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B-SEMIPREINVEX FUNCTIONS AND VECTOR OPTIMIZATION PROBLEMS IN BANACH SPACES

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Abstract. In this paper, we extend the scalar-valued B-semipreinvex functions and vector-valued preinvex functions to the cases of vector-valued B-semipreinvex functions in Banach spaces. We investigate some properties for the vector-valued B-semipreinvex functions and consider a new class of vector-valued nonsmooth programming problems in which functions are locally Lipschitz. In terms of the Ralph vector sub-gradient, we obtain the generalized Kuhn-Tucker type sufficient optimality conditions and saddle point condition. Also, a generalized Mond-Weir type dual is formulated and some duality theorems are established involving locally Lipschitz B-semipreinvex functions for the pair of primal and dual programming. The results presented in this paper generalize some main results of Kuang and Batista Dos Santoset, Osuna-Gomez, Rojas-Medar and Rufian-Lizana.

1. INTRODUCTION

It is well known that convex functions play an important role in optimization theory. A meaningful generalization of convex functions is the introduction of B-vex functions, which was given by Bector and Singh [1]. Later, the concept of B-vexity was extented to B-invex functions by Bector, Suneja, and Alitha [2], and to B-preinvex functions by Suneja, Singh, and Bector [3]. Yang and Chen [4] introduced and studied a wider class of nonconvex functions which is called the semipreinvex functions. In 2004, Kuang [5] defined the concept of B-semipreinvex functions as a generalization of the above functions, discussed some properties of B-semipreinvex

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functions, and proved the necessary and sufficient optimality theorems, weak duality for nonsmooth programs under the locally Lipschitz B-semipreinvexity assumptions.

On the other hand, vectorial problems on Banach spaces have many applications in mathematical economies and engineering such as optimal control, the optimum of which is described by a curve instead of a finite vector. In the past and recent years, some related works have been done for developing the vector optimization theory in finite or infinite dimensional spaces, which involve differentiable or non-differentiable convex, or invex functions (see, for example, [6-13] and the references therein).

In this paper, we extend the scalar-valued B-semipreinvex functions [5] and vector-valued preinvex functions [6] to the cases of vector-valued B-semipreinvex functions in Banach spaces. We investigate some properties for the vector-valued B-semipreinvex functions and consider a new class of vector-valued nonsmooth programming problems in which functions are locally Lipschitz. In terms of the Ralph vector sub-gradient, we obtain the generalized Kuhn-Tucker type sufficient optimality conditions and saddle point condition. Also, a generalized Mond-Weir type dual is formulated and some duality theorems are established involving locally Lipschitz B-semipreinvex functions for the pair of primal and dual programming. The results presented in this paper generalize some main results of [5] and [6].

2. Preliminaries

Let X, Y, Z be real Banach spaces. A nonempty subset Q of X is called a pointed closed convex cone if Q is closed and the following conditions hold:

$$\text{(i)}\, Q + Q \subseteq Q, \qquad \text{(ii)}\, \lambda Q \subseteq Q, \forall\, \lambda \geq 0, \qquad \text{(iii)}\, Q \cap (-Q) = 0.$$

Let $P \subset Y$ and $\Omega \subset Z$ be closed convex cones with nonempty interiors, i.e., $\text{int}P \neq \emptyset, \text{int}\Omega \neq \emptyset$, and $P \neq Y, \Omega \neq Z$. We assume that Y and Z are ordered Banach spaces whose partial order is induced by P and Ω, respectively.

Let Y^* be the dual of Y and $P^* = \{p^* \in Y^* : \langle p^*, y\rangle \geq 0, \forall\, y \in P\}$ be the dual cone of P. Let K be a nonempty subset of X and the functions $f : K \to Y$ and $g : K \to Z$ be given functions.

Now we consider the following vector-valued optimization problem:

$$\text{(VOP)} \quad \min \quad f(x) \qquad \text{s.t.} \quad x \in K.$$

We say that (VOP) has a weakly efficient solution at $x_0 \in K$ iff there exists no $x \in K$ such that

$$f(x_0) - f(x) \in \text{int}P.$$

Definition 2.1. Let X be real Banach space, K be a subset of X, and P be a closed convex cone in Y. A function $f : K \to Y$ is said to be P-B-semipreinvex (PBS) with respect to η and b at $y \in K$, if there exists a mapping $\eta : K \times K \times [0,1] \to X$ such that for all $x \in K$ and $\lambda \in [0,1]$,

$$\lambda b(x,y,\lambda)f(x) + (1 - \lambda b(x,y,\lambda))f(y) - f(y + \lambda\eta(x,y,\lambda)) \in P$$

with $\lim_{\lambda \to 0^+} \lambda\eta(x,y,\lambda) = 0$ where $b : K \times K \times [0,1] \to R_+$ (the set of nonnegative real numbers) with $\lambda b(x,y,\lambda) \in [0,1]$ for all $x, y \in K$ and $\lambda \in [0,1]$.

