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ON GENERALIZED VECTOR IMPLICIT VARIATIONAL INEQUALITIES AND COMPLEMENTARITY PROBLEMS

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Abstract. In this paper, two generalized vector implicit variational inequalities and three generalized vector implicit complementarity problems are introduced with a general order in ordered Banach spaces and the equivalence between them is studied under certain assumptions. Furthermore, some existence theorems for the generalized vector implicit variational inequalities are derived by using Lin's result (1986) and Nadler's result (1969). We affirmatively answer the open question proposed by Rapcsak (2000). Our results are the extension and improvement of the corresponding results in Huang and Li (2006).

1. INTRODUCTION

It is well known that Giannessi (1980) first introduced and studied the vector variational inequality in a finite-dimensional Euclidean space, which is the vector-valued version of the variational inequality of Hartman and Stampacchia. Throughout over last 20 years development, existence theorems for solutions of vector variational inequalities in various situations have been extensively studied by many authors. The reader is referred to Chen and Craven (1989), Chen and Yang (1990), Giannessi (2000), Huang, Li and Thompson (2003), Rapcsak (2000), Yang (1993), Yang (1997), Yang and Goh (1997), Huang and Li (2006), Konnov and Yao (1997), Zeng and Yao (2007) and the references therein. Vector variational inequalities have many important applications in various problems, for example, vector optimization (see Yang (1997)), approximate vector optimization (see Chen and Craven (1989)),

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vector equilibria (see Giannessi (2000) and Huang, Li and Thompson (2003)), vector traffic equilibria (see Yang and Goh (1997)) and abstract economical equilibria (see Shen (2001)).

On one hand, Chen and Yang (1990) defined a vector variational inequality and three vector complementarity problems, i.e., the weak, positive and strong vector complementarity problems in ordered Banach spaces and proved the existence theorem for them. And also, the equivalence between them was established under some additional assumptions. Subsequently, Yang (1993) analyzed the relations among vector variational inequalities, vector complementarity problems, minimal element problems, and fixed point problems. Recently, Rapcsak (2000) introduced the weak order in Banach spaces. By virtue of this new order, Rapcsak (2000) introduced a vector variational inequality and three vector complementarity problems and discussed some relations between them. At the end of the paper, Rapcsak (2000) put forth an open question, i.e., in the case of ordering (2.1), the existence of a solution to (VVIP) or (WVCP) (see Section 3).

On the other hand, Huang and Li (2006) introduced two vector implicit variational inequalities and three vector implicit complementarity problems, i.e., the weak, positive and strong vector implicit complementarity problems with a general order in ordered Banach spaces, and established the equivalence between them under certain conditions. Furthermore, they proved some existence theorems for two vector implicit variational inequalities. There is no doubt that they answered the open question proposed by Rapcsak (2000).

In this paper, two generalized vector implicit variational inequalities and three generalized vector implicit complementarity problems are introduced with a general order in ordered Banach spaces and the equivalence between them is studied under certain assumptions. Furthermore, some existence theorems for two generalized vector implicit variational inequalities are derived by using Lin's result (1986) and Nadler's result (1969). We affirmatively answer the open question proposed by Rapcsak (2000). Our results are the extension and improvement of the corresponding results in Huang and Li (2006).

2. Preliminaries

In this section, we recall some notations, definitions and results, which are essential for our main results.

Let Y be a real Banach space. A nonempty subset P of Y is said to be a convex cone if (i) $P + P \subseteq P$; (ii) $\lambda P \subseteq P$ for all $\lambda > 0$. A cone P is called pointed if P is convex and $P \cap (-P) = \{0\}$. An ordered Banach space (Y, P) is a real Banach space Y with an order defined by a closed convex cone $P \subseteq Y$ with apex at the origin, in the form of

$$y \leq_P x \;\Leftrightarrow\; x - y \in P, \quad \forall x, y \in Y$$

and

$$y \not\leq_P x \;\Leftrightarrow\; x - y \notin P, \quad \forall x, y \in Y.$$

If the interior of P, say $\mathrm{int}P$, is nonempty, then a weak order in Y is also defined by

$$y \not\leq_{\mathrm{int}P} x \;\Leftrightarrow\; x - y \notin \mathrm{int}P, \quad \forall x, y \in Y.$$

We will use the following notation to denote the same order relation:

$$y \leq_P x \;\Leftrightarrow\; x \geq_P y$$

and

$$y \not\leq_P x \;\Leftrightarrow\; x \not\geq_P y, \quad y \not\leq_{\mathrm{int}P} x \;\Leftrightarrow\; x \not\geq_{\mathrm{int}P} y.$$

Let (X, K) and (Y, P) be two ordered Banach spaces with $\mathrm{int}P \neq \emptyset$. Denote by $L(X, Y)$ the set of all linear continuous mappings from X to Y. The weak $(\mathrm{int}P)$-dual cone $K^{w^+}_{\mathrm{int}P}$ of K is defined by

$$K^{w^+}_{\mathrm{int}P} = \{q \in L(X, Y) : \langle q, x\rangle \not\leq_{\mathrm{(int)P}} 0, \ \forall x \in K\},$$

where the subscript $\mathrm{int}P$ means that the weak order $\not\leq_{\mathrm{int}P}$ is defined by $\mathrm{int}P$, and the value of $q \in L(X, Y)$ at $x \in K$ is denoted by $\langle q, x\rangle$. The strong (P)-dual cone $K^{s^+}_P$ of K is defined by

$$K^{s^+}_P = \{q \in L(X, Y) : 0 \leq_P \langle q, x\rangle, \ \forall x \in K\}.$$

