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# RELATIONS BETWEEN DISTRIBUTION COSINE FUNCTIONS AND ALMOST-DISTRIBUTION COSINE FUNCTIONS

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**Abstract.** In this paper, we give connections between distribution cosine functions (defined in [10]) and almost-distribution cosine functions (introduced in [13]). We prove several equalities involving trigonometric convolution products and distribution cosine functions as well as some relations between distribution cosine functions and ultradistribution semigroups.

## 1. Introduction and Preliminaries

The class of distribution cosine functions is introduced in [10] as a unification of the concept of (local) $\alpha-$times integrated cosine functions, $\alpha \geq 0$. By applying fractional integration and derivation, several results on equivalence between almost-distribution cosine functions and global $\alpha-$times integrated cosine functions with corresponding growth order are proved in [13]. In this paper, we obtain necessary and sufficient conditions under a closed linear operator $A$ generates an almost-distribution cosine function in terms of distribution cosine functions. In order to do that, we employ our results from [13] and [10]. In the last section, we relate distribution cosine functions to ultradistribution semigroups and prove an extension of [6, Theorem 3.1] obtained by V. Keyantuo. The paper is illustrated by some examples and can be viewed as a continuation of [13] and [10].

Let us introduce the terminology of distribution spaces used in the paper and the basic definitions from [13] and [10].

The space of all compactly supported $C^\infty$-functions from $\mathbb{R}$ into $\mathbb{C}$ is denoted by $\mathcal{D}$. It is equipped with the usual inductive limit topology. Its dual is $\mathcal{D}'$. We assume that $\mathcal{D}'$ is supplied with strong topology; $\mathcal{D}_0$ is the subspace of $\mathcal{D}$ which

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consists of the elements supported by $[0,\infty)$. In the sequel, we assume that $E$ is a complex Banach space in this paper. Further on, $\mathcal{D}'(L(E)) = L(\mathcal{D}, L(E))$ is the space of continuous linear functions from $\mathcal{D}$ into $L(E)$ equipped with the topology of uniform convergence on bounded subsets of $\mathcal{D}$; $\mathcal{D}'_0(L(E))$ is the subspace of $\mathcal{D}'(L(E))$ consisted of elements supported by $[0,\infty)$.

Let $K \subset \mathbb{R}$ and $\mathcal{D}_K := \{\varphi \in \mathcal{D} : \text{supp}\varphi \subset K\}$. Recall, if $k \in \mathbb{N}_0$, then the distribution $\delta^{(k)}$ is defined by $\delta^{(k)}(\varphi) = (-1)^k \varphi^{(k)}(0),\ \varphi \in \mathcal{D}$.

Let $\mathcal{D}_+ := \{f \in C^\infty([0,\infty)) : f \text{ is compactly supported}\}$. Define $\mathcal{K} : \mathcal{D} \to \mathcal{D}_+$ by $\mathcal{K}(\varphi)(t) = \varphi(t),\ t \geq 0,\ \varphi \in \mathcal{D}$. We know that $\mathcal{D}_+$ is an (LF) space and due to the theorem of R. T. Seeley [15], there exists a linear continuous operator $\Lambda : \mathcal{D}_+ \to \mathcal{D}$ satisfying $\mathcal{K}\Lambda = I_{\mathcal{D}_+}$.

We use the convolution product $*_0$ of measurable complex valued functions $f$ and $g$ : $f *_0 g(t) =: \int\limits_0^t f(t-s)g(s)ds$. If $f,\ g \in \mathcal{D}_+$, put $f \circ g(t) := \int\limits_t^\infty f(s-t)g(s)ds,\ t \geq 0$. Clearly, $f \circ g \in \mathcal{D}_+$. The *cosine convolution product* $f *_c g$ is defined by $f *_c g := \frac{1}{2}(f *_0 g + f \circ g + g \circ f)$; the *sine convolution product* by $f *_s g := \frac{1}{2}(f *_0 g - f \circ g - g \circ f)$ and the *sine-cosine convolution product* by $f *_{sc} g := \frac{1}{2}(f *_0 g - f \circ g + g \circ f)$. Notice, $f *_c g,\ f *_s g,\ f *_{sc} g \in \mathcal{D}_+$, see for example [16].

Hereafter we assume that $A$ is a closed linear operator. Its domain, range and null space are denoted by $D(A),\ R(A)$ and Ker$(A)$, respectively; $[D(A)]$ denotes the Banach space $D(A)$ equipped with the graph norm.

We need the next short review from [10]. Let $\alpha \in \mathcal{D}_{[-2,-1]}$ be a fixed test function with $\int\limits_{-\infty}^{\infty} \alpha(x)dx = 1$. Then, with $\alpha$ chosen in this way, for every fixed $\varphi \in \mathcal{D}$ we define $I(\varphi) \in \mathcal{D}$ by

$$I(\varphi)(x) := \int\limits_{-\infty}^{x} [\varphi(t) - \alpha(t) \int\limits_{-\infty}^{\infty} \varphi(u)du]dt,\ x \in \mathbb{R}.$$

Recall, $I(\varphi) \in \mathcal{D}$, $I(\varphi') = \varphi$ and $\frac{d}{dx} I(\varphi)(x) = \varphi(x) - \alpha(x) \int\limits_{-\infty}^{\infty} \varphi(u)du,\ x \in \mathbb{R}$. Next, we define a primitive of $G \in \mathcal{D}'(L(E))$, $G^{-1}$, by $G^{-1}(\varphi) := -G(I(\varphi)),\ \varphi \in \mathcal{D}$. We have $G^{-1} \in \mathcal{D}'(L(E))$ and $(G^{-1})' = G$; more precisely, $-G^{-1}(\varphi') = G(I(\varphi')) = G(\varphi),\ \varphi \in \mathcal{D}$. We know that supp$G \subset [0,\infty)$ implies supp$G^{-1} \subset [0,\infty)$.

