TAIWANESE JOURNAL OF MATHEMATICS
Vol. 11, No. 3, pp. 717-728, August 2007
This paper is available online at http://www.math.nthu.edu.tw/tjm/

ITERATIVE ALGORITHMS AND CONVERGENCE THEOREMS FOR SOLVING F-IGVIP AND F-IGCP

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Abstract. In this paper, we study the iterative algorithm and convergence theorems for F-implicit generalized variational inequalities problem (F-IGVIP). By employing our earlier works ([6], Theorem 2.2), we establish several iterative convergence results for F-IGVIP. The algorithm and convergence results are new for solving the strong solution of F-IGVIP. Furthermore, new algorithms and convergence theorems for F-implicit generalized complementarity problem (F-IGCP) are also discussed.

1. Introduction and Preliminaries

In very recent years, iterative algorithms have been established for solving variational inequalities. Ding el al. [1, 3] present a predictor-corrector iterative algorithms for solving generalized mixed variational-like problems.

Motivated and inspired by the above works, the purpose of this paper is to establish the predictor-corrector iterative algorithms and discuss the convergence theorems for solving the strong solution of F-implicit generalized variational inequalities problem (F-IGVIP) which is discussed by Zeng et al. [6].

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $C(H)$ be the family of all nonempty compact subsets of H. Let $T : H \to C(H)$ be a set-valued mapping, $F : H \to \mathbb{R}$, $g : H \to H$ be two single-valued mappings. In very recent year, Zeng et al. [6] consider the following F-implicit generalized variational inequalities problem (F-IGVIP) is to find an $\bar{x} \in H$ with an $\bar{s} \in T(\bar{x})$ such that

$$\langle \bar{s}, x - g(\bar{x})\rangle \geq F(g(\bar{x})) - F(x) \tag{1.1}$$

for all $x \in H$, and we say a solution of (1.1) is a *strong solution* of F-IGVIP (we refer to [6]).

There are some special cases of (F-IGVIP):

Received December 14, 2006.
Communicated by J. C. Yao.
2000 *Mathematics Subject Classification*: 90C33, 49J40.
Key words and phrases: Iterative algorithm, Partially relaxed g-strongly monotone, F-IGVIP, F-IGCP.

(1) If T is a single valued mapping, then the (F-IGVIP) is equivalent to the (F-IVIP) which is to find an $\bar{x} \in H$ such that

$$\langle T(\bar{x}), x - g(\bar{x}) \rangle \geq F(g(\bar{x})) - F(x) \tag{1.2}$$

for all $x \in H$. This problem was introduced and studied in [2]

(2) If H is a Hilbert space, T is a single valued mapping and g is an identity mapping, then the (F-IGVIP) is equivalent to find an $\bar{x} \in H$ such that

$$\langle T(\bar{x}), x - \bar{x} \rangle \geq F(\bar{x}) - F(x) \tag{1.3}$$

for all $x \in H$. This problem is known as a variational inequality introduced by Stampacchia [4].

(3) If $H = \mathbb{R}^n$ and $F \equiv 0$, g is an identity mapping, then the (F-IGVIP) is equivalent to find $\bar{x} \in H$ and $\bar{s} \in T(\bar{x})$ such that

$$\langle \bar{s}, x - \bar{x} \rangle \geq 0 \tag{1.4}$$

for all $x \in H$. This problem was introduced and studied by Fang and Peterson[2], Yao and Guo[5].

For the detail, we refer to [6]. We first give some definitions which will use in the sequel.

Definition 1.1. Let $T : H \to C(H)$ be a set-valued mapping.

(1) T is a partially relaxed strongly monotone w.r.t. g if there is a constant $\alpha > 0$ such that

$$\langle u_1, g(v_2) - g(z) \rangle + \langle u_2, g(z) - g(v_2) \rangle \leq \alpha \|g(v_1) - g(z)\|^2, \ \forall v_1, v_2, z \in H, \ u_i \in T(v_i), \ i = 1, 2.$$

(2) T is D-continuous on H if $\{x_n\} \subset H$ and $x_n \to x$, then $T(x_n) \to T(x)$ under the Hausdorff metric D on $C(H)$. T is D-uniformly continuous on H if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $x, y \in H$ with $\|x - y\| < \delta$, then $D(T(x), T(y)) < \epsilon$.

