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LIMITING BEHAVIORS OF WEIGHTED SUMS FOR LINEARLY NEGATIVE QUADRANT DEPENDENT RANDOM VARIABLES

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Abstract. In this paper the strong convergence for weighted sums of linearly negative quadrant dependent(LNQD) arrays is discussed. The central limit theorem for weighted sums of LNQD variables and linear process based on LNQD variables is also considered. Finally the results on i.i.d. of Li et al. ([7]) in LNQD setting are obtained.

1. Introduction

Many useful linear statistics based on random samples are weighted sums of i.i.d. random variables. Examples include least-square estimators, nonparametric regression function estimators and jacknife estimates, among others.

In this respect, studies of strong convergence for these weighted sums have demonstrated significant progress in probability theory with applications in mathematical statistics. Up to now, various limit properties for i.i.d. random variables have been studied by many authors. The most commonly studied method is Cesàro summation. Set, for $\alpha > -1$,

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!},\ n = 1, 2, \dots \text{ and } A_0^\alpha = 1$$

and note that $A_n^\alpha \sim n^\alpha/[\Gamma(\alpha+1)]$ as $n \to \infty$, where $\sim$ denotes that the limit as $n \to \infty$ of the ratio between the members on either side equals 1. Let $\{X, X_n, n \geq$

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$1\}$ be a sequence of i.i.d. random variables. One says that X satisfies Cesàro law of large numbers of order $\alpha,\ 0<\alpha<1$, if and only if

$$\frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}X_k \text{ converges } a.s. \text{ as } n\to\infty.$$

It is well known that

$$\lim_{n\to\infty}\frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}X_k=\mu\ a.s.$$

if and only if $E\mid X\mid^{1/\alpha}<\infty$ and $EX=\mu$.

For $\alpha=1$ this result is, of course, the classical Kolmogorov strong law. For $1/2<\alpha<1$ the proof is due to Lorentz ([8]); for $0<\alpha<1/2$ it follows from Chow and Lai ([2]). The case $\alpha=1/2$ was treated by Déniel and Derriennic([3]). Li et al.([7]) proved the following result on Cesàro summation of i.i.d. random variables.

Theorem A. *Let* $\{X, X_n, n\ge 1\}$ *be a sequence of i.i.d. random variables.*
(i) *For* $0<\alpha<1/2,$ *if* $Ee^{t|X|}<\infty$ *for all* $t>0$, *then*

$$\frac{1}{A_n^\alpha}\sum_{k=0}^{n} A_{n-k}^{\alpha-1}(X_k-EX_k)=o(n^{-\alpha}\log n),\ a.s.$$

(ii) *For* $1/2<\alpha<1$*, if* $E(X-EX)^2=1$*, then*

$$\alpha(2\alpha-1)^{1/2}\Gamma^2(\alpha)n^{1/2}(1/A_n^\alpha)\sum_{k=0}^{n} A_{n-k}^{\alpha-1}(X_k-EX_k)\xrightarrow{D}N(0,1).$$

However, many variables are dependent in actual problems. We first recall Lehmann's definition([6]) of positive and negative quadrant dependent(PQD and NQD) random variables. X_1 and X_2 are said to be PQD if $P(X_1>x_1, X_2>x_2)\ge P(X_1>x_1)P(X_2>x_2)$ for all $x_1,x_2\in R$ and they are said to be NQD if $P(X_1>x_1,X_2>x_2)\le P(X_1>x_1)P(X_2>x_2)$.

The random variables $X_j^{'}$s are said be linearly positive quadrant dependent(LPQD) if for any disjoint sets A,B and positive $r_j^{'}$s, $\sum_{k\in A}r_kX_k$ and $\sum_{j\in B}r_jX_j$ are PQD; they are said to be linearly negative quadrant dependent(LNQD) if for any disjoint subsets A,B and positive $r_j^{'}$s, $\sum_{k\in A}r_kX_k$ and $\sum_{j\in B}r_jX_j$ are NQD. This definition was introduced by Newman([5], [11]).