Now we repeat the definition of a distribution cosine function, (DCF) in short, and its generator ([10]). An element $G \in \mathcal{D}'_0(L(E))$ is called a pre-(DCF) if it satisfies

$$(DCF_1): \qquad G^{-1}(\varphi *_0 \psi) = G^{-1}(\varphi)G(\psi) + G(\varphi)G^{-1}(\psi),\ \varphi,\ \psi \in \mathcal{D},$$

and it is called a distribution cosine function, in short (DCF), if it additionally satisfies

$(DCF_2)$ : $\quad x = y = 0$ if and only if $G(\varphi)x + G^{-1}(\varphi)y = 0$ for all $\varphi \in \mathcal{D}_0$.

If $G$ is a (DCF), then its generator $(A, D(A))$ is defined by

(1) $$\{(x, y) \in E^2 : \ G^{-1}(\varphi'')x = G^{-1}(\varphi)y \ \text{ for all } \varphi \in \mathcal{D}_0\},$$

where $x \in D(A)$ and $Ax := y$. Because of $(DCF_2)$, $A$ is a function and it is easy to see that $A$ is a closed linear operator in $E$. Moreover, if $\psi \in \mathcal{D}$ and $x \in E$, then $G(\psi)A \subset AG(\psi)$, $G(\psi)x \in D(A)$, $G^{-1}(\psi)x \in D(A)$ and the next equalities are valid: $AG(\psi)x = G(\psi'')x + \psi'(0)x$, $AG^{-1}(\psi)x = -G(\psi')x - \psi(0)x$, see [10, Proposition 2.7]. The exponential region $E(\alpha, \beta) := \{\eta + i\xi : \eta \geq \beta, \ |\xi| \leq e^{\alpha\eta}\}$ is introduced in [1]. If $n \in \mathbb{N}$, then we define $E^n(\alpha, \beta)$ by $E^n(\alpha, \beta) := \{z^n : z \in E(\alpha, \beta)\}$. Recall [10], a closed linear operator $A$ generates a (DCF) iff there are constants $\alpha, \ \beta, \ M > 0$ and $n \in \mathbb{N}_0$ so that

$$E^2(\alpha, \beta) \subset \rho(A) \text{ and } \|R(\lambda : A)\| \leq M(1 + |\lambda|)^n \ , \ \lambda \in E^2(\alpha, \beta).$$

If G is a (DCF), then we know that $\varphi(t) = \psi(t), \ t \geq 0$, for some $\varphi, \ \psi \in \mathcal{D}$, implies $G(\varphi) = G(\psi)$.

Operator cosine functions in any Banach space define distribution cosine functions. Differential operators in Euclidean spaces generate (global) $\alpha$-times integrated cosine functions which define distribution cosine functions, see examples in [5]. More elaborate examples appear in [10, Section 6] and in the forth section of this paper.

## 2. Convolution Products and Distribution Cosine Functions

We starts proving an analogue of a formula $\cos(t + s) = \cos(t)\cos(s) - \sin(t)\sin(s)$ for distribution cosine functions.

**Proposition 2.1.** *Let $G$ be a distribution cosine function generated by $(A, D(A))$. Then*

$$G(\varphi *_0 \psi)x = G(\varphi)G(\psi)x + AG^{-1}(\varphi)G^{-1}(\psi)x, \qquad \varphi, \ \psi \in \mathcal{D}, \ x \in E.$$

*Proof.* Notice, if $\varphi, \ \psi \in \mathcal{D}$, then $(\varphi *_0 \psi)'(t) = \varphi' *_0 \psi(t) + \varphi(0)\psi(t), \ t \in \mathbb{R}$. Since $A$ generates the distribution cosine function $G$ and $G(\varphi) = -G^{-1}(\varphi'), \ \varphi \in \mathcal{D}$, we have

$$\begin{aligned} G(\varphi *_0 \psi)x &= -\varphi(0)G^{-1}(\psi)x - G^{-1}(\varphi' *_0 \psi)x \\ &= G(\varphi)G(\psi)x + (-\varphi(0) - G(\varphi'))G^{-1}(\psi)x \\ &= G(\varphi)G(\psi)x + AG^{-1}(\varphi)G^{-1}(\psi)x, \end{aligned}$$

for any $x \in E$. ■

In the next theorem we characterize pre-distribution cosine functions by convolution products.