(3) T is D-convergent preserving set-valued mapping if $\{a_n\}, \ \{b_n\} \subset H$ and $D(T(a_n), T(b_n)) \to 0$ as $n \to \infty$ under the Hausdorff metric D on $C(X)$ implies the sequence $\|a_n - b_n\| \to 0$ as $n \to \infty$.

We note that if $z = v_1$ and g is an identity mapping, then a partially relaxed strongly monotone w.r.t. g is a monotone mapping.

We need the following theorem which we can directly derive from Theorem 2.2 and Theorem 2.3[6].

Theorem A. *Let the mapping $F : H \to \mathbb{R}$ be lower semicontinuous and convex, $g : H \to H$ be continuous and $T : H \to 2^H$ be upper semicontinuous with nonempty compact convex values. Suppose that*

(1) *for each $x \in H$, there is an $s \in T(x)$ such that $\langle s, g(x) - x\rangle \leq F(x) - F(g(x))$,*

(2) *there is a nonempty compact convex subset C of H, such that for each $x \in H \setminus C$, there is a $y \in C$ such that for some $s \in T(x)$, $\langle s, y - g(x)\rangle < F(g(x)) - F(y)$.*

Then there is a strong solution of F-IGVIP.

2. Iterative Algorithm and Convergence Theorems

In this section, we first consider the auxiliary F-implicit generalized variational inequalities problems as follows:

For any given $\bar{x} \in H$, $\bar{s} \in T(\bar{x})$, to find a $w \in H$ such that

$$\langle g(w) - g(\bar{x}), x - g(w)\rangle + \rho\langle \bar{s}, x - g(w)\rangle + \rho F(x) - \rho F(g(w)) \geq 0, \tag{2.1}$$

for all $x \in H$, where $\rho > 0$ is a constant. We note that if $w = \bar{x}$, then $\bar{x}$ is a strong solution of F-IGVIP. This observation enables us to suggest the following new predictor-corrector method for solving the strong solution of F-IGVIP.

Algorithm 2.1. For given $x_0 \in H$, $s_0 \in T(x_0)$, compute the approximate solution x_n of F-IGVIP with $s_n \in T(x_n)$ by the following iterative schemes.

$$\begin{aligned}&\langle g(y_n) - g(x_n), x - g(y_n)\rangle + \mu\langle s_n, x - g(y_n)\rangle + \mu F(x) - \mu F(g(y_n))\\ &\quad \geq 0,\ \forall x \in H,\end{aligned} \tag{2.2}$$

$$\begin{aligned}&\langle g(w_n) - g(y_n), x - g(w_n)\rangle + \beta\langle \xi_n, x - g(w_n)\rangle + \beta F(x) - \beta F(g(w_n))\\ &\quad \geq 0,\ \forall x \in H,\end{aligned} \tag{2.3}$$

$$\begin{aligned}&\langle g(x_{n+1}) - g(w_n), x - g(x_{n+1})\rangle + \rho\langle \eta_n, x - g(x_{n+1})\rangle + \rho F(x)\\ &\quad -\rho F(g(x_{n+1})) \geq 0,\ \forall x \in H,\end{aligned} \tag{2.4}$$

$$s_n \in T(x_n) : \|s_{n+1} - s_n\| \leq D(T(x_{n+1}), T(x_n)), \tag{2.5}$$

(2.6) $$\xi_n \in T(y_n) : \|\xi_{n+1} - \xi_n\| \leq D(T(y_{n+1}), T(y_n)),$$

(2.7) $$\eta_n \in T(w_n) : \|\eta_{n+1} - \eta_n\| \leq D(T(w_{n+1}), T(w_n)),$$

where $\mu,\ \beta,\ \rho > 0$ are constants, and D is the Hausdorff metric on $C(H)$.