In order to extend Theorem A to LNQD setting, in this paper, we will discuss the strong convergence and central limit theorem for weighted sums of LNQD random variables.

2. Statements of the Main Results

Theorem 2.1. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise LNQD random variables with $EX_{ni} = 0$ and let there exist a positive constant C and a random variable X such that $P(|X_{ni}| > x) = CP(|X| > x)$ for all $x > 0$ and for all $1 \leq i \leq k_n$ and $Ee^{t|X|} < \infty$ for all $t > 0$, where k_n is a sequence of positive integers. If $\{a_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ is an array of real numbers satisfying*

$$\text{(i)} \max_{1 \leq i \leq k_n} |a_{ni}| = \mathrm{O}((\log n)^{-1}) \qquad \text{(ii)} \sum_{i=1}^{k_n} a_{ni}^2 = \mathrm{o}((\log n)^{-1}),$$

then

$$\sum_{n=1}^{\infty} n^{r-2} P(|\sum_{i=1}^{k_n} a_{ni} X_{ni}| > \epsilon) < \infty \quad \text{for all } \epsilon > 0 \text{ and all } r \geq 2.$$

Corollary 2.1. *Let $\{X_i, i \geq 0\}$ be a sequence of LNQD random variables. If there exist a positive constant C and a random variable X such that $P(|X_i| > x) = CP(|X| > x)$ for all $i \geq 0$ and $x > 0$ and $Ee^{t|X|} < \infty$ for all $t > 0$, then, for $0 < \alpha < 1/2$*

$$\frac{1}{A_n^{\alpha}} \sum_{k=0}^{n} A_{n-k}^{\alpha-1}(X_k - EX_k) = \mathrm{o}(n^{-\alpha} \log n) \ a.s.$$

Theorem 2.2. *Let $\{X_i, -\infty < i < \infty\}$ be a sequence of mean zero LNQD random variables satisfying*

$$\sum_{j:|k-j| \geq u} |\mathrm{cov}(X_k, X_j)| \to 0 \text{ uniformly as } u \to \infty \text{ for } k \geq 1. \tag{2.1}$$

Assume that $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of positive numbers satisfying

$$\sup_n \sum_{i=1}^{n} a_{ni}^2 < \infty \text{ and } \max_{1 \leq i \leq n} a_{ni} \to 0 \text{ as } n \to \infty \tag{2.2}$$

and that $Var(\sum_{i=1}^n a_{ni} X_i) \to 1$ as $n \to \infty$.

(a) If X_i^2 is uniformly integrable, then

$$\sum_{i=1}^{n} a_{ni} X_i \xrightarrow{D} N(0, 1),$$

(b) Put $\xi_t = \sum_{j=0}^{\infty} c_j X_{t-j}$. Here $\{c_j\}$ is a sequence of positive numbers with $C(1) = \sum_{j=0}^{\infty} c_j$ and $\sum_{j=1}^{\infty} jc_j < \infty$. If X_i^2 is uniformly integrable and $\max_{1\le i\le n} a_{ni} = \mathrm{O}(n^{-1/2})$, then

$$\sum_{i=1}^{n} a_{ni}\xi_i \xrightarrow{D} N(0, C^2(1)),$$

(c) Put $\eta_t = \sum_{j=-\infty}^{\infty} c_j X_{t-j}$. Here $\{c_j\}$ is a sequence of positive numbers with $D(1) = \sum_{j=-\infty}^{\infty} c_j$ and $\sum_{j=-\infty}^{\infty} j^2 c_j^2 < \infty$. If $\sup_i E|X_i|^{2+\delta} < \infty$ for any $\delta > 0$ and $\max_{1\le i\le n} a_{ni} = \mathrm{O}(n^{-1/2})$, then

$$\sum_{i=1}^{n} a_{ni}\eta_i \xrightarrow{D} N(0, D^2(1)).$$

Corollary 2.2 *Let $\{X_i, i \ge 1\}$ be a sequence of LNQD random variables. For $1/2 < \alpha < 1$, if X_i^2 is uniformly integrable, then*

$$R_n = \alpha(2\alpha-1)^{1/2}\Gamma^2(\alpha)n^{1/2}(1/A_n^{\alpha})\sum_{k=0}^{n} A_{n-k}^{\alpha-1}(X_k - EX_k) \xrightarrow{D} N(o, \sigma^2),$$

where $\sigma^2 = \lim_{n\to\infty}$ VarR_n.