It is obvious that $K^{w^+}_{\mathrm{int}P}$ and $K^{s^+}_P$ are nonempty, since the null linear mapping belongs to $K^{w^+}_{\mathrm{int}P}$ and $K^{s^+}_P$. It is easy to see that $K^{s^+}_P \subseteq K^{w^+}_{\mathrm{int}P}$ if P is pointed. If $Y = R$ and $P = R^+$, then the weak $(\mathrm{int}P)$ and strong (P)-dual cones of K are reduced to the usual dual cone K^* of K given by

$$K^{w^+}_{\mathrm{int}P} = K^{s^+}_P = K^* = \{q \in L(X, Y), \ \forall x \in K\}.$$

Let $D \subseteq Y$ be a convex cone. Rapcsak (2000) introduced the following order

$$y \leq_D x \;\Leftrightarrow\; x - y \in D, \quad \forall x, y \in Y. \tag{2.1}$$

Since $\mathrm{cl}D$ is a closed convex cone, we can define the order induced by $\mathrm{cl}D$;

$$y \not\leq_{\mathrm{cl}D} x \;\Leftrightarrow\; x - y \notin \mathrm{cl}D, \quad \forall x, y \in Y$$

where $\mathrm{cl}D$ denotes the closure of D. This order satisfies the following properties:

$$y \nleq_D x \;\Leftrightarrow\; y + w \nleq_D x + w, \quad \forall x, y, w \in Y;$$

$$y \nleq_D x \;\Leftrightarrow\; \lambda y \nleq_D \lambda x, \quad \forall x, y \in Y \text{ and } \lambda > 0.$$

Similarly, a weak (D)-dual $K_D^{w^+}$ of K is defined by

$$K_D^{w^+} = \{q \in L(X, Y) : \langle q, x\rangle \nleq_D 0, \ \forall x \in K\}.$$

A strong $(\mathrm{cl}D)$-dual cone $K_{\mathrm{cl}D}^{s^+}$ of K is defined by

$$K_{\mathrm{cl}D}^{s^+} = \{q \in L(X, Y) : 0 \leq_{\mathrm{cl}D} \langle q, x\rangle, \ \forall x \in K\},$$

where $\mathrm{cl}D$ denotes the closure of D. Moreover, Rapcsak (2000) introduced the following binary relation:

$$(2.2) \qquad y =_D 0, \ \in Y \;\Leftrightarrow\; \begin{cases} y \in \mathrm{Fr}(\mathrm{cl}D) \cup \mathrm{Fr}(-\mathrm{cl}D) \setminus (-D) \text{and} \\ -y \in \mathrm{Fr}(\mathrm{cl}D) \cup \mathrm{Fr}(-\mathrm{cl}D) \setminus (-D), \end{cases}$$

where $\mathrm{Fr}A$ denotes the frontier of A. By this relation, the set of zero points with respect to a convex cone D is nonempty if $\mathrm{Fr}(\mathrm{cl}D) \setminus D$ is nonempty. We note that

$$y =_D 0, \quad y \in Y \;\Leftrightarrow\; \lambda y =_D 0, \quad y \in Y, \ \lambda > 0.$$

3. Generalized Vector Implicit Variational Inequalities and Complementarity Problems

Let (X, K) and (Y, P) be two ordered Banach spaces with $\mathrm{int}P \neq \emptyset$, and D a convex cone in Y. Let $T : X \to 2^{L(X,Y)}$, $A : L(X, Y) \to L(X, Y)$ and $g : K \to K$ be a mapping. In this paper, we consider two kinds of generalized vector implicit variational inequality problems and three kinds of generalized vector implicit complementarity problems as follows:

1. Generalized weak vector implicit variational inequality problem (in short, GWVIVIP): find $x^* \in K$ such that $\langle As^*, y - g(x^*)\rangle \nleq_D 0, \ \forall y \in K$, for some $s^* \in Tx^*$.
2. Generalized strong vector implicit variational inequality problem (in short, GSVIVIP): find $x^* \in K$ such that $\langle As^*, y - g(x^*)\rangle \geq_{\mathrm{cl}D} 0, \ \forall y \in K$, for some $s^* \in Tx^*$.
3. Generalized weak vector implicit complementarity problem (in short, GWVICP): find $x^* \in K$ such that $As^* \in K_D^{w^+}$ and $\langle As^*, g(x^*)\rangle =_D 0$, for some $s^* \in Tx^*$.

4. Generalized positive vector implicit complementarity problem (in short, GPVICP): find $x^* \in K$ such that $As^* \in K^{s^+}_{\mathrm{cl}D}$ and $\langle As^*, g(x^*)\rangle =_D 0$, for some $s^* \in Tx^*$.

5. Generalized strong vector implicit complementarity problem (in short, GSVICP): find $x^* \in K$ such that $As^* \in K^{s^+}_{\mathrm{cl}D}$ and $\langle As^*, g(x^*)\rangle = 0$, for some $s^* \in Tx^*$.

Special Cases. (I) If $A = I$ the identity mapping of $L(X, Y)$, and $T = f$ a single valued mapping from X into $L(X, Y)$, then the above (GWVIVIP), (GSVIVIP), (GWVICP), (GPVICP) and (GSVICP) reduce to the following (WVIVIP), (SVIVIP), (WVICP), (PVICP) and (SVICP), respectively:

1. Weak vector implicit variational inequality problem (in short, WVIVIP): find $x^* \in K$ such that $\langle f(x^*), y - g(x^*)\rangle \not\leq_D 0,\ \forall y \in K$.