In order to obtain the convergence theorem, we need the following lemma:

Lemma 2.1. *Let $\bar{x}$ be the strong solution of F-IGVIP, $\bar{s} \in T(\bar{x})$ and $\{x_n\}$, $\{w_n\}$, $\{y_n\}$ be the sequences of approximate solutions of F-IGVIP generated by the Algorithm 2.1. Suppose that T is a partially relaxed strongly monotone w.r.t. g with constant $\alpha > 0$. Then*

(2.8) $$\|g(x_{n+1}) - g(\bar{x})\|^2 \leq \|g(x_n) - g(\bar{x})\|^2 - (1-2\rho\alpha)\|g(x_{n+1}) - g(w_n)\|^2,$$

(2.9) $$\|g(w_n) - g(\bar{x})\|^2 \leq \|g(w_{n-1}) - g(\bar{x})\|^2 - (1-2\beta\alpha)\|g(w_n) - g(y_n)\|^2,$$

(2.10) $$\|g(y_n) - g(\bar{x})\|^2 \leq \|g(y_{n-1}) - g(\bar{x})\|^2 - (1-2\mu\alpha)\|g(y_n) - g(x_n)\|^2,$$

where $0 < \rho, \beta, \mu < \frac{1}{2\alpha}$.

Proof. The conclusion can be derived by using the technique of Lemma 3.1[1]. For the sake of completeness, we give the proof as follows.

For the constants μ, β, ρ with $0 < \rho, \beta, \mu < \frac{1}{2\alpha}$. Let $\bar{x}$ be the strong solution of F-IGVIP and $\bar{s} \in T(\bar{x})$. Then

(2.11) $$\mu\langle \bar{s}, x - g(\bar{x})\rangle - \mu F(g(\bar{x})) + \mu F(x) \geq 0$$

(2.12) $$\beta\langle \bar{s}, x - g(\bar{x})\rangle - \beta F(g(\bar{x})) + \beta F(x) \geq 0$$

(2.13) $$\rho\langle \bar{s}, x - g(\bar{x})\rangle - \rho F(g(\bar{x})) + \rho F(x) \geq 0$$

for all $x \in H$.

Taking $x = g(x_{n+1})$ in (2.13) and $x = g(\bar{x})$ in (2.4), we have

(2.14) $$\rho\langle \bar{s}, g(x_{n+1}) - g(\bar{x})\rangle - \rho F(g(\bar{x})) + \rho F(g(x_{n+1})) \geq 0,$$

(2.15) $$\begin{aligned}&\langle g(x_{n+1}) - g(w_n), g(\bar{x}) - g(x_{n+1})\rangle + \rho\langle \eta_n, g(\bar{x}) - g(x_{n+1})\rangle \\ &+\rho F(g(\bar{x})) - \rho F(g(x_{n+1})) \geq 0.\end{aligned}$$

Adding (2.14) and (2.15), we have

$$\begin{aligned} \langle g(x_{n+1}) - g(w_n), g(\bar{x}) - g(x_{n+1})\rangle + \rho\langle \eta_n, g(\bar{x}) - g(x_{n+1})\rangle \\ +\rho\langle \bar{s}, g(x_{n+1}) - g(\bar{x})\rangle \geq 0. \end{aligned} \tag{2.16}$$

Since T is a partially relaxed strongly monotone w.r.t. g with constant $\alpha > 0$, we get

$$\langle g(x_{n+1}) - g(w_n), g(\bar{x}) - g(x_{n+1})\rangle \geq -\rho\alpha\|g(x_{n+1}) - g(w_n)\|^2. \tag{2.17}$$

Since

$$\begin{aligned} \|g(\bar{x}) - g(w_n)\|^2 &= \|(g(\bar{x} - g(x_{n+1}) + (g(x_{n+1} - g(w_n))\|^2 \\ &= \|g(\bar{x}) - g(x_{n+1})\|^2 + \|g(x_{n+1}) - g(w_n)\|^2 + 2\langle g(\bar{x}) \\ &\quad -g(x_{n+1}), g(x_{n+1}) - g(w_n)\rangle, \end{aligned}$$

we have

$$\begin{aligned} -\rho\alpha\|g(x_{n+1}) - g(w_n)\|^2 &\leq \langle g(x_{n+1}) - g(w_n), g(\bar{x}) - g(x_{n+1})\rangle \\ &= \frac{1}{2}[\|g(\bar{x}) - g(w_n)\|^2 \\ &\quad -\|g(\bar{x}) - g(x_{n+1})\|^2 - \|g(x_{n+1}) - g(w_n)\|^2]. \end{aligned}$$