**Theorem 2.2.** *Let* $G \in \mathcal{D}_0'(L(E))$ *satisfy* $G(\varphi)G(\psi) = G(\psi)G(\varphi),\ \varphi,\ \psi \in \mathcal{D}$. *Then the following are equivalent:*

(i) $G$ *is a pre-*(DCF) *and* $G^{-1}(\Lambda(f \circ g - g \circ f)) = G(\Lambda(f))G^{-1}(\Lambda(g)) - G^{-1}(\Lambda(f))G(\Lambda(g))$, *for all* $f,\ g \in \mathcal{D}_+$.

(ii) $G^{-1}(\Lambda(f *_{sc} g)) = G^{-1}(\Lambda(f))G(\Lambda(g)),\ for\ all\ f,\ g \in \mathcal{D}_+$.

*Proof.* (i) $\Rightarrow$ (ii). Note, $f *_0 g(t) = (g *_{sc} f + f *_{sc} g)(t)$, $(f \circ g - g \circ f)(t) = (g *_{sc} f - f *_{sc} g)(t)$ and $\Lambda(f *_0 g)(t) = \Lambda(f) *_0 \Lambda(g)(t)$, for $t \geq 0$ and $f,\ g \in \mathcal{D}_+$. Moreover, $G^{-1}(\varphi) = 0$ if $\varphi \in \mathcal{D}_{(-\infty,0]}$ and we obtain

$$G^{-1}(\Lambda(g *_{sc} f + f *_{sc} g)) = G^{-1}(\Lambda(f))G(\Lambda(g)) + G(\Lambda(f))G^{-1}(\Lambda(g)),$$

$$G^{-1}(\Lambda(g *_{sc} f - f *_{sc} g)) = G(\Lambda(f))G^{-1}(\Lambda(g)) - G^{-1}(\Lambda(f))G(\Lambda(g)).$$

It, in turn, implies $G^{-1}(\Lambda(f *_{sc} g)) = G^{-1}(\Lambda(f))G(\Lambda(g))$ for all $f,\ g \in \mathcal{D}_+$. (In this direction, we do not use $G(\varphi)G(\psi) = G(\psi)G(\varphi),\ \varphi,\ \psi \in \mathcal{D}$).

(ii) $\Rightarrow$ (i). Fix $\varphi,\ \psi \in \mathcal{D}$. Since $G(\varphi)G(\psi) = G(\psi)G(\varphi)$, we have $G^{-1}(\varphi)G(\psi) = G(\psi)G^{-1}(\varphi)$. As $\mathcal{K}(\varphi) *_0 \mathcal{K}(\psi)(t) = (\mathcal{K}(\psi) *_{sc} \mathcal{K}(\varphi) + \mathcal{K}(\varphi) *_{sc} \mathcal{K}(\psi))(t)$ for $t \geq 0$, then

$$G^{-1}(\varphi *_0 \psi) = G^{-1}(\Lambda(\mathcal{K}(\varphi) *_0 \mathcal{K}(\psi))) = G^{-1}(\Lambda(\mathcal{K}(\psi) *_{sc} \mathcal{K}(\varphi) + \mathcal{K}(\varphi) *_{sc} \mathcal{K}(\psi)))$$

$$= G^{-1}(\Lambda\mathcal{K}(\varphi))G(\Lambda\mathcal{K}(\psi)) + G^{-1}(\Lambda\mathcal{K}(\psi))G(\Lambda\mathcal{K}(\varphi)) = G^{-1}(\varphi)G(\psi) + G^{-1}(\psi)G(\varphi)$$

$$= G^{-1}(\varphi)G(\psi) + G(\varphi)G^{-1}(\psi).$$

Hence, $G$ is a pre-(DCF). Since $G(\varphi)G(\psi) = G(\psi)G(\varphi),\ \varphi,\ \psi \in \mathcal{D}$, the second equality follows from the assumption (ii):

$$G^{-1}(\Lambda(f \circ g - g \circ f)) = G^{-1}(\Lambda(g *_{sc} f - f *_{sc} g))$$

$$= G(\Lambda(f))G^{-1}(\Lambda(g)) - G^{-1}(\Lambda(f))G(\Lambda(g)),$$

for all $f,\ g \in \mathcal{D}_+$. ■

## 3. Distribution Cosine Functions and Almost-distribution Cosine Functions

We need the definition of an almost-distribution cosine function and its generator, see [13]. An element $G \in L(\mathcal{D}_+, L(E))$ is called an *almost-distribution cosine function* if it satisfies

(i) $G(f *_c g) = G(f)G(g),\ f,\ g \in \mathcal{D}_+$, and

(ii) $\bigcap_{f \in \mathcal{D}_+} \mathrm{Ker}(G(f)) = \{0\}$.