2. Strong vector implicit variational inequality problem (in short, SVIVIP): find $x^* \in K$ such that $\langle f(x^*), y - g(x^*)\rangle \geq_{\mathrm{cl}D} 0,\ \forall y \in K$.

3. Weak vector implicit complementarity problem (in short, WVICP): find $x^* \in K$ such that $f(x^*) \in K^{w^+}_D$ and $\langle f(x^*), g(x^*)\rangle =_D 0$.

4. Positive vector implicit complementarity problem (in short, PVICP): find $x^* \in K$ such that $f(x^*) \in K^{s^+}_{\mathrm{cl}D}$ and $\langle f(x^*), g(x^*)\rangle =_D 0$.

5. Strong vector implicit complementarity problem (in short, SVICP): find $x^* \in K$ such that $f(x^*) \in K^{s^+}_{\mathrm{cl}D}$ and $\langle f(x^*), g(x^*)\rangle = 0$.

The above (WVIVIP), (SVIVIP), (WVICP), (PVICP) and (SVICP) have been considered and studied by Huang and Li (2006).

(II) If g is the identity mapping on K, then the above (WVIVIP), (WVICP), (PVICP) and (SVICP) reduce to the following (VVIP), (WVCP), (PVCP) and (SVCP), respectively:

1. Vector variational inequality problem (in short, VVIP): find $x^* \in K$ such that $\langle f(x^*), y - x^*\rangle \not\leq_D 0,\ \forall y \in K$.

2. Weak vector complementarity problem (in short, WVCP): find $x^* \in K$ such that $f(x^*) \in K^{w^+}_D$ and $\langle f(x^*), x^*\rangle =_D 0$.

3. Positive vector complementarity problem (in short, PVCP): find $x^* \in K$ such that $f(x^*) \in K^{s^+}_{\mathrm{cl}D}$ and $\langle f(x^*), x^*\rangle =_D 0$

4. Strong vector complementarity problem (in short, SVCP): find $x^* \in K$ such that $f(x^*) \in K^{s^+}_{\mathrm{cl}D}$ and $\langle f(x^*), x^*\rangle = 0$

The above (VVIP), (WVCP), (PVCP) and (SVCP) have been considered and studied by Rapcsak (2000). At the end of Rapcsak (2000), an open question is proposed, i.e., in the case of ordering (2.1), the existence problem of a solution to (VVIP) or (WVCP). Furthermore, if D is an open convex cone, then (WVIVIP), (WVICP), (PVICP) and (SVICP) reduce to (VVIP), (WVCP), (PVCP) and (SVCP), respectively, which have been considered and studied by Chen and Yang (1990) and Yang (1993).

Definition 3.1. See Rapcsak (2000). A convex cone D is acute if $\mathrm{cl}D$ is pointed. A convex cone D is correct if

$$\mathrm{cl}D + \{D \setminus (D \cap (-D))\} \subseteq D.$$

Lemma 3.1. *See Rapcsak (2000). If $D \subseteq Y$ is a convex cone, then for any $x, y \in Y$,*

(1) $0 \not\leq_D y$ and $x \leq_D y$ imply that $0 \not\leq_D x$;

(2) $y \not\leq_D 0$ and $y \leq_D x$ imply that $x \not\leq_D 0$.

Lemma 3.2. *See Huang and Li (2006). If $D \subseteq Y$ is an acute cone and $0 \notin -D$, then*

(1) $y \geq_{\mathrm{cl}D} 0$ implies that $y \not\leq_D 0$, $\forall y \in Y$;

(2) $K_{\mathrm{cl}D}^{s^+} \subseteq K_D^{w^+}$.

Theorem 3.1. *Let $D \subseteq Y$ be an acute convex cone with apex at the origin.*

(i) *If x^* solves (GSVIVIP) and $0 \notin -D$, then x^* solves (GWVIVIP).*

(ii) *If x^* solves (GWVIVIP) and $\langle As^*, g(x^*)\rangle \in \mathrm{Fr}(\mathrm{cl}D) \cup \mathrm{Fr}(-\mathrm{cl}D)$ where $s^* \in Tx^*$ is an element associated with x^* in the definition of (GWVIVIP), then x^* solves (GWVICP).*

(iii) *If x^* solves (GWVICP), $\langle As^*, g(x^*)\rangle \in \mathrm{Fr}(-\mathrm{cl}D) \setminus (-D)$, $0 \notin -D$ and $-D$ is correct where $s^* \in Tx^*$ is an element associated with x^* in the definition of (GWVICP), then x^* solves (GWVIVIP).*

(iv) *If x^* solves (GPVICP) and $0 \notin -D$, then x^* solves (GWVICP).*

(v) *If x^* solves (GSVICP) and $0 \notin -D$, then x^* solves (GPVICP).*

(vi) *If x^* solves (GSVICP) and $0 \notin -D$, then x^* solves (GWVIVIP).*

Proof. (i) Let $x^* \in K$ be a solution of (GSVIVIP), i.e., $x^* \in K$ such that

$$\langle As^*, y - g(x^*)\rangle \geq_{\mathrm{cl}D} 0, \ \forall y \in K,$$

for some $s^* \in Tx^*$. Since D is an acute cone and $0 \notin -D$, from Lemma 3.2 we derive

$$\langle As^*, y - g(x^*)\rangle \not\leq_D 0, \quad \forall y \in K,$$

which implies that x^* is a solution of (GWVIVIP).