Thus, $\|g(\bar{x}) - g(x_{n+1})\|^2 \leq \|g(\bar{x}) - g(w_n)\|^2 - (1-2\rho\alpha)\|g(x_{n+1}) - g(w_n)\|^2$ and this prove (2.8). Similarly, we have (2.9) and (2.10). ■

Now, we deduce the convergence theorem for the iterative algorithm we constructed by Algorithm 2.1. We denote the strong solution set Ω of the F-IGVIP as follows:

$$\Omega = \{\bar{x} \in H : \exists \ an\ \bar{s} \in T(\bar{x}) \text{ with } \langle \bar{s}, x - g(\bar{x})\rangle + F(x) - F(g(\bar{x})) \geq 0 \ \forall x \in H\}.$$

Theorem 2.1. *Let H be a finite-dimensional Hilbert space, $g : H \to H$ be continuous, $F : H \to \mathbb{R}$ be lower semi-continuous and convex, $g^{-1} : g(H) \to C(H)$ be D-uniformly continuous and bounded set-valued mapping where g^{-1} is bounded means the image of a bounded set under the mapping g^{-1} is bounded, $g^{-1} \circ g : H \to C(H)$ be D-convergent preserving set-valued mapping and $T : H \to C(H)$ be D-continuous set-valued mapping such that T is partially relaxed strongly monotone w.r.t. g with constant $\alpha > 0$ Suppose that the solution set Ω of the F-IGVIP is nonempty. Then for any given $x_0 \in H$, $s_0 \in T(x_0)$, the iterative* *sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{w_n\}$ *defined by Algorithm 2.1 with* $0 < \rho,\ \mu,\ \beta < \frac{1}{2\alpha}$ *converge strongly to an* $\hat{x} \in \Omega$ *which is a strong solution of the* F*-IGVIP.*

Proof. For any $\bar{x} \in \Omega$ with an $\bar{s} \in T(\bar{x})$ such that $\langle \bar{s}, x - g(\bar{x}) \rangle + F(x) - F(g(\bar{x})) \geq 0\ \forall x \in H$. From (2.8)-(2.10) in Lemma 2.1 it follows that the sequences $\{\|g(x_n) - g(\bar{x})\|\}$, $\{\|g(w_n) - g(\bar{x})\|\}$ and $\{\|g(y_n) - g(\bar{x})\|\}$ are non-increasing and hence $\{g(y_n)\}$, $\{g(w_n)\}$ and $\{g(x_n)\}$ are bounded. Furthermore, we have

$$\sum_{n=0}^{\infty}(1 - 2\rho\alpha)\|g(x_{n+1}) - g(w_n)\|^2 \leq \|g(x_0) - g(\bar{x})\|^2,$$

$$\sum_{n=0}^{\infty}(1 - 2\beta\alpha)\|g(w_n) - g(y_n)\|^2 \leq \|g(w_0) - g(\bar{x})\|^2,$$

$$\sum_{n=0}^{\infty}(1 - 2\mu\alpha)\|g(y_n) - g(x_n)\|^2 \leq \|g(y_0) - g(\bar{x})\|^2.$$

From these inequalities, we have $\|g(x_{n+1}) - g(w_n)\| \to 0$, $\|g(w_n) - g(y_n)\| \to 0$ and $\|g(y_n) - g(x_n)\| \to 0$ as $n \to \infty$. Since g^{-1} is D-uniformly continuous, we have

$$D(g^{-1}(g(x_{n+1})), g^{-1}(g(w_n))) \to 0,$$

$$D(g^{-1}(g(w_n)), g^{-1}(g(y_n))) \to 0$$

and

$$D(g^{-1}(g(y_n)), g^{-1}(g(x_n))) \to 0$$

as $n \to \infty$. Since $g^{-1} \circ g$ is D-convergent preserving set-valued mapping, we can deduce that $\|x_{n+1} - w_n\| \to 0$, $\|w_n - y_n\| \to 0$ and $\|y_n - x_n\| \to 0$ as $n \to \infty$. Therefore we have

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - w_n\| + \|w_n - y_n\| + \|y_n - x_n\| \to 0, \text{ as } n \to \infty. \tag{2.18}$$

Since $\{g(y_n)\}$, $\{g(w_n)\}$ and $\{g(x_n)\}$ are bounded, from the boundedness of g^{-1}, we have the sequences $\{y_n\}$, $\{w_n\}$ and $\{x_n\}$ are bounded. Hence there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to \hat{x}$ and hence $y_{n_i} \to \hat{x}$. Since T is D-continuous on H, by using the same argument of Theorem 2.1 in [1] that there is a subsequence $\{s_{n_{i_j}}\}$ of $\{s_{n_i}\}$ such that $s_{n_{i_j}} \to \hat{s}$ and $\hat{s} \in T(\hat{x})$.