The generator $A$ of $G$ is defined by $A := \{(x, y) \in E^2 : G(f)y = G(f'')x + f'(0)x, \text{ for all } f \in \mathcal{D}_+\}$. It is known that $A$ is a closed linear operator. Further on, $G(f)A \subset AG(f),\ G(f)x \in D(A)$ and $AG(f)x = G(f'')x + f'(0)x,\ f \in \mathcal{D}_+$. Recall, a global $n$-times integrated cosine function $(C_n(t))_{t\geq 0}$ defines an almost-distribution cosine functions $G$ (cf. [13]) by

$$G(f)x = (-1)^n \int_0^\infty f^{(n)}(t)C_n(t)x dt, \qquad x \in E, \quad f \in \mathcal{D}_+.$$

**Theorem 3.1.** *Let $G$ be a* (DCF) *generated by $A$. Then $G\Lambda$ is an almost-distribution cosine function generated by $A$.*

*Proof.* First of all, we have $G\Lambda \in L(\mathcal{D}_+, L(E))$. Since $G$ is a (DCF) generated by $A$, it follows $\bigcap_{\varphi \in \mathcal{D}_0} \mathrm{Ker}(G(\varphi)) = \{0\}$ ([10]). Hence, the condition (ii) in the definition of an almost-distribution cosine function is fulfilled. In order to prove (i), let us fix $f,\ g \in \mathcal{D}_+$. Suppose

(2) $$\mathrm{supp} f \cup \mathrm{supp} g \cup (\mathrm{supp} f + \mathrm{supp} g) \cup \mathrm{supp}(f \circ g) \cup \mathrm{supp}(g \circ f) \subset [0, a],$$

for some $a \in (0, \infty)$. This implies $\mathrm{supp}(f *_c g) \subset [0, a]$ and $\mathrm{supp}(\Lambda(f *_c g)) \subset (-\infty, a]$. Due to [10, Theorem 3.6], there exists an $n \in \mathbb{N}$ such that $A$ is the generator of an $n$-times integrated cosine function $(C_n(t))_{t\in[0,2a)}$. Then the proof of [10, Theorem 3.2] and [10, Corollary 3.11] imply

$$G(\varphi)x = (-1)^n \int_0^\infty \varphi^{(n)}(t)C_n(t)x dt,\ x \in E,\ \varphi \in \mathcal{D}_{(-\infty,2a)}.$$

Therefore,

$$G\Lambda(f *_c g) = (-1)^n \int_0^\infty (\Lambda(f *_c g))^{(n)}(t)C_n(t)x dt = (-1)^n \int_0^\infty (f *_c g)^{(n)}(t)C_n(t)x dt.$$

Clearly, $G\Lambda(f) = (-1)^n \int_0^\infty f^{(n)}(t)C_n(t)x dt$. Hence, we have to prove

(3) $$(-1)^n \int_0^\infty (f *_c g)^{(n)}(t)C_n(t)x dt = (-1)^n \int_0^\infty f^{(n)}(t)C_n(t) \int_0^\infty g^{(n)}(s)C_n(s)x ds dt.$$

This can be obtained as in the proof of [13, Theorem 4] with $\alpha = n \in \mathbb{N}$. We want only to notice that (2) implies that Fubini theorem can be applied in the proofs of [14, Proposition 1.1] and [13, Theorem 4]. Let $B$ be the generator of $G\Lambda$. We will prove $A = B$. Suppose $(x, y) \in A$. Then $G^{-1}(\varphi'')x = G^{-1}(\varphi)y$, for all $\varphi \in \mathcal{D}_0$. We will show

$$G\Lambda(f)y = G\Lambda(f'')x + f'(0)x, \text{ for all } f \in \mathcal{D}_+, \tag{4}$$

which implies $(x, y) \in B$ and $A \subset B$. Fix an $f \in \mathcal{D}_+$. Taking into account [10, Proposition 2.7], we have

$$G\Lambda(f)y = G\Lambda(f)Ax = AG(\Lambda(f))x = G((\Lambda(f))'')x + (\Lambda(f))'(0)x.$$

Since $(\Lambda(f))''(t) = \Lambda(f'')(t),\ t \geq 0$, one can continue as follows

$$= G(\Lambda(f''))x + f'(0)x,$$

and (4) holds. Suppose now $(x, y) \in B$. Then we know

$$G\Lambda(f)y = G\Lambda(f'')x + f'(0)x,\ \forall f \in \mathcal{D}_+. \tag{5}$$

One must prove that $G^{-1}(\varphi'')x = G^{-1}(\varphi)y,\ \varphi \in \mathcal{D}_0$. Suppose supp$\varphi \subset [0, b]$, for some $b > 0$. An analysis made in Introduction of [10] gives that supp$I(\varphi) \subset [-2, b]$. Note, $\frac{d^2}{dt^2}I(\varphi)(t) = \varphi'(t) - \alpha'(t)\int\limits_{-\infty}^{\infty}\varphi(u)du,\ t \in \mathbb{R}$, and consequently, $\frac{d^2}{dt^2}I(\varphi)(t) = \varphi'(t),\ t \geq 0$. Then $I(\varphi)(t) = \Lambda(\mathcal{K}(I(\varphi)))(t)$, $t \geq 0$, $(\mathcal{K}(I(\varphi)))'(0) = \varphi(0) - \alpha(0)\int\limits_{-\infty}^{\infty}\varphi(u)du = \varphi(0) = 0$ and $\Lambda((\mathcal{K}(I(\varphi)))'')(t) = (I(\varphi))''(t) = \varphi'(t),\ t \geq 0$. Now one obtains from (5):

$$\begin{aligned} G^{-1}(\varphi)y &= -G(I(\varphi))y = -G(\Lambda(\mathcal{K}(I(\varphi))))y \\ &= -(G\Lambda((\mathcal{K}(I(\varphi)))'')x + (\mathcal{K}(I(\varphi)))'(0)x) \\ &= -G\Lambda((\mathcal{K}(I(\varphi)))'')x = -G(\varphi')x = G^{-1}(\varphi'')x, \end{aligned}$$

which gives $(x, y) \in A$ and ends the proof. ■

**Corollary 3.2.** *Let $G$ be a* (DCF) *generated by $A$. Then*

$$G(\Lambda(f *_s g)) = AG^{-1}(\Lambda(f))G^{-1}(\Lambda(g)), \qquad f,\ g \in \mathcal{D}_+.$$