(ii) Let $x^* \in K$ be a solution of (GWVIVIP), i.e., $x^* \in K$ such that

$$\langle As^*, y - g(x^*)\rangle \not\leq_D 0, \ \forall y \in K, \tag{3.1}$$

for some $s^* \in Tx^*$. Let $y = x + g(x^*)$ for any $x \in K$. Since K is a convex cone, we know that $y \in K$ and thus

$$\langle As^*, x\rangle \not\leq_D 0,$$

which implies that $As^* \in K_D^{w^+}$.

Since $y = (1+\lambda)g(x^*)$, $\lambda > -1$, belongs to K, it follows from (3.1) that

$$\lambda\langle As^*, g(x^*)\rangle = \langle As^*, \lambda g(x^*)\rangle \not\leq_D 0, \quad \lambda \in (-1, +\infty),$$

from which it follows that

$$\langle As^*, g(x^*)\rangle \notin D \quad \text{and} \quad \langle As^*, g(x^*)\rangle \notin -D. \tag{3.2}$$

By the assumption that $\langle As^*, g(x^*)\rangle \in \mathrm{Fr}(\mathrm{cl}D) \cup \mathrm{Fr}(-\mathrm{cl}D)$, we have $-\langle As^*, g(x^*)\rangle \in \mathrm{Fr}(\mathrm{cl}D) \cup \mathrm{Fr}(-\mathrm{cl}D)$. It follows from (2.2) and (3.2) that

$$\langle As^*, g(x^*)\rangle =_D 0,$$

which shows that x^* is a solution of (GWVICP).

(iii) Let $x^* \in K$ be a solution of (GWVICP), i.e., $x^* \in K$ such that

$$As^* \in K_D^{w^+} \quad \text{and} \quad \langle As^*, g(x^*)\rangle =_D 0 \ \text{ for some } \ s^* \in Tx^*. \tag{3.3}$$

Since $As^* \in L(X, Y)$ and $K \subseteq X$, we have

$$\langle As^*, y - g(x^*)\rangle = \langle As^*, y\rangle - \langle As^*, g(x^*)\rangle, \quad \forall y \in K \tag{3.4}$$

and from assumption, we obtain

$$0 \leq_{\mathrm{cl}D} -\langle As^*, g(x^*)\rangle. \tag{3.5}$$

If $\langle As^*, g(x^*)\rangle = 0$, then it follows from (3.3) and (3.4) that the conclusion holds. Suppose that $\langle As^*, g(x^*)\rangle \neq 0$, and

$$\langle As^*, y\rangle \not\leq_{\mathrm{cl}D} 0, \quad \forall y \in K. \tag{3.6}$$

Then, from (3.5) we obtain

$$\langle As^*, y\rangle \leq_{\mathrm{cl}D} \langle As^*, y - g(x^*)\rangle. \tag{3.7}$$

Thus, by Lemma 3.1, combining (3.6) with (3.7) yields that

$$\langle As^*, y - g(x^*)\rangle \not\leq_{\mathrm{cl}D} 0, \quad \forall y \in K. \tag{3.8}$$

A consequence of (3.8) is that

$$\langle As^*, y - g(x^*)\rangle \not\leq_D 0, \quad \forall y \in K,$$

i.e., x^* is a solution of (GWVIVIP). Assume $\langle As^*, g(x^*)\rangle \neq 0$ and there exists an $y \in K$ such that $\langle As^*, y\rangle \leq_{\mathrm{cl}D} 0$. Clearly $\langle As^*, y\rangle \notin -D$ because $As^* \in K_D^{w^+}$. If $\langle As^*, y - g(x^*)\rangle \in -D$, then from assumptions $\langle As^*, g(x^*)\rangle \in \mathrm{Fr}(-\mathrm{cl}D) \setminus (-D)$, $0 \notin -D$ (i.e., $D \cap (-D) = \emptyset$), and the correctness of the cone $-D$, we have

$$\langle As^*, y\rangle = \langle As^*, y - g(x^*)\rangle + \langle As^*, g(x^*)\rangle \in -D - \mathrm{cl}D \subseteq -D,$$

which is a contradiction. Thus $\langle As^*, y - g(x^*)\rangle \not\leq_D 0$ for an $y \in K$ satisfying $\langle As^*, y\rangle \leq_{\mathrm{cl}D} 0$. For an $y \in K$ such that $\langle As^*, y\rangle \not\leq_{\mathrm{cl}D} 0$, using the argument appeared in (3.6), (3.7) and (3.8), we see that $\langle As^*, y - g(x^*)\rangle \not\leq_D 0$. This implies that x^* is a solution of (GWVIVIP).

(iv) Let $x^* \in K$ be a solution of (GPVICP), i.e., $x^* \in K$ such that

$$As^* \in K_{\mathrm{cl}D}^{s^+} \quad \text{and} \quad \langle As^*, g(x^*)\rangle =_D 0 \;\; \text{for some} \;\; s^* \in Tx^*.$$

Since D is an acute cone and $0 \notin -D$, then it follows from Lemma 3.2 that

$$As^* \in K_D^{w^+},$$

which shows that x^* is also a solution of (GWVICP).