A function $f : K \to Y$ is said to be PBS on K with respect to (w.r.t.) η and b if it is PBS w.r.t. η and b at each point of K.

It should be noted that the set K in Definition 2.1 must be assumed to have semi-connectedness property, that is,

$$y + \lambda\eta(x,y,\lambda) \in K, \qquad \forall\, x, y \in K \text{ and } \forall\, \lambda \in [0,1].$$

Example 2.1. Let $f = (f_1, f_2) : R^2 \to R^2$ be a function where f_1 and f_2 are given by

$$f_1(x_1,x_2) = |x_1| + |x_2|, \quad f_2(x_1,x_2) = |x_1| - x_2.$$

Taking $P = R^2_+$, it is easy to see that f is PBS w.r.t. η and b where $\eta(x,y,\lambda) = x-y$ and $b(x,y,\lambda) = 1$, respectively.

Example 2.2. Let $P = R^2_+$ and $f = (f_1, f_2) : R^2 \to R^2$ be defined by

$$f_1(x) = x_1, \quad f_2(x) = x_2.$$

Then f is PBS on its domain with respect to $\eta = (\eta_1, \eta_2)$ and b where

$$\eta_1(x,y,\lambda) = \begin{cases} y_1 - x_1, & x_1 \geq y_1, \\ x_1 - y_1, & x_1 < y_1; \end{cases}$$

$$\eta_2(x,y,\lambda) = \begin{cases} y_2 - x_2, & x_2 \geq y_2, \\ x_2 - y_2, & x_2 < y_2; \end{cases}$$

and

$$b(x,y,\lambda) = \begin{cases} \lambda, & x_1 \geq y_1,\ x_2 \geq y_2 \text{ or } x_1 < y_1,\ x_2 < y_2; \\ 1 - \lambda, & x_1 \geq y_1,\ x_2 < y_2 \text{ or } x_1 < y_1,\ x_2 \geq y_2. \end{cases}$$

Remark 2.1. If $X = R^n, Y = R$ and $P = R_+$, then Definition 2.1 reduces to the definition of scalar-valued B-semipreinvex function introduced by Kuang [5].

Remark 2.2. If $\eta(x,y,\lambda) = \eta(x,y)$ and $b(x,y,\lambda) = 1$ for all $x, y \in K$, then Definition 2.1 reduces to the definition of vector-valued preinvex function discussed in [6].

Lemma 2.1. ([8]) *Let X be a Banach space ordered by the cone $Q \subset X$, with Q convex and closed. Let Q^* be the dual cone of Q. If there exists $x \in X$ such that $\langle x^*, x\rangle \geq 0$ for all $x^* \in Q^*$, then $x \in Q$.*

Lemma 2.2. *A mapping $f : K \to Y$ is PBS w.r.t. η and b iff for each $p^* \in P^*$, the composition function $p^* f$ is scalar-valued B-semipreinvex w.r.t. η and b.*

Proof. The necessity is obvious, it suffices therefore to prove the sufficiency. Let $x, y \in K$, $\lambda \in (0,1)$, and $b := b(x,y,\lambda)$. We have

$$p^* f(y + \lambda\eta(x,y,\lambda) \leq \lambda b(x,y,\lambda) p^* f(x) + (1 - \lambda b(x,y,\lambda)) p^* f(y), \quad \forall p^* \in P^*.$$

This implies that

$$p^*(\lambda b f(x) + (1-\lambda b) f(y) - f(y + \lambda\eta(x,y,\lambda))) \geq 0, \quad \forall p^* \in P^*$$

and so it follows from Lemma 2.1 that

$$\lambda b f(x) + (1-\lambda b) f(y) - f(y + \lambda\eta(x,y,\lambda)) \in P.$$

This completes the proof. ■

We suppose further that K is an open semi-connected subset of X. Recall (see [9]) that a function $f : K \to R$ is said to be locally Lipschitz if for each $x \in K$, there exist a neighborhood $N(x)$ of x and a constant $k_x > 0$ such that for any $y, z \in N(x)$,

$$|f(y) - f(z)| \leq k_x \|y - z\|.$$

If f is locally Lipschitz at x, then Clarke generalized directional derivative of f at x in the direction of $v \in X$ is defined as follows:

$$f^\circ(x;v) = \limsup_{y \to x, t \downarrow 0} \frac{f(y+tv) - f(y)}{t}.$$

It then follows that for any $v \in X$,

$$f^\circ(x;v) = \max\{v^T \xi \mid \xi \in \partial f(x)\},$$

where $\partial f(\cdot)$ denotes the Clarke sub-differential (see [9]). Let f be locally Lipschitz. A function f is said to be regular at x in the sense of Clarke (see [9]) if it is directionally differentiable at x and $f'(x;v) = f^\circ(x;v)$ for any $v \in X$.