*Proof.* Take $f,\ g \in \mathcal{D}_+$. Since $f *_0 g = f *_c g + f *_s g$, we apply Proposition

2.1 and Theorem 3.1 to obtain the equality. ■

Relations between distribution cosine functions and equations of convolution type are analyzed in [10]. The use of [10, Theorem 3.10] enables one to briefly prove the following and to show directly some other results (see for example [10, Proposition 4.10]):

**Theorem 3.3.** *Let $G_1$ be an almost-distribution cosine function generated by $A$. Then $A$ is the generator of a* (DCF) *$G$ given by $G(\varphi) = G_1(\mathcal{K}(\varphi)),\ \varphi \in \mathcal{D}$.*

*Proof.* Note, if supp$\varphi \subset (-\infty, 0)$, then $\mathcal{K}(\varphi) = 0$ in $\mathcal{D}_+$, which clearly implies supp$G \subset [0, \infty)$ and $G \in \mathcal{D}_0'(L(E))$. Recall, $G(f)A \subset AG(f),\ G(f)x \in D(A)$, and $AG(f)x = G(f'')x + f'(0)x,\ f \in \mathcal{D}_+$; see [13, p. 178]. We want to prove that

$$(6)\qquad \begin{aligned} &AG(\varphi)x = G(\varphi'')x + \varphi'(0)x,\ x \in E,\ \varphi \in \mathcal{D},\ \text{and}\\ &G(\varphi)Ax = G(\varphi'')x + \varphi'(0)x,\ x \in D(A),\ \varphi \in \mathcal{D}. \end{aligned}$$

Let $x \in E$ and $\varphi \in \mathcal{D}$. Then $AG(\varphi)x = AG_1(\mathcal{K}(\varphi))x = G_1((\mathcal{K}(\varphi))'')x + \varphi'(0)x = G_1(\mathcal{K}(\varphi''))x + \varphi'(0)x = G(\varphi'')x + \varphi'(0)x$. Since $G_1 A \subset AG_1$, the second equality in (6) can be proved similarly. It is evident that (6) implies $G \in \mathcal{D}_0'(L(E, [D(A)]))$. Moreover,

$$G * P = \delta' \otimes Id_{[D(A)]} \text{ and } P * G = \delta' \otimes Id_E,$$

where we use the terminology given in [10, Section 3]: $P = \delta'' \otimes I - \delta \otimes A \in \mathcal{D}_0'(L([D(A)], E))$, $Id_{[D(A)]}$ denotes the inclusion $D(A) \to E$ and $(\delta^{(k)} \otimes Id_{[D(A)]})(\varphi)x = (-1)^k \varphi^{(k)}(0)x$, $(\delta^{(k)} \otimes I)(\varphi)x = (-1)^k \varphi^{(k)}(0)x$, $(\delta \otimes A)(\varphi)x = \varphi(0)Ax,\ \varphi \in \mathcal{D},\ x \in D(A),\ k \in \mathbb{N}_0$ and $(\delta' \otimes Id_E)(\varphi)x = -\varphi'(0)x,\ \varphi \in \mathcal{D},\ x \in E$. An application of [10, Theorem 3.10] gives that $G$ is a (DCF) generated by $A$. ■

**Corollary 3.4.** *Every almost-distribution cosine function is uniquely determined by its generator.*

*Proof.* Suppose $G_1$ and $G_2$ are almost-distribution cosine functions generated by a closed linear operator $A$. Put $\mathcal{G}_i(\varphi) := G_i(\mathcal{K}(\varphi)),\ \varphi \in \mathcal{D},\ i = 1, 2$. Due to Theorem 3.3, $\mathcal{G}_1$ and $\mathcal{G}_2$ are distribution cosine functions generated by $A$ and one can use [10, Corollary 3.11] in order to obtain that $\mathcal{G}_1 = \mathcal{G}_2$, i.e., $G_1(\mathcal{K}(\varphi)) = G_2(\mathcal{K}(\varphi))$, $\varphi \in \mathcal{D}$. Since $\mathcal{K} : \mathcal{D} \to \mathcal{D}_+$ is a surjective mapping, we have $G_1 = G_2$. This ends the proof. ■

We note that the definition of a (local) $\alpha$-times integrated cosine function $(C_\alpha(t))_{t\in[0,\tau)}$, $0<\tau\leq\infty$, is taken in the sense of [10, Definition 1.1]. The next result follows from [10, Theorem 3.6, Proposition 3.12], Theorem 3.1 and Theorem 3.3:

**Theorem 3.5.** *Let $A$ be a closed linear operator. Then the next assertions are equivalent:*

(i) *$A$ is the generator of a* (DCF).