(v) Let $x^* \in K$ be a solution of (GSVICP), i.e., $x^* \in K$ such that

$$As^* \in K_{\mathrm{cl}D}^{s^+} \quad \text{and} \quad \langle As^*, g(x^*)\rangle = 0 \;\; \text{for some} \;\; s^* \in Tx^*.$$

Note that D is an acute cone and $0 \notin -D$, then it follows from relation (2.2) that

$$\langle As^*, g(x^*)\rangle =_D 0,$$

which shows that x^* is also a solution of (GPVICP).

(vi) Let $x^* \in K$ be a solution of (GSVICP), i.e., $x^* \in K$ such that

$$As^* \in K_{\mathrm{cl}D}^{s^+} \quad \text{and} \quad \langle As^*, g(x^*)\rangle = 0 \;\; \text{for some} \;\; s^* \in Tx^*.$$

Since D is an acute cone and $0 \notin -D$, then by Lemma 3.2 we have that

$$\langle As^*, y - g(x^*)\rangle = \langle As^*, y\rangle - \langle As^*, g(x^*)\rangle \not\leq_D 0,$$

which implies that x^* is also a solution of (GWVIVIP). This completes the proof.■

If $A = I$ the identity mapping of $L(X, Y)$, and $T = f$ a single valued mapping from X into $L(X, Y)$, then the following corollary follows immediately from Theorem 3.1.

Corollary 3.1. (Huang and Li [12, Theorem 3.1]). *Let $D \subseteq Y$ be an acute convex cone with apex at the origin.*

(i) *If x^* solves (SVIVIP) and $0 \notin -D$, then x^* solves (WVIVIP).*

(ii) *If x^* solves (WVIVIP) and $\langle f(x^*), g(x^*)\rangle \in \mathrm{Fr}(\mathrm{cl}D) \cup \mathrm{Fr}(-\mathrm{cl}D)$, then x^* solves (WVICP).*

(iii) *If x^* solves (WVICP), $\langle f(x^*), g(x^*)\rangle \in \mathrm{Fr}(-\mathrm{cl}D) \setminus (-D)$, $0 \notin -D$ and $-D$ is correct, then x^* solves (WVIVIP).*

(iv) *If x^* solves (PVICP) and $0 \notin -D$, then x^* solves (WVICP).*

(v) *If x^* solves (SVICP) and $0 \notin -D$, then x^* solves (PVICP).*

(vi) *If x^* solves (SVICP) and $0 \notin -D$, then x^* solves (WVIVIP).*

If $A = I$ the identity mapping of $L(X, Y)$, and $T = f$ a single valued mapping from X into $L(X, Y)$, and g is the identity mapping on K, then the following corollary follows immediately from Theorem 3.1.

Corollary 3.2. (Huang and Li [12, Remark 3.2]). *Let $D \subseteq Y$ be an acute convex cone with apex at the origin.*

(i) *If x^* solves (SVVIP) and $0 \notin -D$, then x^* solves (WVVIP).*

(ii) *If x^* solves (WVVIP) and $\langle f(x^*), x^*\rangle \in \mathrm{Fr}(\mathrm{cl}D) \cup \mathrm{Fr}(-\mathrm{cl}D)$, then x^* solves (WVCP).*

(iii) *If x^* solves (WVCP), $\langle f(x^*), x^*\rangle \in \mathrm{Fr}(-\mathrm{cl}D) \setminus (-D)$, $0 \notin -D$ and $-D$ is correct, then x^* solves (WVVIP).*

(iv) *If x^* solves (PVCP) and $0 \notin -D$, then x^* solves (WVCP).*

(v) *If x^* solves (SVCP) and $0 \notin -D$, then x^* solves (PVCP).*

(vi) *If x^* solves (SVCP) and $0 \notin -D$, then x^* solves (WVVIP).*

Lemma 3.3. (Lin [6]). *Let K be a nonempty, convex subset of a Hausdorff topological vector space X, and A be a nonempty subset of $K \times K$. Suppose the following assumptions hold:*

(i) *for each* $x \in K$, $(x, x) \in A$;
(ii) *for each* $y \in K$, $A_y = \{x \in K : (x, y) \in A\}$ *is closed in* K;
(iii) *for each* $x \in K$, $A_x = \{y \in K : (x, y) \notin A\}$ *is convex or empty;*
(iv) *there exists a nonempty compact convex subset* C *of* K *such that* $B = \{x \in K : (x, y) \in A, \forall y \in C\}$ *is compact in* K.

Then there exists $x^* \in K$ *such that* $\{x^*\} \times K \subseteq A$.

Lemma 3.4. (Nadler [13]). *Let* $(X, \|\cdot\|)$ *be a normed space and* H *be the Hausdorff metric on the collection* $CB(X)$ *of all nonempty, closed and bounded subsets of* X, *induced by the metric* $d(x, y) = \|x - y\|$, *which is defined by*

$$H(U, V) = \max\{\sup_{x\in U} \inf_{y\in V} \|x - y\|, \sup_{y\in V} \inf_{x\in U} \|x - y\|\}),$$

for U and V in $CB(X)$. If U and V are compact sets in X, then for each $x \in U$, there exists $y \in V$ such that

$$\|x - y\| \leq H(U, V).$$

Definition 3.2. Let $T : X \to 2^{L(X,Y)}$ be a nonempty compact-valued multifunction.