Definition 2.3. (14) A mapping $f : K \to R$ is said to be arcwise directionally differentiable at x if the following limit exists

$$f^*(x; v) = \lim_{t \to 0^+} \frac{f(x + \omega(t)) - f(x)}{t}$$

for each continuous arc $\omega : [0, 1] \to K$ such that $\omega(0) = 0$ and $\omega'(0^+) = v$.

Lemma 2.3. ([5]) *Suppose that $f : K \to R$ is locally Lipschitz and regular at $x \in K$. Then f is arcwise differentiable at x and*

$$f^*(x; v) = f^\circ(x; v) = f'(x; v).$$

Definition 2.4. ([10]) Let $x \in X$. The Ralph vector sub-gradient of a locally Lipschitz function $f : K \to Y$ at x is defined by

$$\partial f(x) := \{A \in L(K, Y) \mid (v, \lambda) \in K \times Y^*,\ (\lambda f)^\circ(x; v) \geq \lambda A v\},$$

where $L(K, Y)$ is the space of all linear and continuous operators from K to Y and $(\lambda f)^\circ(x; \cdot)$ denotes the Clarke generalized directional derivative of the scalar locally Lipschitz function λf at the point $x \in K$.

Lemma 2.4. ([7]) *Let x be a fixed point in K. If $f : K \to Y$ is continuously $G\hat{a}teaux$ differentiable at x, then for any $\lambda \in Y^*$, λf is regular at x. Moreover, f is locally Lipschitz at x and $\partial f(x) = \{f'(x)\}$ where $f'(x)$ denotes the $G\hat{a}teaux$ derivative of f at x.*

3. Weakly Efficient Solutions of (VOP)

Theorem 3.1. Let K be a semi-connected subset of X and $f : K \to Y$ be PBS w.r.t. η and b, where $P \subset Y$ is a closed pointed and convex cone with nonempty interior. Suppose that $b(x, y, \lambda) > 0$ for $0 < \lambda < 1$ with fixed $x, y \in K$. Then any locally weakly efficient solution of (VOP) is a globally weakly efficient solution.

Proof. Let $\bar{x} \in K$ be a local weakly efficient solution of (VOP). Then there is a neighborhood U of $\bar{x}$ such that

(1) $$f(\bar{x}) - f(y) \notin \text{int}P, \qquad \forall y \in U \cap K.$$

Suppose that $\bar{x}$ is not global. It follows that there exists $x_0 \in K$ such that

(2) $$f(\bar{x}) - f(x_0) \in \text{int}P.$$

Since f is PBS and K is semi-connected, we know that $\bar{x}+\lambda\eta(x_0,\bar{x},\lambda))\in K$ for all $\lambda\in(0,1)$ and

$$\lambda bf(x_0)+(1-\lambda b)f(\bar{x})-f(\bar{x}+\lambda\eta(x_0,\bar{x},\lambda)\in P,$$

i.e.,

$$f(\bar{x})-f(\bar{x}+\lambda\eta(x_0,\bar{x},\lambda)+\lambda b(f(x_0)-f(\bar{x}))\in P, \qquad \forall\lambda\in(0,1), \tag{3}$$

where $b:=b(x,y,\lambda)$. Since $\lim\limits_{\lambda\to 0^+}\lambda\eta(x_0,\bar{x},\lambda)=0$, for a sufficiently small enough $\bar{\lambda}$, $\bar{x}\neq\bar{x}+\bar{\lambda}\eta(x_0,\bar{x},\bar{\lambda})\in U\cap K$. By (1),

$$f(\bar{x})-f(\bar{x}+\bar{\lambda}\eta(x_0,\bar{x},\bar{\lambda}))\notin \text{int}P.$$

It follows from (2) and (3) that

$$\begin{aligned} f(\bar{x})-f(\bar{x}+\bar{\lambda}\eta(x_0,\bar{x},\bar{\lambda})) &\in P+\bar{\lambda}b(x_0,\bar{x},\bar{\lambda})(f(\bar{x})-f(x_0)) \\ &\subset P+\text{int}P \\ &\subset \text{int}P, \end{aligned}$$

which is a contradiction. This completes the proof. ■

Theorem 3.2. *Suppose that Y is reflexive. Let x be a fixed point in K, P be a closed convex cone such that $P\neq Y$ and int$P\neq\emptyset$, and $f:K\to Y$ be a locally Lipschitz PBS function w.r.t. η and b at x where K is an open semi-connected subset of X w.r.t. η. Assume that*

(*i*) *for any $p^*\in P^*$, p^*f is regular at x where P^* is the dual cone of P;*

(*ii*) *$\frac{d}{d\lambda}[\lambda\eta(y,x,\lambda)]\big|_{\lambda=0}=\bar{\eta}(y,x)$ and $\lim\limits_{\lambda\to 0^+}b(y,x,\lambda)=\bar{b}(y,x)$ for all $y\in K$.*