By (2.2), the continuity of g and the lower semi-continuity of F, we have

$$\langle \hat{s}, x - g(\hat{x}) \rangle + F(x) - F(g(\hat{x})) \geq 0\ \forall x \in H.$$

Hence $\hat{x} \in \Omega$ is a strong solution of the F-IGVIP.

By (2.18), we have $x_n \to \hat{x}$ as $n \to \infty$ and this also implies that $y_n \to \hat{x}$ and $w_n \to \hat{x}$ as $n \to \infty$. Since T is D-continuous on H, by (2.5), we have $\|s_{n+1} - s_n\| \leq D(T(x_{n+1}), T(x_n)) \to 0$ as $n \to \infty$. It follows that $s_n \to \hat{s}$ as $n \to \infty$. This complete the proof. ∎

The following result, we combine the results of Theorem A and Theorem 2.1 to develop the both existence result and efficient iterative convergence theorem in order to approach to the strong solution of F-IGVIP.

Theorem 2.2. *Let H be a finite-dimensional Hilbert space, $g : H \to H$ be continuous, $F : H \to \mathbb{R}$ be lower semi-continuous and convex, $g^{-1} : g(H) \to C(H)$ be D-uniformly continuous and bounded set-valued mapping, $g^{-1} \circ g : H \to C(H)$ be D-convergent preserving set-valued mapping and $T : H \to C(H)$ be upper semi-continuous and D-continuous set-valued mapping with convex values such that T is partially relaxed strongly monotone w.r.t. g with constant $\alpha > 0$. Suppose that*

(1) *for each $x \in H$, there is an $s \in T(x)$ such that $\langle s, g(x) - x\rangle \leq F(x) - F(g(x))$,*
(2) *there is a nonempty compact convex subset C of H, such that for each $x \in H \setminus C$, there is a $y \in C$ such that for some $s \in T(x)$, $\langle s, y - g(x)\rangle < F(g(x)) - F(y)$.*

Then for any given $x_0 \in H$, $s_0 \in T(x_0)$, the iterative sequences $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$ defined by Algorithm 2.1 with $0 < \rho,\ \mu,\ \beta < \frac{1}{2\alpha}$ converge strongly to a strong solution $\hat{x}$ of the F-IGVIP.

Proof. The existence result for solving F-IGVIP follows from Theorem A. Hence the solution set of the F-IGVIP is nonempty. Applying Theorem 2.1, we know that for any given $x_0 \in H$, $s_0 \in T(x_0)$, the iterative sequences $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$ defined by Algorithm 2.1 with $0 < \rho,\ \mu,\ \beta < \frac{1}{2\alpha}$ converge strongly to a strong solution $\hat{x}$ of the F-IGVIP. ∎

We note that if g is injection in Theorem 2.1, then the D-convergent preservation of $g^{-1} \circ g$ is fulfilled. Hence we have the following results.

Theorem 2.3. *Let H be a finite-dimensional Hilbert space, $g : H \to H$ be continuous injection and $g^{-1} : g(H) \to H$ is D-uniformly continuous, $F : H \to \mathbb{R}$ be lower semi-continuous and convex and $T : H \to C(H)$ be D-continuous set-valued mapping such that T is partially relaxed strongly monotone w.r.t. g with constant $\alpha > 0$. Suppose that the solution set Ω of the F-IGVIP is nonempty. Then for any given $x_0 \in H$, $s_0 \in T(x_0)$, the iterative sequences $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$*

defined by Algorithm 2.1 with $0 < \rho,\ \mu,\ \beta < \frac{1}{2\alpha}$ *converge strongly to an* $\hat{x} \in \Omega$ *which is a strong solution of the* F*-IGVIP.*