(i) T is said to be H-continuous at a point $x_0 \in X$, i.e., $\lim_{x\to x_0} H(Tx, Tx_0) = 0$, if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $H(Tx, Tx_0) < \varepsilon$ whenever $\|x - x_0\| < \delta$. T is said to be H-continuous if T is H-continuous at each point $x \in X$.
(ii) T is said to be H-Lipschitz continuous if there exists $\eta > 0$ such that

$$H(Tx, Ty) \leq \eta\|x - y\|, \quad \forall x, y \in X.$$

Remark 3.1. It is well known that the concept of H-Lipschitz continuity for vector valued multifunctions is usually used for guaranteeing the convergence of iterative algorithms for set-valued variational inequalities and set-valued variational inclusions; see Chang (2000), Zeng, Guu and Yao (2005), and Zeng, Schaible and Yao (2005). It is easy to see that the H-Lipschitz continuity implies the H-continuity. Moreover, it is clear that if T is a single valued mapping from X into $L(X, Y)$ then the concept of H-continuity coincides with the one of usual continuity.

In order to establish the existence theorem of (GWVIVIP), we first prove the following existence theorem of (GSVIVIP).

Theorem 3.2. *Let* $A : L(X, Y) \to L(X, Y)$ *and* $g : K \to K$ *be continuous, and* $T : X \to 2^{L(X,Y)}$ *be a nonempty compact-valued multifunction which is* H-*continuous. Assume that for each* $x \in K$, *the set* $\{y \in K : \langle As, y - g(x)\rangle \notin$

$\mathrm{cl}D,\ \forall s \in Tx\}$ *is convex or empty, and* $\langle As, x - g(x)\rangle \in \mathrm{cl}D$ *for all* $x \in K$ *and* $s \in Tx$. *Further assume that there exists a nonempty, compact and convex subset* C *of* K*, such that for each* $x \in K \setminus C$ *there exists* $y \in C$ *satisfying*

$$\langle As, y - g(x)\rangle \notin \mathrm{cl}D,$$

for all $s \in Tx$. *Then (GSVIVIP) has a solution. In particular, the solutions set of (GSVIVIP) is closed.*

Proof. Set

$$A = \{(x, y) \in K \times K : \langle As, y - g(x)\rangle \in \mathrm{cl}D \text{ for some } s \in Tx\}.$$

We divide the proof of the theorem into four steps.

Step 1. For each $x \in K$, $(x, x) \in A$, since $\langle As, x - g(x)\rangle \in \mathrm{cl}D$ for all $x \in K$ and $s \in Tx$.

Step 2. $A_y = \{x \in K : (x, y) \in A\}$ is closed in K for all $y \in K$. In fact, let $\{x_\alpha\}$ be a net in A_y such that $x_\alpha \to x_0 \in K$. Since $x_\alpha \in A_y$, we know that $\langle As_\alpha, y - g(x_\alpha)\rangle \in \mathrm{cl}D$ for some $s_\alpha \in Tx_\alpha$. Since T is a nonempty compact-valued multifunction, Tx_α and Tx_0 are compact subsets of $L(X, Y)$. Hence by Lemma 3.4 for $s_\alpha \in Tx_\alpha$ we can find an $\bar{s}_\alpha \in Tx_0$ such that

$$\|s_\alpha - \bar{s}_\alpha\| \le H(Tx_\alpha, Tx_0).$$

Since Tx_0 is compact, without loss of generality we may assume that $\bar{s}_\alpha \to s_0 \in Tx_0$. Moreover, we have

$$\begin{aligned} \|s_\alpha - s_0\| &\le \|s_\alpha - \bar{s}_\alpha\| + \|\bar{s}_\alpha - s_0\| \\ &\le H(Tx_\alpha, Tx_0) + \|\bar{s}_\alpha - s_0\|. \end{aligned}$$

Note that T is H-continuous. Thus we have $s_\alpha \to s_0$. Also, observe that

$$\begin{aligned} &\|\langle As_\alpha, y - g(x_\alpha)\rangle - \langle As_0, y - g(x_0)\rangle\| \\ &\le \|\langle As_\alpha, y - g(x_\alpha)\rangle - \langle As_\alpha, y - g(x_0)\rangle\| \\ &\quad + \|\langle As_\alpha, y - g(x_0)\rangle - \langle As_0, y - g(x_0)\rangle\| \\ &= \|\langle As_\alpha, g(x_\alpha) - g(x_0)\rangle\| + \|\langle As_\alpha - As_0, y - g(x_0)\rangle\| \\ &\le \|As_\alpha\|\|g(x_\alpha) - g(x_0)\| + \|As_\alpha - As_0\|\|y - g(x_0)\|. \end{aligned} \tag{3.9}$$

Since $A : L(X, Y) \to L(X, Y)$ and $g : K \to K$ are continuous, $As_\alpha \to As_0$ and $g(x_\alpha) \to g(x_0)$. Hence from (3.9) it follows that

$$\langle As_\alpha, y - g(x_\alpha)\rangle \to \langle As_0, y - g(x_0)\rangle.$$

From the closedness of $\mathrm{cl}D$ we obtain $\langle As_0, y - g(x_0)\rangle \in \mathrm{cl}D$ for some $s_0 \in Tx_0$, and hence $x_0 \in A_y$.

Step 3. From assumption, for each $x \in K$ we have

$$\begin{aligned} A_x &= \{y \in K : (x, y) \notin A\} \\ &= \{y \in K : \langle As, y - g(x)\rangle \notin \mathrm{cl}D,\ \forall s \in Tx\} \end{aligned}$$

is convex or empty.