Then for any $y\in K$ and $A\in\partial f(x)$,

$$\bar{b}(y,x)(f(y)-f(x))-\langle A,\bar{\eta}(y,x)\rangle\in P. \tag{4}$$

Proof. Since Y is reflexive, it follows from the result of ([11]) that $\partial f(x)\neq\emptyset$. For any $p^*\in P^*$, it follows from the assumptions and Lemma 2.2 that p^*f is locally Lipschitz, B-semipreinvex and regular at x. So we have for any $y\in K$ and $A\in\partial f(x)$,

$$p^*f(x+\lambda\eta(y,x,\lambda))-p^*f(x)\le\lambda b(p^*f(y)-p^*f(x)),$$

where $b := b(x, y, \lambda)$ and $\lambda \in (0, 1]$. Now dividing across by λ and letting $\lambda \to 0^+$, it follows from Lemma 2.3 that

$$\begin{aligned}\bar{b}(y, x)(p^* f(y) - p^* f(x)) &\geq (p^* f)^*((x; \bar{\eta}(y, x)) \\ &= (p^* f)^\circ((x; \bar{\eta}(y, x)) \\ &\geq p^* A\bar{\eta}(y, x).\end{aligned}$$

By Lemma 2.1, we have

$$\bar{b}(y, x)(f(y) - f(x)) - \langle A, \bar{\eta}(y, x)\rangle \in P.$$

This completes the proof. ∎

We can get the following corollary with a stronger assumption on f.

Corollary 3.1. *Let Y be a real Banach space, $x \in K$ be a given point, and $f : K \to Y$ be PBS w.r.t. η and b at x. If f is continuously $G\hat{a}teaux$ differentiable at x, and the assumption (ii) of Theorem 3.2 is satisfied for all $y \in K$, then*

$$\bar{b}(y, x)(f(y) - f(x)) - \langle f'(x), \bar{\eta}(y, x)\rangle \in P, \quad \forall\, y \in K,$$

where $f^{'}(x)$ denotes the $G\hat{a}teaux$ derivative of f at x.

Proof. It can be easily verified by invoking Lemma 2.4 and Theorem 3.2. ∎

Theorem 3.3. *Suppose that Y is reflexive and $P \subset Y$ is a closed convex cone with nonempty interior. Let $f : K \to Y$ be a locally Lipschitz PBS function w.r.t. η and b at $\bar{x} \in K$. If in Theorem 3.2, we assume further that $\bar{b}(x, \bar{x}) > 0$ for all $x \in K$, then $\langle A, \bar{\eta}(x, \bar{x})\rangle \notin -intP$ implies that $\bar{x}$ is the weakly efficient solution of (VOP).*

Proof. Suppose that $\bar{x} \in K$ is not a weakly efficient solution of (VP). Then there exists $x^* \in K$ such that

$$f(x^*) - f(\bar{x}) \in -\text{int}P. \tag{5}$$

Since f is PBS, it follows from Theorem 3.2 that for any $x \in K$ and $A \in \partial f(x)$,

$$\langle A, \bar{\eta}(x, \bar{x})\rangle + \bar{b}(x, \bar{x})(f(\bar{x}) - f(x)) \in -P. \tag{6}$$

From (5) and (6), for any $A \in \partial f(x)$,

$$\begin{aligned}\langle A, \bar{\eta}(x^*, \bar{x})\rangle &\in \bar{b}(x^*, \bar{x})(f(x^*) - f(\bar{x})) - P \\ &\subset -\text{int}P - P \\ &\subset -\text{int}P,\end{aligned}$$

which is a contradiction. This completes the proof. ■

Remark 3.1. Theorems 3.1 and 3.2 extend Theorem 4.1 of [6] and Theorem 4.2 of [5], respectively.

Remark 3.2. If we further assume that $b(x, y, \lambda)$ is continuous at $\lambda = 0$ and $\eta(x, y, \lambda)$ is continuous differentiable at $\lambda = 0$, then Theorem 3.2 is also a generalization of Theorems 4.3-4.5 of [5].

Remark 3.3. In Theorem 3.3, it is not necessarily true that the condition $\bar{b}(x, \bar{x}) > 0$ for all $x \in K$. To see this, let $f(x) = \sin x$ for all $[0, \pi)$. Obviously, 0 is the optimal solution of f. However, f is PBS w.r.t. η and b at $y = 0$ defined by $\eta(x, y, \lambda) = \lambda \sin(x - y)$ and $b(x, y, \lambda) = \lambda$ for all $x, y \in R$.