Proof. For any $\bar{x} \in \Omega$ with $\bar{s} \in T(\bar{x})$ such that $\langle \bar{s}, x - g(\bar{x})\rangle + F(x) - F(g(\bar{x})) \geq 0\ \forall x \in H$. From (2.8)-(2.10) in Lemma 2.1 it follows that the sequences $\{\|g(x_n) - g(\bar{x})\|\}$, $\{\|g(w_n) - g(\bar{x})\|\}$ and $\{\|g(y_n) - g(\bar{x})\|\}$ are non-increasing and hence $\{g(y_n)\}$, $\{g(w_n)\}$ and $\{g(x_n)\}$ are bounded. Furthermore, we have

$$\sum_{n=0}^{\infty}(1 - 2\rho\alpha)\|g(x_{n+1}) - g(w_n)\|^2 \leq \|g(x_0) - g(\bar{x})\|^2,$$

$$\sum_{n=0}^{\infty}(1 - 2\beta\alpha)\|g(w_n) - g(y_n)\|^2 \leq \|g(w_0) - g(\bar{x})\|^2,$$

$$\sum_{n=0}^{\infty}(1 - 2\mu\alpha)\|g(y_n) - g(x_n)\|^2 \leq \|g(y_0) - g(\bar{x})\|^2.$$

From these inequalities, we have $\|g(x_{n+1}) - g(w_n)\| \to 0$, $\|g(w_n) - g(y_n)\| \to 0$ and $\|g(y_n) - g(x_n)\| \to 0$ as $n \to \infty$. Since g is injection, we can deduce that $\|x_{n+1} - w_n\| \to 0$, $\|w_n - y_n\| \to 0$ and $\|y_n - x_n\| \to 0$ as $n \to \infty$. Therefore we have (2.18). From the same technique of Theorem 2.1, we have

$$\langle \hat{s}, x - g(\hat{x})\rangle + F(x) - F(g(\hat{x})) \geq 0\ \forall x \in H.$$

Hence $\hat{x} \in \Omega$ is a strong solution of the F-IGVIP. This complete the proof. ■

Theorem 2.4. *Let* H *be a finite-dimensional Hilbert space,* $g : H \to H$ *be continuous injection and* $g^{-1} : g(H) \to H$ *is* D*-uniformly continuous,* $F : H \to \mathbb{R}$ *be lower semi-continuous and convex,* $T : H \to C(H)$ *be upper semi-continuous and* D*-continuous set-valued mapping with convex values such that* T *is partially relaxed strongly monotone w.r.t.* g *with constant* $\alpha > 0$*. Suppose that*

(1) *for each* $x \in H$*, there is an* $s \in T(x)$ *such that* $\langle s, g(x) - x\rangle \leq F(x) - F(g(x))$*,*
(2) *there is a nonempty compact convex subset* C *of* H*, such that for each* $x \in H \setminus C$*, there is a* $y \in C$ *such that for some* $s \in T(x)$*,* $\langle s, y - g(x)\rangle < F(g(x)) - F(y)$*.*

Then for any given $x_0 \in H$*,* $s_0 \in T(x_0)$*, the iterative sequences* $\{x_n\}$*,* $\{y_n\}$ *and* $\{w_n\}$ *defined by Algorithm 2.1 with* $0 < \rho,\ \mu,\ \beta < \frac{1}{2\alpha}$ *converge strongly to a strong solution* $\hat{x}$ *of the* F*-IGVIP.*

Proof. The existence result for solving F-IGVIP follows from Theorem A. Hence the solution set of the F-IGVIP is nonempty. Applying Theorem 2.3, we know that for any given $x_0 \in H$, $s_0 \in T(x_0)$, the iterative sequences $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$ defined by Algorithm 2.1 with $0 < \rho,\ \mu,\ \beta < \frac{1}{2\alpha}$ converge strongly to a strong solution $\hat{x}$ of the F-IGVIP. ■

3. Algorithm and Convergent Theorem for F-Implicit Generalized Complementarity Problem

In this section, we consider the F-implicit generalized complementarity problem (F-IGCP):Find $\bar{x} \in H$ with an $\bar{s} \in T(\bar{x})$ such that

$$(3.1)\qquad \langle \bar{s}, g(\bar{x})\rangle + F(g(\bar{x})) = 0 \text{ and } \langle \bar{s}, y\rangle + F(y) \geq 0,\ \forall y \in H.$$

From Theorem 3.1[6], we know that a strong solution of (F-IGVIP) is also a solution of (F-IGCP) if the function $F : H \to \mathbb{R}$ is positive homogeneous and convex. Furthermore, a solution of (F-IGCP) is a strong solution of (F-IGVIP).

