\lambda}$ in (WMD). Suppose that*

- (i) *$\nabla_{yy}(\overline{\lambda}^T f)$ is non-singular, and*
- (ii) *the matrix $\frac{\partial}{\partial y_i}(\nabla_{yy}(\overline{\lambda}^T f))$ is positive or negative definite, for some $i \in \{1, 2, \cdots, m\}$,*

where $f = f(\overline{x}, \overline{y})$.
Then there exists $\overline{z} \in B_i (i = 1, 2, \cdots, k)$ *such that* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r} = 0)$ *is a feasible solution for* (WMD) *and* $K_W(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w}, \overline{p}) = G_W(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r})$.

Moreover, if the hypotheses of Theorem 4.4.(Weak Duality) are satisfied for all feasible solutions of (WMP) *and* (WMD), *then* $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{z}, \overline{r})$ *is a properly efficient solution for* (WMD).

Theorem 4.6. (Converse Duality) *Let* f *be a thrice differentiable function from* $\mathbb{R}^n \times \mathbb{R}^m$ *to* $\mathbb{R}^k$ *and* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{z}, \overline{r})$ *be an efficient solution for* (WMD); *fix* $\lambda = \overline{\lambda}$ *in* (WMP). *Suppose that*

(i) $\nabla_{xx}(\overline{\lambda}^T f)$ *is non-singular, and*

(ii) *the matrix* $\frac{\partial}{\partial x_i}(\nabla_{xx}(\overline{\lambda}^T f))$ *is positive or negative definite, for some* $i \in \{1, 2, \cdots, n\}$,

where $f = f(\overline{u}, \overline{v})$.
Then there exists $\overline{w} \in C_i (i = 1, 2, \cdots, k)$ *such that* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p} = 0)$ *is a feasible solution for* (WMP) *and* $G_W(\overline{u}, \overline{v}, \overline{\lambda}, \overline{z}, \overline{r}) = K_W(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p})$.

Moreover, if the hypotheses of Theorem 4.4(Weak Duality) are satisfied for all feasible solutions of (WMD) *and* (WMP), *then* $(\overline{u}, \overline{v}, \overline{\lambda}, \overline{w}, \overline{p})$ *is a properly efficient solution for* (WMP).

The proof of above duality theorems for Wolfe type models can be connected with Theorems 3.1, 3.2 and 3.3.

4.3. Remarks and Example

We give some special cases of our symmetric duality.

(1) If $k = 1$, then (MMP) and (MMD) are reduced to the second order symmetric dual programs in Hou and Yang [8].

(2) Let $D \in \mathbb{R}^n \times \mathbb{R}^n$ and $E \in \mathbb{R}^m \times \mathbb{R}^m$ are positive semidefinite symmetric matrices. If $s(x|B) = (x^T Dx)^{\frac{1}{2}}$ where $B = \{Dz | z^T Dz \leqq q\,1\}$ and $s(y|C) = (y^T Ey)^{\frac{1}{2}}$ where $C = \{Ew | w^T Ew \leqq q\,1\}$, then the pair of programs (MMP) and (MMD) is reduced to the nondifferentiable second order symmetric duality in multiobjective programs, which is different from the problems in Ahmad and Husain [1].

(3) If $B = C = \{0\}$, then (MMP) and (MMD) are reduced to the second order multiobjective symmetric dual programs, which is different from the programs in Suneja et al. [18].

(4) If $B = C = \{0\}$, $p = r = 0$, and $k = 1$, then we get the first order symmetric dual programs which studied by Chandra et al. [4].

(5) If $k = 1$, then (WMP) and (WMD) are reduced to the second order symmetric dual programs studied by Yang et al. [22].

(6) Let $D \in \mathbb{R}^n \times \mathbb{R}^n$ and $E \in \mathbb{R}^m \times \mathbb{R}^m$ are positive semidefinite symmetric matrices. If $s(x|B) = (x^T Dx)^{\frac{1}{2}}$ where $B = \{Dz | z^T Dz \leqq q\,1\}$ and $s(y|C) = (y^T Ey)^{\frac{1}{2}}$ where $C = \{Ew | w^T Ew \leqq q\,1\}$, then the pair of programs (WMP) and (WMD) is reduced to the nondifferentiable second order symmetric duality in multiobjective programming. In addition, if $k = 1$, then we get the second order symmetric dual programs on nondifferentiable studied by Ahmad and Husain [2].

(7) If $B = C = \{0\}$, then (WMP) and (WMD) are reduced to the second order multiobjective symmetric dual programs studied by Yang et al. [23].

(8) If $B = C = \{0\}$, $p^T(\nabla_{yy}(\lambda^T f)(x, y))p = 0$ for some $p \neq 0$ and $r^T(\nabla_{xx}(\lambda^T f)(u, v))r = 0$ for some $r \neq 0$, then (WMP) and (WMD) are reduced to the second order symmetric dual programs studied by Kim et al. [10].