4. Optimality Conditions and Duality Theorems

We are concerned with the following constrained vector optimization problem:

$$\text{(CVOP)} \qquad \begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & -g(x) \in \Omega, \\ & x \in K \subset X, \end{aligned}$$

where $f : K \to Y$ and $g : K \to Z$ are two mappings, K and Ω are subsets of X and Z, respectively. We assume that the spaces Y and Z are ordered by cones $P \subset Y$ and $\Omega \subset Z$, respectively, and the cones P and Ω are closed convex with nonempty interior.

We denote by $K_0 = \{x \in K \mid -g(x) \in \Omega\}$ the feasible set of (CVOP).

In this section, we assume that Y and Z are reflexive Banach spaces and all the functions, unless otherwise stated, are locally Lipschitz on K where K is an open semi-connected subset of X w.r.t. a function $\eta : K \times K \times [0, 1] \to X$.

Theorem 4.1.(Sufficient condition) *Let $\bar{x}$ be a feasible point in K, $f : K \to Y$ be PBS function w.r.t. η and b at $\bar{x}$, and $g : K \to Z$ be ΩBS function w.r.t. η and b' at $\bar{x}$. Assume that there exists $(\lambda^*, \mu^*) \in P^* \times \Omega^*$ with $\lambda^* \neq 0$ such that*

$$0 \in \partial(\lambda^* f + \mu^* g)(\bar{x}), \tag{7}$$

$$\mu^* g(\bar{x}) = 0 \tag{8}$$

and

$$\lim_{\lambda \to 0^+} b(x, \bar{x}, \lambda) = \bar{b}(x, \bar{x}) > 0, \quad \lim_{\lambda \to 0^+} b'(x, \bar{x}, \lambda) = \bar{b}'(x, \bar{x}), \quad \forall x \in K_0, \tag{9}$$

(10) $$\frac{d}{d\lambda}[\lambda\eta(x,\bar{x},\lambda)]\Big|_{\lambda=0} = \bar{\eta}(x,\bar{x}), \quad \forall x \in K_0.$$

Further suppose that $\lambda^* f$ and $\mu^* g$ are regular at $\bar{x}$. Then $\bar{x}$ is a weakly efficient solution of $(CVOP)$.

Proof. First from (7) and the properties of the Clark sub-differential, we have

$$0 \in \partial(\lambda^* f + \mu^* g)(\bar{x}) \subset \partial(\lambda^* f)(\bar{x}) + \partial(\mu^* g)(\bar{x}).$$

It follows that there exists $A \in \partial f(\bar{x})$ and $B \in \partial g(\bar{x})$ such that

$$\lambda^* A + \mu^* B = 0.$$

By hypotheses (9) and (10), and Theorem 3.2, for any $x \in K_0$,

(11) $$\bar{b}(x,\bar{x})(\lambda^* f(x) - \lambda^* f(\bar{x})) \geq \langle \lambda^* A, \bar{\eta}(x,\bar{x})\rangle$$

and

(12) $$\bar{b}'(x,\bar{x})(\mu^* g(x) - \mu^* g(\bar{x})) \geq \langle \mu^* B, \bar{\eta}(x,\bar{x})\rangle.$$

Since $-g(x) \in \Omega$, it follows from (8)-(9) and (11)-(12) that

$$\bar{b}(x,\bar{x})(\lambda^* f(x) - \lambda^* f(\bar{x})) \geq 0 - \bar{b}'(x,\bar{x})\mu^* g(x) \geq 0$$

and

$$\lambda^*(f(x) - f(\bar{x})) \geq 0.$$

Therefore, $f(x) - f(\bar{x}) \notin -\text{int}P$, i.e., $\bar{x}$ is a weakly sufficient solution of (CVOP). This completes the proof. ∎

From Theorem 4.1, we can get the following corollary for differentiable vector optimization problem.

Corollary 4.1. *Let Y and Z be real Banach spaces, f and g be the same as in Theorem 4.1. If f and g are continuously $G\hat{a}teaux$ differentiable at $\bar{x}$, then the condition (7) can be simplified to the following form*

$$\lambda^* f'(\bar{x}) + \mu^* g'(\bar{x}) = 0,$$

where $f'(\bar{x})$ and $g'(\bar{x})$ denote the $G\hat{a}teaux$ derivatives of f and g at $\bar{x}$, respectively.

Remark 4.1. The conditions (7) and (8) are called the generalized Kuhn-Tucker conditions of problem (CVOP).

Remark 4.2. A Lagrangian function $L : K_0 \times P^* \times \Omega^* \to R$ associated with the problem (CVOP) can be defined as follows

$$L(x, \lambda^*, \mu^*) = \lambda^* f(x) + \mu^* g(x)$$

for all $x \in K_0$ and $(\lambda^*, \mu^*) \in P^* \times \Omega^*$.