We first consider the auxiliary F-implicit generalized complementarity problems as follows:

For any given $\bar{x} \in H$, $\bar{s} \in T(\bar{x})$, to find a $w \in H$ such that

$$(3.1)\qquad \frac{1}{2}\langle g(w) - g(\bar{x}), x - g(w)\rangle + \rho\langle \bar{s}, x\rangle + \rho F(x) \geq 0,$$

and

$$(3.2)\qquad \frac{1}{2}\langle g(w) - g(\bar{x}), x - g(w)\rangle - \rho\langle \bar{s}, g(w)\rangle - \rho F(g(w)) = 0,$$

for all $x \in H$, where $\rho > 0$ is a constant.

We note that if $w = \bar{x}$, then $\bar{x}$ is a solution of F-IGCP. This observation enables us to suggest the following new predictor-corrector method for solving the solution of F-IGCP.

We consider the algorithm for F-implicit generalized complementarity problem as follows. From this algorithm, we can direct to approximate a solution of F-IGCP.

Algorithm 3.1. For given $x_0 \in H$, $s_0 \in T(x_0)$, compute the approximate solution x_n of F-IGCP with $s_n \in T(x_n)$ by the following iterative schemes.

$$\frac{1}{2}\langle g(y_n) - g(x_n), x - g(y_n)\rangle - \mu\langle s_n, g(y_n)\rangle - \mu F(g(y_n)) = 0,\ \forall x \in H,$$

$$\frac{1}{2}\langle g(y_n) - g(x_n), x - g(y_n)\rangle + \mu\langle s_n, x\rangle + \mu F(x) \geq 0,\ \forall x \in H,$$

$$\frac{1}{2}\langle g(w_n) - g(y_n), x - g(w_n)\rangle - \beta\langle \xi_n, g(w_n)\rangle - \beta F(g(w_n)) = 0,\ \forall x \in H,$$

$$\frac{1}{2}\langle g(w_n) - g(y_n), x - g(w_n)\rangle + \beta\langle \xi_n, x\rangle + \beta F(x) \geq 0,\ \forall x \in H,$$

$$\frac{1}{2}\langle g(x_{n+1})-g(w_n), x-g(x_{n+1})\rangle-\rho\langle\eta_n, g(x_{n+1})\rangle-\rho F(g(x_{n+1}))=0, \forall x\in H,$$

$$\frac{1}{2}\langle g(x_{n+1})-g(w_n), x-g(x_{n+1})\rangle+\rho\langle\eta_n, x\rangle+\rho F(x)\geq 0,\ \forall x\in H,$$

$$s_n\in T(x_n):\|s_{n+1}-s_n\|\leq D(T(x_{n+1}),T(x_n)),$$

$$\xi_n\in T(y_n):\|\xi_{n+1}-\xi_n\|\leq D(T(y_{n+1}),T(y_n)),$$

$$\eta_n\in T(w_n):\|\eta_{n+1}-\eta_n\|\leq D(T(w_{n+1}),T(w_n)),$$

where $\mu,\ \beta,\ \rho>0$ are constants and D is the Hausdorff metric on $C(H)$.

It follows from Theorem 3.1 [6] and Theorem 2.1, we have the convergent result for F-IGCP as follows.

Theorem 3.1. *Let H be a finite-dimensional Hilbert space, $g:H\to H$ be continuous, $F:H\to\mathbb{R}$ be lower semi-continuous, positive homogeneous and convex, $g^{-1}:g(H)\to C(H)$ be D-uniformly continuous and bounded set-valued mapping, $g^{-1}\circ g:H\to C(H)$ be D-convergent preserving set-valued mapping and $T:H\to C(H)$ be D-continuous set-valued mapping such that T is partially relaxed strongly monotone w.r.t. g with constant $\alpha>0$. Suppose that the solution set of the F-IGCP is nonempty. Then for any given $x_0\in H$, $s_0\in T(x_0)$, the iterative sequences $\{x_n\}$, $\{y_n\}$ and $\{w_n\}$ defined by Algorithm 3.1 with $0<\rho,\ \mu,\ \beta<\frac{1}{2\alpha}$ converge strongly to an $\hat{x}$ which is a solution of the F-IGCP.*

Combine Theorem 3.3[6] and Theorem 2.2, we have the following convergent theorem for F-implicit generalized complementarity problem.