Remark 2.2. In Corollary 2.2, if $\{X, X_n, n \ge 1\}$ is a sequence of i.i.d. random variables and $E(X-EX)^2 = 1$, then $\sigma^2 = 1$. Since independent random variables are a special case of LNQD random variables, Corollarys 2.1 and 2.2 extend Theorem A to the LNQD case.

3. Proofs of the Main Results

In this section, $a^+ = \max(0, a)$ and $a^- = \max(0, -a)$. Let C and c denote positive constants whose values are unimportant and may vary at different place. We start with Newman's inequality([11]).

Lemma 3.1. *Suppose $X_1, \cdots, X_n$ are LNQD. Then*

$$|E\exp(i\sum_{j=1}^{n} r_jX_j) - \prod_{j=1}^{n} E\exp(ir_jX_j)| \le \sum_{1\le i<j\le n} |r_ir_jCov(X_i, X_j)|.$$

Lemma 3.2. *Suppose $X_1, \cdots, X_n$ are LNQD. Then*

$$E(\exp\sum_{i=1}^{n} X_i) \le \prod_{i=1}^{n} E\exp(X_i).$$

Proof. Sine X_i and $\sum_{j=i+1}^n X_j$ are NQD, $\exp(X_i)$ and $\exp(\sum_{j=i+1}^n X_j)$ are also NQD for $i = 1, 2, \cdots, n-1$ by Lemma A1 in the Appendix. Thus, by induction we have

$$E(\exp\sum_{i=1}^n X_i = E[\exp(X_1)\cdot\exp(\sum_{i=2}^n X_i)]$$

$$\leq E(\exp X_1)\cdot E(\exp\sum_{i=2}^n X_i) \leq \prod_{i=1}^n [E\exp(X_i)].$$

Proof of Theorem 2.1. Since $a_{ni} = a_{ni}^+ - a_{ni}^-$, it suffices to show

$$\sum_{n=1}^\infty n^{r-2}P(|\sum_{i=1}^{k_n} a_{ni}^+ X_{ni}| > \epsilon) < \infty \text{ for any } \epsilon > 0,\ r \geq 2, \tag{3.1}$$

$$\sum_{n=1}^\infty n^{r-2}P(|\sum_{i=1}^{k_n} a_{ni}^- X_{ni}| > \epsilon) < \infty \text{ for any } \epsilon > 0,\ r \geq 2. \tag{3.2}$$

We prove only (3.1), since the proof of (3.2) is analogous. To prove (3.1), we need to prove

$$\sum_{n=1}^\infty n^{r-2}P(\sum_{i=1}^{k_n} a_{ni}^+ X_{ni} > \epsilon) < \infty \text{ for any } \epsilon > 0, \tag{3.3}$$

$$\sum_{n=1}^\infty n^{r-2}P(\sum_{i=1}^{k_n} a_{ni}^+ X_{ni} < -\epsilon) < \infty \text{ for any } \epsilon > 0. \tag{3.4}$$

We first prove (3.3). From the definition of LNQD variables, we know that $\{a_{ni}^+ X_{ni} 1 \leq i \leq k_n, n \geq 1\}$ is still an array of rowwise LNQD random variables. Since $e^x \leq 1 + x + \frac{1}{2}x^2 e^{|x|}$ for all $x \in R$, by Markov inequality and Lemma 3.2, we get for $t = M\log n/\epsilon$, where M is a large constant and will be specified later on,