Theorem 4.2. *If $(\bar{x}, \lambda^*, \mu^*)$ with $\lambda^* \neq 0$ is a point satisfying the generalized Kuhn-Tucker conditions, then Theorem 4.1 still holds in the case when the Lagrangian $L(x, \lambda^*, \mu^*)$ is scalar-valued B-semipreinvex w.r.t. η and b at $\bar{x}$ on K_0 where*

$$\frac{d}{d\lambda}[\lambda\eta(x, \bar{x}, \lambda)]\Big|_{\lambda=0} = \bar{\eta}(x, \bar{x})$$

and

$$\lim_{\lambda \to 0^+} b(x, \bar{x}, \lambda) = \bar{b}(x, \bar{x}) > 0$$

for all $x \in K_0$.

Proof. From Theorem 3.2 and Lemma 2.3, we have the following inequalities

$$\begin{aligned} \bar{b}(x, \bar{x})((\lambda^* f + \mu^* g)(x) - (\lambda^* f + \mu^* g)(\bar{x})) &\geq ((\lambda^* f + \mu^* g)^*(\bar{x}; \bar{\eta}(x, \bar{x})) \\ &= (\lambda^* f + \mu^* g)^\circ(\bar{x}; \bar{\eta}(x; \bar{x})) \\ &\geq T\bar{\eta}(x, \bar{x}) \end{aligned}$$

for all $T \in \partial(\lambda^* f + \mu^* g)(\bar{x})$. In particular, we can choose $T = 0 \in \partial(\lambda^* f + \mu^* g)(\bar{x})$ such that

$$\bar{b}(x, \bar{x})(\lambda^*(f(x) - f(\bar{x}))) \geq -\bar{b}(x, \bar{x})(\mu^*(g(x) - g(\bar{x}))) \geq 0.$$

Since $\bar{b}(x, \bar{x}) > 0$, it follows that

$$f(x) - f(\bar{x}) \notin -\text{int}P.$$

This completes the proof. ∎

Example 4.1. Consider the following nonsmooth programming problem:

$$\begin{aligned} \min f(x) &= \frac{1}{2}x - \cos x, \\ g_1(x) &= |x| - \frac{\pi}{3}x, \\ g_2(x) &= x - \frac{\pi}{2}, \end{aligned}$$

where $f : R \to R$, $g = (g_1, g_2) : R \to R^2$. The feasible region is $K_0 = [0, \frac{\pi}{2}]$. If we take $\mu = (1, 0)$, then $\mu g(x) = |x| - \frac{\pi}{3}x$. In this case, we can easily check that $\mu g(x)$ is regular at 0, and $\partial(\mu g)(0) = [-1 - \frac{\pi}{3}, 1 - \frac{\pi}{3}]$. We also note that λf is regular at 0 for all $\lambda \geq 0$ and $\partial(\lambda f)(0) = \{\frac{\lambda}{2}\}$. Thus we know that the conditions (7) and (8) are satisfied at 0 for $\lambda = 2$ and $\mu = (1, 0)$. It follows from Theorem 4.1 that 0 is the weakly efficient solution of f.

By Theorem 4.1, it is easy to prove the following result.

Theorem 4.3. (Generalized Kuhn-Tucker saddle point condition) *Suppose that $(\bar{x}, \bar{\lambda}^*, \bar{\mu}^*)$ with $\bar{\lambda}^* \neq 0$ is a generalized Kuhn-Tucker point of (CVOP). Let f and g be PBS functions w.r.t. the same η and b. Then under the assumption of Theorem 4.1, $\bar{x}$ is a point of weakly efficient solution for (CVOP) if and only if $(\bar{x}, \bar{\lambda}^*, \bar{\mu}^*)$ is a saddle point of the Lagrangian function, i.e.,*

$$L(\bar{x}, \bar{\lambda}^*, \mu^*) \leq L(\bar{x}, \bar{\lambda}^*, \bar{\mu}^*) \leq L(x, \bar{\lambda}^*, \bar{\mu}^*), \quad \forall\, x \in K_0 \text{ and } \forall\, (\lambda^*, \mu^*) \in P^* \times \Omega^*,$$

where $L(x, \lambda^*, \mu^*)$ is defined as in Remark 4.2.

We now consider the dual problem for (CVOP). Let $r \in \text{int}P$ be a fixed point. Following the approaches of Mond-Weir [11], we formulate the dual problem for (CVOP) as follows

$$\begin{aligned} \text{(D)} \qquad & \max \quad f(u) \\ & \text{s.t. } \ 0 \in \partial(\lambda^* f + \mu^* g)(u), \quad \forall u \in K, \\ & \qquad \mu^* g(u) \geq 0, \quad \forall u \in K, \\ & \qquad \lambda^* \in P^*,\ \mu^* \in \Omega^*,\ \lambda^* r = 1. \end{aligned}$$

Let

$$\begin{aligned} W = \{(u, \lambda^*, \mu^*) \in K \times P^* \times \Omega^* :\ & 0 \in \partial(\lambda^* f + \mu^* g)(u),\ \mu^* g(u) \geq 0, \\ & \lambda^* \in P^*,\ \mu^* \in \Omega^*,\ \lambda^* r = 1\} \end{aligned}$$

denote the set of all feasible points of (D). We denote by $pr_K W$ the projection of the set W on K, that is, $pr_K W = \{u \in K : (u, \lambda^*, \mu^*) \in W\}$.