Step 4. Let $B = \{x \in K : (x, y) \in A,\ \forall y \in C\}$. We show that B is compact in C. By assumption, for each $x \in K \setminus C$, there exists $y \in C$ such that $\langle As, y - g(x)\rangle \notin \mathrm{cl}D$ for all $s \in Ax$, that is, $(x, y) \notin A$, which implies $x \notin B$. Thus, we have $B \subseteq C$. Since $B = \cap_{y\in C} A_y$, A_y is closed, and C is compact, B is a compact subset of K.

From the above four steps and Lemma 3.3, there exists $x^* \in K$ such that $\{x^*\} \times K \subseteq A$, that is,

$$\langle As^*, y - g(x^*)\rangle \in \mathrm{cl}D, \quad \forall y \in K,$$

for some $s^* \in Tx^*$.

We claim that the solutions set of (GSVIVIP) is closed. Indeed, the proof of the assertion is very similar to that in Step 2. However, for the sake of completeness, we still present its proof. Let $\{x_n^*\} \subseteq K$ be a sequence of solutions of (GSVIVIP) such that $x_n^* \to \hat{x} \in K$ as $n \to \infty$. Then for each $n \geq 1$ there exists some $s_n^* \in Tx_n^*$ such that

$$\langle As_n^*, y - g(x_n^*)\rangle \in \mathrm{cl}D, \quad \forall y \in K.$$

Since T is a nonempty compact-valued multifunction, Tx_n^* and $T\hat{x}$ are compact subsets of $L(X, Y)$. Hence by Lemma 3.4 for $s_n^* \in Tx_n^*$ we can find an $\hat{s}_n \in T\hat{x}$ such that

$$\|s_n^* - \hat{s}_n\| \leq H(Tx_n^*, T\hat{x}).$$

Since $T\hat{x}$ is compact, without loss of generality we may assume that $\hat{s}_n \to \hat{s} \in T\hat{x}$. Moreover, we have

$$\begin{aligned} \|s_n^* - \hat{s}\| &\leq \|s_n^* - \hat{s}_n\| + \|\hat{s}_n - \hat{s}\| \\ &\leq H(Tx_n^*, T\hat{x}) + \|\hat{s}_n - \hat{s}\|. \end{aligned}$$

Note that T is H-continuous. Thus we have $s_n^* \to \hat{s}$. Also, observe that

$$\begin{aligned}
&\|\langle As_n^*, y - g(x_n^*)\rangle - \langle A\hat{s}, y - g(\hat{x})\rangle\| \\
&\leq \|\langle As_n^*, y - g(x_n^*)\rangle - \langle As_n^*, y - g(\hat{x})\rangle\| \\
&\ +\|\langle As_n^*, y - g(\hat{x})\rangle - \langle A\hat{s}, y - g(\hat{x})\rangle\| \\
&= \|\langle As_n^*, g(x_n^*) - g(\hat{x})\rangle\| + \|\langle As_n^* - A\hat{s}, y - g(\hat{x})\rangle\| \\
&\leq \|As_n^*\|\|g(x_n^*) - g(\hat{x})\| + \|As_n^* - A\hat{s}\|\|y - g(\hat{x})\|.
\end{aligned}$$

Since $A : L(X,Y) \to L(X,Y)$ and $g : K \to K$ are continuous, $As_n^* \to A\hat{s}$ and $g(x_n^*) \to g(\hat{x})$. Hence from the last inequality it follows that

$$\langle As_n^*, y - g(x_n^*)\rangle \to \langle A\hat{s}, y - g(\hat{x})\rangle.$$

From the closedness of $\mathrm{cl}D$ we obtain $\langle A\hat{s}, y - g(\hat{x})\rangle \in \mathrm{cl}D$ for some $\hat{s} \in T\hat{x}$, and hence $\hat{x}$ is a solution of (GSVIVIP). This shows that the solutions set of (GSVIVIP) is closed. ■

If $A = I$ the identity mapping of $L(X,Y)$, and $T = f$ a single valued mapping from X into $L(X,Y)$, then the following corollary follows immediately from Theorem 3.2.

Corollary 3.3. *See Theorem 3.2 in Huang and Li (2006). Assume that $f : X \to L(X,Y)$ and $g : K \to K$ are continuous, and the set $\{y \in K : \langle f(x), y - g(x)\rangle \notin \mathrm{cl}D\}$ is convex or empty for each $x \in K$, and assume that $\langle f(x), x - g(x)\rangle \in \mathrm{cl}D$ for all $x \in K$. If there exists a nonempty, compact and convex subset C of K, such that for each $x \in K \setminus C$ there exists $y \in C$ such that*

$$\langle f(x), y - g(x)\rangle \notin \mathrm{cl}D,$$

then, (SVIVIP) has a solution. Furthermore, the solutions set of (SVIVIP) is closed.

Theorem 3.3. *Let D be an acute cone with apex at the origin and $0 \notin -D$. Let $A : L(X,Y) \to L(X,Y)$ and $g : K \to K$ be continuous, and $T : X \to 2^{L(X,Y)}$ be a nonempty compact-valued multifunction which is H-continuous. If all assumptions in Theorem 3.2 hold, then (GWVIVIP) has a solution.*

Proof. It follows from Theorems 3.1 (i) and 3.2 that the conclusion holds. This completes the proof. ■

If $A = I$ the identity mapping of $L(X,Y)$, and $T = f$ a single valued mapping from X into $L(X,Y)$, then the following corollary follows immediately from Theorem 3.3.