Theorem 3.2. *Let H be a finite-dimensional Hilbert space, $g:H\to H$ be continuous, $F:H\to\mathbb{R}$ be lower semi-continuous, positive homogeneous and convex, $g^{-1}:g(H)\to C(H)$ be D-uniformly continuous and bounded set-valued mapping, $g^{-1}\circ g:H\to C(H)$ be D-convergent preserving set-valued mapping and $T:H\to C(H)$ be upper semi-continuous and D-continuous set-valued mapping with convex values such that T is partially relaxed strongly monotone w.r.t. g with constant $\alpha>0$. Suppose that*

(1) *for each $x\in H$, there is an $s\in T(x)$ such that $\langle s,g(x)\rangle+F(g(x))=0$, $\langle s,x\rangle+F(x)\geq 0$; and*

(2) *there is a nonempty compact convex subset C of H, such that for each $x\in H\setminus C$, there is a $y\in C$ such that for some $s\in T(x)$, $\langle s,g(x)\rangle+F(g(x))=0$ and $\langle s,y\rangle+F(y)<0$.*

Then for any given $x_0 \in H$, $s_0 \in T(x_0)$, *the iterative sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{w_n\}$ *defined by Algorithm 3.1 with* $0 < \rho,\ \mu,\ \beta < \frac{1}{2\alpha}$ *converge strongly to a solution* $\hat{x}$ *of the* F*-IGCP.*

Furthermore, if g is injection in Theorem 3.1 and Theorem 3.2, then the D-convergent preservation of $g^{-1} \circ g$ is fulfilled. Hence we have the following results.

Theorem 3.3. *Let* H *be a finite-dimensional Hilbert space,* $g : H \to H$ *be continuous injection and* $g^{-1} : g(H) \to H$ *is* D*-uniformly continuous,* $F : H \to \mathbb{R}$ *be lower semi-continuous, positive homogeneous and convex and* $T : H \to C(H)$ *be* D*-continuous set-valued mapping such that* T *is partially relaxed strongly monotone w.r.t.* g *with constant* $\alpha > 0$. *Suppose that the solution set of the* F*-IGCP is nonempty. Then for any given* $x_0 \in H$, $s_0 \in T(x_0)$, *the iterative sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{w_n\}$ *defined by Algorithm 3.1 with* $0 < \rho,\ \mu,\ \beta < \frac{1}{2\alpha}$ *converge strongly to an* $\hat{x}$ *which is a solution of the* F*-IGCP.*

Theorem 3.4. *Let* H *be a finite-dimensional Hilbert space,* $g : H \to H$ *be continuous injection and* $g^{-1} : g(H) \to H$ *is* D*-uniformly continuous,* $F : H \to \mathbb{R}$ *be lower semi-continuous, positive homogeneous and convex,* $T : H \to C(H)$ *be upper semi-continuous and* D*-continuous set-valued mapping with convex values such that* T *is partially relaxed strongly monotone w.r.t.* g *with constant* $\alpha > 0$. *Suppose that*

(1) *for each* $x \in H$, *there is an* $s \in T(x)$ *such that* $\langle s, g(x)\rangle + F(g(x)) = 0$, $\langle s, x\rangle + F(x) \geq 0$*; and*

(2) *there is a nonempty compact convex subset* C *of* H, *such that for each* $x \in H \setminus C$, *there is a* $y \in C$ *such that for some* $s \in T(x)$, $\langle s, g(x)\rangle + F(g(x)) = 0$ *and* $\langle s, y\rangle + F(y) < 0$.

Then for any given $x_0 \in H$, $s_0 \in T(x_0)$, *the iterative sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{w_n\}$ *defined by Algorithm 3.1 with* $0 < \rho,\ \mu,\ \beta < \frac{1}{2\alpha}$ *converge strongly to a solution* $\hat{x}$ *of the* F*-IGCP.*

ACKNOWLEDGMENT

The author thanks the referees for useful comments. The research was supported by grant from CMU94-051 of China Medical University in Taiwan.

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