(ii) *$A$ is the generator of an almost-distribution cosine function.*

(iii) *There exist $\tau>0$ and $n\in\mathbb{N}$ such that $A$ is the generator of an $n$-times integrated cosine function on $[0,\tau)$.*

(iv) *For every $\tau>0$ there is an $n\in\mathbb{N}$ such that $A$ is the generator of an $n$-times integrated cosine function on $[0,\tau)$.*

(v) *$\rho(A)\neq\emptyset$ and there exist $\lambda\in\rho(A)$, $n\in\mathbb{N}$ and $\tau\in(0,\infty]$ such that $A$ is the generator of an $R(\lambda:A)^n$-cosine function on $[0,\tau)$.*

(vi) *There are constants $\alpha,\ \beta,\ M>0$ and $n\in\mathbb{N}_0$ so that*

$$E^2(\alpha,\beta)\subset\rho(A)\ and\ \|R(\lambda:A)\|\leq M(1+|\lambda|)^n\ ,\ \lambda\in E^2(\alpha,\beta).$$

Some other equivalent conditions can be found in [10].

## 4. An Application

The main aim of this section is to present some new relations between distribution cosine functions and ultradistribution semigroups of [2, 3] and [9]. We recall the basic notions and notations from the theory of ultradistribution spaces and ultradistribution semigroups. In this section, we always assume that $(M_p)_p$ is a sequence of positive numbers, $M_0=1$, such that the following conditions are fulfilled:

(M.1) $M_p^2\leq M_{p+1}M_{p-1},\ p\in\mathbb{N}$,

(M.2) $M_p\leq AH^p\min_{0\leq i\leq p}M_iM_{p-i},\ p\in\mathbb{N}$, for some $A,\ H>0$,

(M.3') $\sum_{p=1}^{\infty}\frac{M_{p-1}}{M_p}<\infty$.

Let $s>1$. The Gevrey sequences $(p!^s),\ (p^{ps})$ or $(\Gamma(1+ps))$ satisfy the above conditions. The associated function of $(M_p)$ is defined by $M(\rho):=\sup_{p\in\mathbb{N}_0}\ln\frac{\rho^p}{M_p}$, $\rho>0;\ M(0):=0$. We know that there exists a sufficiently small $\varepsilon>0$ so that $M(\rho)=0$ if $\rho\in[0,\varepsilon]$. Furthermore, $M:[0,\infty)\to[0,\infty)$ is increasing, $\lim\limits_{\rho\to\infty}\frac{M(\rho)}{\rho^k}=0$, $\lim\limits_{\rho\to\infty}\frac{(\ln\rho)^k}{M(\rho)}=0$, $k\in\mathbb{N}$ and, for every Gevrey sequence, there exists a constant $C_s>0$ such that the associated function satisfies $\lim\limits_{\rho\to\infty}\frac{M(\rho)}{C_s\rho^{\frac{1}{s}}}=1$.

We refer to [7] for the fundamental facts concerning projective and inductive limits of locally convex spaces. Let us introduce now the basic ultradistribution type spaces used in this paper. For more details, see [7-9]. Let $K$ be a compact subset of $\mathbb{R}$ and $h > 0$. The space $\mathcal{D}_K^{M_p,h}$ is consisted of all functions $\phi \in C^\infty(\mathbb{R})$ with supp$\phi \subset K$ and $||\phi||_{M_p,h} := \sup\{\frac{h^p|\phi^{(p)}(t)|}{M_p} : t \in K,\ p \in \mathbb{N}_0\} < \infty$. Recall, $(\mathcal{D}_K^{M_p,h}, ||\phi||_{M_p,h})$ is a Banach space and the spaces $\mathcal{D}_K^{(M_p)}$ and $\mathcal{D}_K^{\{M_p\}}$ are defined as follows: $\mathcal{D}_K^{(M_p)}$ :=proj $\lim_{h\to\infty}\mathcal{D}_K^{M_p,h}$ and $\mathcal{D}_K^{\{M_p\}}$ :=ind $\lim_{h\to 0}\mathcal{D}_K^{M_p,h}$. Let $(K_n)$ be a sequence of compact subsets of $\mathbb{R}$ with smooth boundary such that $\bigcup\limits_{n\in\mathbb{N}} K_n = \mathbb{R}$ and that $K_n \subset (K_{n+1})^\circ$. Then we define the space of Beurling ultradifferentiable functions $\mathcal{D}^{(M_p)} = \mathcal{D}^{(M_p)}(\mathbb{R}) :=$ ind $\lim_{n\to\infty}\mathcal{D}_{K_n}^{(M_p)}$ and the space of Roumieu ultradifferentiable functions $\mathcal{D}^{\{M_p\}} = \mathcal{D}^{\{M_p\}}(\mathbb{R})$ :=ind $\lim_{n\to\infty}\mathcal{D}_{K_n}^{\{M_p\}}$. Note only that these definitions do not depend on the choice of a sequence $(K_n)$. With the notation $*$ for both cases of brackets, we denote by $\mathcal{D}'^*(E) := L(\mathcal{D}^*(\mathbb{R}), E)$ the space of continuous linear functions from $\mathcal{D}^*(\mathbb{R})$ into $E$; $\mathcal{D}_0'^*(E)$ denotes the space of elements in $\mathcal{D}'^*(E)$ which are supported by $[0,\infty)$. We refer to [8] for the definition of convolution of vector valued ultradifferentiable functions and vector valued ultradistributions. The next definition of an ultradistribution semigroup of $*$-class and its generator was employed by H. Komatsu in [9].