$$\begin{aligned}
&\sum_{n=1}^\infty n^{r-2}P(\sum_{i=1}^{k_n} a_{ni}^+ X_{ni} > \epsilon)\\
&\leq \sum_{n=1}^\infty n^{r-2}e^{-\epsilon t}Ee^{t\sum_{i=1}^{k_n} a_{ni}^+ X_{ni}} \text{ by Markov inequality}\\
&\leq \sum_{n=1}^\infty n^{r-2-M}\prod_{i=1}^{k_n} Ee^{ta_{ni}^+ X_{ni}} \text{ by Lemma 3.2}\\
&\leq \sum_{n=1}^\infty n^{r-2-M}\prod_{i=1}^{k_n}[1 + \frac{1}{2}t^2(a_{ni}^+)^2 EX_{ni}^2 e^{ta_{ni}^+|X_{ni}|}]
\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{n=1}^{\infty} n^{r-2-M}\prod_{i=1}^{k_n}[1+c(\log n)^2(a_{ni}^+)^2 Ee^{(1+c)|X|}] \\
&\le C\sum_{n=1}^{\infty} n^{r-2-M}\exp\{c(\log n)^2\sum_{i=1}^{k_n}(a_{ni}^+)^2\} \\
&\le C\sum_{n=1}^{\infty} n^{(r+\epsilon)-(2+M)} < \infty,
\end{aligned}$$

provided $M > (r+\epsilon)-1$. Thus, (3.3) is proved.

By replacing X_{ni} by $-X_{ni}$ from the above statement and noticing $\{a_{ni}^+(-X_{ni}) : 1 \le i \le k_n, n \ge 1\}$ is still an array of rowwise LNQD random variables, we know that (3.4) holds.

Proof of Theorem 2.2. (a) Without loss of generality, we assume that $a_{ni} = 0$ for all $i > n$. Note that, for $1 \le u \le n-1$

$$\sum_{i,j=1;|i-j|\ge u}^{n} |a_{ni}a_{nj}\mathrm{Cov}(X_i, X_j)| \le \sup_k |\sum_{j;|k-j|\ge u} \mathrm{Cov}(X_k, X_j)|(\sum_{i=1}^{n} a_{ni}^2),$$

and hence, by (2.1) and (2.2), for a fixed small $\epsilon > 0$, we can find a positive integer $u = u_\epsilon$ such that, for every $n \ge u+1$

$$0 \le \sum_{i,j=1;|i-j|\ge u}^{n} |a_{ni}a_{nj}\mathrm{Cov}(X_iX_j)| \le \epsilon.$$

By Definition of LNQD, we also have, for every $1 \le a \le b \le n$,

$$Var(\sum_{i=a}^{b} a_{ni}X_i) \le \sup EX_i^2 \sum_{i=a}^{b} a_{ni}^2, \tag{3.5}$$

which is bounded by assumptions.

Denote by $[x]$ the integer part of x and define

$$K = [\frac{1}{\epsilon}],$$

$$Y_{nj} = \sum_{i=uj+1}^{u(j+1)} a_{ni}X_i,\ j = 0,1,2,\ldots,$$

$$A_j = \left\{ i : 2Kj \le i \le 2Kj+K,\ |\mathrm{Cov}(Y_{ni}, Y_{n,i+1})| \le \frac{2}{K}\sum_{m=2Kj}^{2Kj+K} \mathrm{Var}(Y_{nm}) \right\}.$$

Since $2|\mathrm{Cov}(Y_{ni}, Y_{n,i+1})| \le Var(Y_{ni}) + Var(Y_{n,i+1})$, we get that for every j the set A_j is not empty. Now we define the integers $m_1, m_2, \ldots, m_n$ recurrently by $m_0 = 0$;

$$m_{j+1} = \min\{m : m > m_j, m \in A_j\}$$

and put

$$Z_{nj} = \sum_{i=m_j+1}^{m_{j+1}} Y_{ni},\ j = 0, 1, 2, \ldots,$$

$$\triangle_j = \{u(m_j + 1) + 1, \ldots u(m_{j+1} + 1)\}.$$

We observe that

$$Z_{nj} = \sum_{k \in \triangle_j} a_{nk} X_k,\ j = 0, 1, \ldots\ .$$

It is easy to see that every set $\triangle_j$ contains no more than $3Ku$ elements. Thus, by (2.2), we know that the uniform integration of $\{X_i^2, i \geq 1\}$ implies the uniform integration of $\{Z_{ni}, 1 \leq i \leq n, n \geq 1\}$, and hence $\{Z_{ni}, 1 \leq i \leq n, n \geq 1\}$ satisfies the Lindeberg's condition. It remains to observe that by Lemma 3.1, for any real number t