Theorem 4.4. (Weak duality) *Let x and (u, λ^*, μ^*) be feasible points for (CVOP) and (D), respectively. Let f be PBS w.r.t. η and b at u on $K_0 \cup pr_K W$, and g be ΩBS function w.r.t. η and b' at u on $K_0 \cup pr_K W$ where*

$$\frac{d}{d\lambda}[\lambda \eta(x, u, \lambda)]\bigg|_{\lambda=0} = \bar{\eta}(x, u),$$
$$\lim_{\lambda \to 0^+} b(x, u, \lambda) = \bar{b}(x, u) > 0, \ \lim_{\lambda \to 0^+} b'(x, u, \lambda) = \bar{b}'(x, u).$$

Moreover, we assume that $\lambda^* f$ and $\mu^* g$ are regular at u. Then $f(x) - f(u) \notin -\mathrm{int}P$.

Proof. Since $x \in K$ and (u, λ^*, μ^*) are feasible solutions of problem (CVOP) and (D), it follows from Theorem 3.2 and the constraints of problem (D) that

$$\begin{aligned} \bar{b}(x,u)(\lambda^* f(x) - \lambda^* f(u)) &\geq \langle \lambda^* A, \bar{\eta}(x,u)\rangle \\ &= \langle -\mu^* B, \bar{\eta}(x,u)\rangle \\ &\geq \bar{b}'(x,u)(\mu^* g(u) - \mu^* g(x)) \\ &\geq 0, \end{aligned}$$

where $A \in \partial f(u)$ and $B \in \partial g(u)$. Hence we have

$$f(x) - f(u) \notin -\mathrm{int}P.$$

This completes the proof. ∎

We can further weaken the conditions on f and g and have weak duality theorem in the following form.

Theorem 4.5. (Weak duality) *Let x and (u, λ^*, μ^*) be feasible points for (CVOP) and (D), respectively. Suppose that the Lagrangian $\lambda^* f + \mu^* g$ is a scalar-valued B-semipreinvex function w.r.t. η and b at u on $K_0 \cup pr_K W$ where*

$$\frac{d}{d\lambda}[\lambda\eta(x,u,\lambda)]\bigg|_{\lambda=0} = \bar{\eta}(x,u), \quad \lim_{\lambda\to 0^+} b(x,u,\lambda) = \bar{b}(x,u) > 0,$$

and $\lambda^* f$ and $\mu^* g$ are regular at u. Then $f(x) - f(u) \notin -\mathrm{int}P$.

Proof. It is similar to the proof of Theorem 4.2. ∎

Theorem 4.6. (Strong duality) *If $\bar{x}$ is an optimal solution of (CVOP), at which the generalized Kuhn-Tucker conditions (7) and (8) are satisfied, then there exists $(\bar{\lambda}^*, \bar{\mu}^*) \in P^* \times \Omega^*$ with $\lambda^* \neq 0$ such that $(\bar{x}, \bar{\lambda}^*, \bar{\mu}^*)$ is feasible for (D), and the two objective values are equal. Moreover, let f be PBS function w.r.t. η and b at any $u \in pr_K W$ on $K_0 \cup pr_K W$, and g be ΩBS w.r.t. η and b' at any $u \in pr_K W$ on $K_0 \cup pr_K W$, (resp., the Lagrangian function $\bar{\lambda}^* f + \bar{\mu}^* g$ be a scalar-valued B-semipreinvex function w.r.t. η and b at any $u \in pr_K W$ on $K_0 \cup pr_K W$), where*

$$\frac{d}{d\lambda}[\lambda\eta(x,u,\lambda)]\bigg|_{\lambda=0} = \bar{\eta}(x,u),$$

$$\lim_{\lambda\to 0^+} b(x,u,\lambda) = \bar{b}(x,u) > 0, \quad \lim_{\lambda\to 0^+} b'(x,u,\lambda) = \bar{b}'(x,u).$$

If we further suppose that $\bar{\lambda}^* f$ *and* $\bar{\mu}^* g$ *are regular at* u*, then* $(\bar{x}, \bar{\lambda}^*, \bar{\mu}^*)$ *is optimal for (D).*

Proof. By assumption, there exists $(\bar{\lambda}^*, \bar{\mu}^*) \in P^* \times \Omega^*$ with $\bar{\lambda}^* r = 1$ such that $0 \in \partial(\bar{\lambda}^* f + \bar{\mu}^* g)(\bar{x})$ and $\bar{\mu}^* g(\bar{x}) = 0$. This implies that $(\bar{x}, \bar{\lambda}^*, \bar{\mu}^*)$ is feasible for (D). From the weak duality property, for any feasible point (ξ, λ, μ) of (D), we conclude that $f(\bar{x}) - f(\xi) \notin -\text{int}P$. Hence, $(\bar{x}, \bar{\lambda}^*, \bar{\mu}^*)$ is optimal for (D). This completes the proof. ■

Finally, we will show that if P is a pointed closed convex cone in Y, then the following converse duality results hold.