**Definition 4.1.** Let $A$ be a closed linear operator. An element $G \in \mathcal{D}_0'^*(L(E))$ is an ultradistribution semigroup of $*$-class generated by $A$ if $G \in \mathcal{D}_0'^*(L(E, [D(A)]))$ satisfies

$$G * P = \delta' \otimes Id_{[D(A)]} \text{ and } P * G = \delta' \otimes Id_E.$$

If $k > 0$ and $C > 0$, put $\Omega_{k,C} := \{\lambda \in \mathbb{C} : \text{Re}(\lambda) \geq M(k|\lambda|) + C\}$.

**Lemma 4.2.** ([9]) *A closed linear operator $A$ is the generator of an ultradistribution class of the Beurling class (Roumieu class) if and only if there exist $k > 0$ and $C > 0$ (for every $k > 0$ there exists a suitable $C_k > 0$) such that $\Omega_{k,C} \subset \rho(A)$ $(\Omega_{k,C_k} \subset \rho(A))$ and that $||R(\lambda : A)|| \leq Ce^{M(k|\lambda|)},\ \lambda \in \Omega_{k,C}$ $(||R(\lambda : A)|| \leq C_k e^{M(k|\lambda|)},\ \lambda \in \Omega_{k,C_k})$.*

Let $(N_p)$ and $(R_p)$ be sequences of positive numbers which satisfy (M.1). Following Z. Chou (cf. for example [7, Definition 3.9, p. 53]), we write $N_p \prec R_p$ if and only if, for every $\delta \in (0,\infty)$, $\sup\limits_{p\in\mathbb{N}_0}\frac{N_p\delta^p}{R_p} < \infty$.

Now we are able to state the main result of this section. It is a generalization of [6, Theorem 3.1] where the corresponding result is proved for the class of dense exponential distribution cosine functions (see [10]). Furthermore, we want to give more precise information concerning a corresponding sequence $(M_p)$ and to clarify some differences between the Beurling case and the Roumieu case (see Example 4.4 given below).

**Theorem 4.3.** *Suppose that a closed linear operator $A$ generates a distribution cosine function. If $(M_p)$ additionally satisfies $M_p \prec p!^s$, for some $s \in (1,2)$, then $\pm iA$ generate $(M_p)$-ultradistribution semigroups of $*$-class.*

*Proof.* We will prove the assertion only for $iA$ since the same arguments work for $-iA$. Due to Theorem 3.5, there exist $\alpha,\ \beta,\ M > 0$ and $n \in \mathbb{N}$ such that $E^2(\alpha,\beta) \subset \rho(A)$ and that $||R(\lambda : A)|| \leq M(1+|\lambda|)^n,\ \lambda \in E^2(\alpha,\beta)$. Put $\Gamma' := \partial E^2(\alpha,\beta)$ and $\Gamma := i\Gamma'$. Then it can be easily seen that $\Gamma' = \Gamma_1' \cup \Gamma_2' \cup \Gamma_3'$, where:

(1) $\Gamma_1'$ is a part of the parabola $\{x+iy : x = \beta^2 - \frac{y^2}{4\beta^2}\}$; further on, $\Gamma_1'$ is contained in some compact subset of $\mathbb{C}$,

(2) $\Gamma_2' = \{t^2 - e^{2\alpha t} + 2te^{\alpha t}i : t \geq \beta\}$ and $\Gamma_3' = \{t^2 - e^{2\alpha t} - 2te^{\alpha t}i : t \geq \beta\}$.

This implies that, for every $c \in (\frac{1}{2}, 1)$, we have

$$\lim_{\lambda\in\Gamma,\ |\lambda|\to\infty} \frac{|\mathrm{Im}(\lambda)|^c}{|\mathrm{Re}(\lambda)|} = \infty. \tag{7}$$

It is clear that the curve $\Gamma$ divides the complex plane into two disjunct open sets. Denote by $\Omega$ one of such two sets which contains a ray $(\omega,\infty)$, for some $\omega > 0$. Fix a $k > 0$. Since $\Omega \subset \rho(iA)$ and $||R(\cdot : iA)||$ is polynomially bounded on $\Omega$, the proof will be completed if one shows that there exists a suitable $C_k > 0$ with

$$\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq M(k|\lambda|) + C_k\} \subset \Omega. \tag{8}$$

Note, (7) implies that, for every $c \in (\frac{1}{2}, 1)$, there exists a sufficiently large $K_c > 0$ satisfying

$$\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq |\mathrm{Im}(\lambda)|^c + K_c\} \subset \Omega. \tag{9}$$