In particular, our assumptions are more classical and meaningful than ones in [1, 8, 18, 22, 23, 24] in the sense that those are closely related to conditions of the first order symmetric duality for multiobective programming by Mond and Weir [16]. Moreover, our results generalize and improve the corresponding first order works.

Example 1. Let $n = m = 2$, $f_1(x, y) = x_1^2 + x_2^2 - y_1^2 - y_2^2$, $f_2(x, y) = e^{x_1+x_2} - e^{-y_1-y_2}$, $B_1 = C_2 = [0, 1] \times [0, 1]$, $B_2 = C_1 = \{0\}$ and $I = A = \{1\}$, $J = B = \{2\}$. Then $s(x|B_1) = \frac{1}{2}(x_1+x_2+(x_1^2+x_2^2)^{\frac{1}{2}})$, $s(x|B_2) = 0$, $s(v|C_1) = 0$, $s(v|C_2) = \frac{1}{2}(v_1 + v_2 + (v_1^2 + v_2^2)^{\frac{1}{2}})$. Problems (GMP) and (GMD) become

$$
\begin{aligned}
(GMP') \quad \text{Minimize} \quad & K(x, y, \lambda, w, p) \\
& = (x_1^2 + x_2^2 - (1 - 2\lambda_1)y_1^2 - y_2^2 \\
& +\frac{1}{2}(x_1 + x_2 + (x_1^2 + x_2^2)^{\frac{1}{2}}) - w_2 y_2 \\
& +(2p_1 y_1 + p_1^2 + p_2^2)\lambda_1 + [(p_1 + 1)y_1 \\
& +\frac{1}{2}(p_1 + p_2)^2]e^{-y_1-y_2}\lambda_2, \\
& e^{x_1+x_2} - e^{-y_1-y_2} - w_2 y_2 + (2p_1 y_1 + p_1^2 + p_2^2)\lambda_1 \\
& +[(p_1 + p_2 - 1)y_1 + \frac{1}{2}(p_1 + p_2)^2]e^{-y_1-y_2}\lambda_2)
\end{aligned}
$$

$$\text{subject to} \quad -\begin{pmatrix} 2(y_1+p_1)\lambda_1-(1-p_1-p_2)e^{-y_1-y_2}\lambda_2+w_1 \\ 2(y_2+p_2)\lambda_1-(1-p_1-p_2)e^{-y_1-y_2}\lambda_2+w_2 \end{pmatrix} \leqq q\, 0,$$

$$-y_2[2(y_2+p_2)\lambda_1-(1-p_1,-p_2)e^{-y_1,-y_2}\lambda_2+w_2] \geqq 0,$$

$$w \in C_2, \quad \lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_1 + \lambda_2 = 1,$$

(GMD') Maximize $G(u, v, \lambda, z, r)$

$$= ((1-2\lambda_1)u_1^2 + u_2^2 - v_1^2 - v_2^2 + z_2u_2$$

$$-(2r_1u_1 + r_1^2 + r_2^2)\lambda_1 - [(r_1+1)u_1$$

$$+\frac{1}{2}(r_1+r_2)^2]e^{u_1+u_2}\lambda_2,$$

$$e^{u_1+u_2} - e^{-v_1-v_2} - \frac{1}{2}(v_1 + v_2 + (v_1^2+v_2^2)^{\frac{1}{2}}) + z_2u_2$$

$$-(2r_1u_1 + r_1^2 + r_2^2)\lambda_1 - [(r_1+1)u_1$$

$$+\frac{1}{2}(r_1+r_2)^2]e^{u_1+u_2}\lambda_2)$$

$$\text{subject to} \quad \begin{pmatrix} 2(u_1+r_1)\lambda_1 + (1+r_1+r_2)e^{u_1+u_2}\lambda_2 + z_1 \\ 2(u_2+r_2)\lambda_1 + (1+r_1+r_2)e^{u_1+u_2}\lambda_2 + z_2 \end{pmatrix} \geqq q\, 0,$$

$$u_2[2(u_2+r_2)\lambda_1 + (1+r_1+r_2)e^{u_1+u_2}\lambda_2 + z_2] \leqq q\, 0,$$

$$z \in B_1, \quad \lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_1 + \lambda_2 = 1.$$

Our symmetric duality results can be applied to (GMP') and (GMD'). However, the symmetric duality for (GMP') and (GMD') cannot be proven by the results in [1, 2, 4, 8, 18, 22, 23, 24], because our models (GMP') and (GMD') are a pair of generalized multiobjective programming problems with non-differentiable terms $s(x|B)$ or $s(v|C)$.

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