Corollary 3.4. *See Theorem 3.3 in Huang and Li (2006). Let D be an acute cone with apex at the origin and $0 \notin -D$. Let $f : X \to L(X, Y)$ and $g : K \to K$ be continuous. If all assumptions in Corollary 3.3 hold, then $(WVIVIP)$ has a solution.*

Theorem 3.4. *Let D be an acute cone with apex at the origin and $0 \notin -D$. Let $A : L(X, Y) \to L(X, Y)$ and $g : K \to K$ be continuous, and $T : X \to 2^{L(X,Y)}$ be a nonempty compact-valued multifunction which is H-continuous. Suppose that all assumptions in Theorem 3.3 hold and that $\langle As^*, g(x^*)\rangle \in \mathrm{Fr}(\mathrm{cl}D) \cup \mathrm{Fr}(-\mathrm{cl}D)$ for any solution x^* of (GWVIVIP), where $s^* \in Tx^*$ is a corresponding element to x^* in the definition of (GWVIVIP). Then (GWVICP) has a solution.*

Proof. It follows from Theorems 3.1 (ii) and 3.3 that the conclusion holds. This completes the proof. ∎

If $A = I$ the identity mapping of $L(X, Y)$, and $T = f$ a single valued mapping from X into $L(X, Y)$, then the following corollary follows immediately from Theorem 3.4.

Corollary 3.5. *See Theorem 3.4 in Huang and Li (2006). Let D be an acute cone with apex at the origin and $0 \notin -D$. Let $f : X \to L(X, Y)$ and $g : K \to K$ be continuous. If $\langle f(x^*), g(x^*)\rangle \in \mathrm{Fr}(\mathrm{cl}D) \cup \mathrm{Fr}(-\mathrm{cl}D)$ for any solution x^* of (WVIVIP), and all assumptions in Corollary 3.4 hold, then (WVICP) has a solution.*

Remark 3.2. If g is the identity mapping on K in Corollaries 3.4 and 3.5, then we obtain the existence theorems of solutions for (VVIP) and (WVCP). Thus, we give an affirmative answer to the open question proposed by Rapcsak (2000).

References

1. G. Y. Chen and B. D. Craven, Approximate dual and approximate vector variational inequality for multiobjective optimization, *Journal of the Australian Mathematical Society Series A*, **47** (1989), 418-423.

2. G. Y. Chen and X. Q. Yang, The vector complementarity problem and its equivalence with the weak minimal element in ordered space, Journal of Mathematical Analysis and Applications, **153** (1990), 136-158.

3. F. Giannessi, Theorems of alternative, quadratic programmes and complementarity problems. In: R. W. Cottle, F. Giannessi and J .L. Lions, (eds.), *Variational Inequalities and Complementarity Problems*, Wiley & Sons, New York. 1980.

4. F. Giannessi, *Vector Variational Inequalities and Vector Equilibria*, Kluwer Academic Publisher, Dordrechet, Holland., 2000,

5. N. J. Huang, J. Li and H. B. Thompson, Implicit vector equilibrium problems with applications, *Mathematical and Computer Modelling.*, **37** (2003), 1343-1356.

6. T. C. Lin, Convex sets, fixed points, variational and minimax inequalities, *Bulletin of the Australian Mathematical Society Series*, **34** (1986), 107-117.

7. Tamas Rapcsak, On vector complementarity systems and vector variational inequalities. Vector variational inequalities and vector equilibria. In: *Nonconvex Optim. Appl.*, Vol. 38, pp. 371-380. Kluwer Acad. Publ., Dordrechet., 2000.

8. X. Q. Yang, Vector complementarity and minimal element problems, *Journal of Optimization Theory and Applications*, **77** (1993), 483-495.

9. X. Q. Yang, Vector variational inequality and vector pseudolinear optimization, *Journal of Optimization Theory and Applications,* **95** (1997), 729-734.

10. X. Q. Yang and C. J. Goh, On vector variational inequality and its application in traffic equilibria, *Journal of Optimization Theory and Applications*, **95(2)** (1997), 431-443.

11. Z. F. Shen, Maximal element theorems of H-majorized correspondence and existence of equilibrium for abstract economics, *Journal of Mathematical Analysis and Applications*, **256** (2001), 67-79.

12. N. J. Huang and J. Li, On vector implicit variational inequalities and complementarity problems, *Journal of Global Optimization*, (2006), (to appear).

13. S. B. Jr. Nadler, Multi-valued contraction mappings, *Pacific Journal of Mathematics*, **30** (1969), 475-488.

14. S. S. Chang, Set-valued variational inclusions in Banach spaces, *Journal of Mathematical Analysis and Applications*, **248** (2000), 438-454.

15. L. C. Zeng, S. Schaible and J. C. Yao, Iterative algorithm for generalized set-valued strongly nonlinear mixed variational-like inequalities, *Journal of Optimization Theory and Applications*, **124(3)** (2005), 725-738.

16. L. C. Zeng, S. M. Guu and J. C. Yao, Iterative algorithm for completely generalized set-valued strongly nonlinear mixed variational-like inequalities, *Computers and Mathematics with Applications*, **50** (2005), 935-945.

17. I. V. Konnov and J. C. Yao, On the generalized vector variational inequality problems, *Journal of Mathematical Analysis and Applications,* **206** (1997), 42-58.

18. L. C. Zeng and J. C. Yao, Existence of solutions of generalized vector variational inequalities in reflexive Banach spaces, *Journal of Global Optimization*, **36** (2006), 483-497,

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