Choose an $s \in (1,2)$ with $M_p \prec p!^s$. Then an application of [7, Lemma 3.10] gives that there exists a constant $C_{k,s} > 0$ with $\rho^{\frac{1}{s}} \leq M(k\rho) + \ln C_{k,s},\ \rho \geq 0$. Moreover, there exists a suitable $K_{\frac{1}{s}} > 0$ such that (9) holds with $c = \frac{1}{s}$. Now it is straightforward to see that (8) is valid with $C_k = \ln C_{k,s} + K_{\frac{1}{s}}$. Indeed, if $\lambda \in \mathbb{C}$ and $\mathrm{Re}(\lambda) \geq M(k|\lambda|) + \ln C_{k,s} + K_{\frac{1}{s}}$, then $\mathrm{Re}(\lambda) \geq |\lambda|^{\frac{1}{s}} + K_{\frac{1}{s}}$, and due to (9), $\lambda \in \Omega$. ∎

Since

$$\lim_{\xi\to+\infty} \frac{\Gamma(\xi)}{\xi^{\xi-\frac{1}{2}}e^{-\xi}} = \sqrt{2\pi}, \tag{10}$$

a Gevrey type sequence $(M_p)$ fulfills the assumption of Theorem 4.3 if and only if $s \in (1, 2)$. The next illustrative example shows that Theorem 4.3 does not hold in the case of a general sequence $(M_p)$.

**Example 4.4.** Let $E := L^p(\mathbb{R}),\ 1 \leq p < \infty$ and $m(x) = (1 - \frac{x^2}{4}) + ix,\ x \in \mathbb{R}$. Define a closed linear operator $A$ on $E$ by: $Af(x) = m(x)f(x),\ x \in \mathbb{R},\ D(A) := \{f \in E : mf \in E\}$. As a matter of routine, one can check that $A$ generates a dense exponential (DCF) (cf. [10]) and that $\sigma(iA) = \{x + (1 - \frac{x^2}{4})i : x \in \mathbb{R}\}$. Suppose now $M_p = p!^2$. We will show that $iA$ generates an ultradistribution semigroup of the Beurling class and that $iA$ is not the generator of an ultradistribution semigroup of the Roumieu class. First of all, we know that there exist constants $\omega > 0$, $a > 0$ and $b > 0$ with $a\rho^{1/2} \leq M(\rho),\ \rho \geq \omega$ and $M(\rho) \leq b\rho^{1/2},\ \rho \geq 0$. The consideration is over if one shows that

$$\partial\Omega_{k,C} \cap\ \sigma(iA) = \emptyset,\ \text{for every } k \in (\tfrac{4}{a^2}, \infty) \text{ and a sufficiently large } C > 0, \text{ and that} \tag{11}$$

$$\partial\Omega_{k,C} \cap\ \sigma(iA) \neq \emptyset,\ \text{for every } k \in (0, \frac{4}{b^2}) \text{ and } C > 0. \tag{12}$$

Let $k \in (\frac{4}{a^2}, \infty)$. Choose a $C \geq \frac{\omega}{k}$. In order to obtain (11), note that, if $x + iy \in \partial\Omega_{k,C}$, then $x \geq C,\ k\sqrt{x^2 + y^2} \geq kx \geq kC \geq \omega$. Thus, $x = M(k\sqrt{x^2 + y^2}) + C \geq a\sqrt{k}\sqrt[4]{x^2 + y^2} + C$. This estimate ensures one to see that for a sufficiently large $C > 0$, the curve $\partial\Omega_{k,C}$ lies above the graph of the function $f(x) = -\sqrt{\frac{(x-C)^4}{a^4k^2} - x^2}$; moreover, $f(x) \sim -\frac{x^2}{a^2k},\ x \to +\infty$. Therefore, the choice of $k$ implies that there exists a suitable $\beta > 0$ such that a part of the parabola $y = -\frac{x^2}{a^2k},\ x \geq \beta$ has the empty intersection with $\sigma(iA)$. It immediately implies (11) while (12) follows similarly from the facts that, for every $k \in (0, \frac{4}{b^2})$ and $C > 0$, the interior of the parabola $y = -\frac{x^2}{b^2k}$ is strictly contained in that of $y = -\frac{x^2}{4}$ and that, for $x + iy \in \partial\Omega_{k,C}$, we have $x = M(k\sqrt{x^2 + y^2}) + C \leq b\sqrt{k}\sqrt[4]{x^2 + y^2} + C$. At the end of this analysis, we point out that the implication: $G$ is an ultradistribution semigroup of $*$-class $\Rightarrow \bigcap_{\varphi \in \mathcal{D}_0^*} \operatorname{Kern}(G(\varphi)) = \{0\}$, is not true in general (see [3] and [11]). In the case of densely defined operators, the concept of regular ultradistribution semigroups of Beurling class was introduced by I. Cioranescu in [3] for this purpose. An application of [3, Proposition 2.6] gives that the operator $iA$, considered above, generates a regular ultradistribution semigroup of $(p^{2p})$-class $G$ (cf. [3] for the notion), i.e., $\bigcap_{\varphi \in \mathcal{D}_0^*} \operatorname{Kern}(G(\varphi)) = \{0\}$ and $\bigcup_{\varphi \in \mathcal{D}_0^*} \operatorname{Im}(G(\varphi))$ is dense in $E$. Similarly, if $M_p = p!^s,\ s > 2$, then it can be proved that $iA$ does not generate an ultradistribution semigroup of the Beurling, resp., Roumieu class. Evidently, the same assertions are valid for $-iA$.

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