Theorem 4.7. (Converse duality) *Let* $P \subset Y$ *be a closed, convex and pointed cone, and let* $(\bar{u}, \bar{\lambda}^*, \bar{\mu}^*)$ *be an optimal solution of (D). Suppose that* f *is PBS w.r.t.* η *and* b *at* $\bar{u}$ *on* $K_0 \cup pr_K W$*, and* g *is* Ω*BS w.r.t.* η *and* b *at* $\bar{u}$ *on* $K_0 \cup pr_K W$ *where*

$$\frac{d}{d\lambda}[\lambda\eta(x, \bar{u}, \lambda)]\Big|_{\lambda=0} = \bar{\eta}(x, \bar{u}),$$

$$\lim_{\lambda \to 0^+} b(x, \bar{u}, \lambda) = \bar{b}(x, \bar{u}) > 0, \qquad \lim_{\lambda \to 0^+} b'(x, \bar{u}, \lambda) = \bar{b}'(x, \bar{u})$$

for all $x \in K_0$*. Further assume that* $\bar{\lambda}^* f$ *and* $\bar{\mu}^* g$ *are regular at* $\bar{u}$*. Then* $\bar{u}$ *is an optimal solution for (CVOP).*

Proof. We proceed by contradiction. If $\bar{u}$ is not an optimal solution for (CVOP), then there exists $\tilde{x} \in K_0$ such that

$$f(\bar{u}) - f(\tilde{x}) \in \text{int}P. \tag{13}$$

From Theorem 3.2, we have

$$\bar{b}(\tilde{x}, \bar{u})(\bar{\lambda}^* f(\tilde{x}) - \bar{\lambda}^* f(\bar{u})) \geq \langle \bar{\lambda}^* A, \bar{\eta}(\tilde{x}, \bar{u})\rangle \tag{14}$$

and

$$\bar{b}'(\tilde{x}, \bar{u})(\bar{\mu}^* g(\tilde{x}) - \bar{\mu}^* g(\bar{u})) \geq \langle \bar{\mu}^* B, \bar{\eta}(\tilde{x}, \bar{u})\rangle, \tag{15}$$

where $A \in \partial f(\bar{u})$, $B \in \partial g(\bar{u})$ and $\bar{\lambda}^* A + \bar{\mu}^* B = 0$. Adding up (14) and (15) yields

$$\begin{aligned} \bar{b}(\tilde{x}, \bar{u})(\bar{\lambda}^*(f(\tilde{x}) - f(\bar{u}))) &\geq -\bar{b}'(\tilde{x}, \bar{u})(\bar{\mu}^*(g(\tilde{x}) - g(\bar{u}))) \\ &\geq -\bar{b}'(\tilde{x}, \bar{u})\bar{\mu}^* g(\tilde{x}) \\ &\geq 0. \end{aligned}$$

Hence $\bar{\lambda}^*(f(\tilde{x}) - f(\bar{u})) \geq 0$, i.e., $f(\bar{u}) - f(\tilde{x}) \in -P$, and this along with (13) imply that

$$f(\bar{u}) - f(\tilde{x}) \in P \cap (-P) = \{0\}.$$

This implies that $f(\bar{u}) - f(\tilde{x}) = 0 \in \text{int}P$ which is a contradiction. This completes the proof. ■

Theorem 4.8. (Converse duality) *Let* $(\bar{u}, \bar{\lambda}^*, \bar{\mu}^*)$ *be an optimal solution for (D). Let the Lagrangian function* $\bar{\lambda}^* f + \bar{\mu}^* g$ *be scalar-valued* B*-semipreinvex w.r.t.* η *and* b *at* $\bar{u}$ *on* $K_0 \cup pr_K W$ *where*

$$\frac{d}{d\lambda}[\lambda\eta(x, \bar{u}, \lambda)]_{\lambda=0} = \bar{\eta}(x, \bar{u}), \qquad \lim_{\lambda \to 0^+} b(x, \bar{u}, \lambda) = \bar{b}(x, \bar{u}) > 0$$

for all $x \in K_0$*. If* $\bar{\lambda}^* f$ *and* $\bar{\mu}^* g$ *are regular at* $\bar{u}$*, then* $\bar{u}$ *is an optimal solution for* $($*CVOP*$)$.

Proof. It is similar to the proof of Theorem 4.7. ■

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