$$\begin{aligned}
&|E\exp(it\sum_{j=1}^{n} Z_{nj}) - \prod_{j=1}^{n} E\exp(itZ_{nj})| \\
&\leq t^2 \sum_{1 \leq i < j \leq n} |\mathrm{Cov}(Z_{ni}, Z_{nj})| \\
&= t^2[\sum_{1 \leq i < j \leq n; |i-j|=1} |\mathrm{Cov}(Z_{ni}, Z_{nj})| + \sum_{1 \leq i < j \leq n; |i-j|>1} |\mathrm{Cov}(Z_{ni}, Z_{nj})|] \\
&\leq Ct^2[\sum_{1 \leq i < j \leq n; |i-j| \geq u} |a_{ni}a_{nj}||\mathrm{Cov}(X_i, X_j)| \\
&+ \sum_{j=1}^{n} |\mathrm{Cov}(Y_{nm_j}, Y_{n,m_{j+1}})|] \\
&\leq Ct^2[\epsilon + \frac{c}{K}\sum_{i=1}^{n} Var(Y_{ni})] \\
&\leq Ct^2[\epsilon + \frac{c}{K}\sum_{i=1}^{n} Var(\sum_{j=ui+1}^{u(i+1)} a_{nj}X_j)] \\
&\leq Ct^2[\epsilon + \frac{c}{K}\sup_j EX_j^2 \sum_{i=1}^{n}(\sum_{j=ui+1}^{u(i+1)} a_{nj}^2)] \text{ by (3.5)}
\end{aligned}$$

$$\le Ct^2\epsilon[1+\sup_n \sum_{i=1}^{n} a_{ni}^2]$$
$$\le Ct^2\epsilon \text{ for every positive } \epsilon \text{ by (2.2).}$$

Thus, the assertion (a) in Theorem 2.2 is valid in view of Lemma A2 in the Appendix.

(b) Note that

$$\xi_k = C(1)X_k + \widetilde{X}_{k-1} - \widetilde{X}_k,$$

where $\widetilde{X}_k = \sum_{j=0}^{\infty} \widetilde{c}_j X_{k-j}$ and $\widetilde{c}_j = \sum_{i=j+1}^{\infty} c_i$. Hence

$$\sum_{i=1}^{n} a_{ni}\xi_i = C(1)\sum_{k=1}^{n} a_{nk}X_k + \sum_{k=1}^{n} a_{nk}(\widetilde{X}_{k-1} - \widetilde{X}_k) := I_n + J_n.$$

By (a), we get $I_n \xrightarrow{D} N(0, C^2(1))$.

To prove $J_n \xrightarrow{P} 0$, we here state the Abelian Inequality (see p.32, Theorem 1 of Mitrinovic ([10])):

Let $A_1, A_2, \ldots, A_n;\ B_1, B_2, \ldots, B_n (B_1 \ge B_2 \ge \cdots \ge B_n \ge 0)$ be two sequences of real numbers, and let $S_k = \sum_{i=1}^{k} A_i,\ M_1 = \min_{1\le k\le n} S_k$ and $M_2 = \max_{1\le k\le n} S_k$. Then

$$B_1 M_1 \le \sum_{k=1}^{n} A_k B_k \le B_1 M_2.$$

Without loss of generality, assume that $a_{n1} \ge a_{n2} \ge \cdots \ge a_{nn}$. Let $B_s = a_{ns} - a_{nn}, 1 \le s \le n-1,\ B_n = 0$. Applying (3.5) we have

$$
\begin{aligned}
|J_n| &\le |\sum_{k=1}^{n}(a_{nk} - a_{nn})(\widetilde{X}_{k-1} - \widetilde{X}_k)| + |\sum_{k=1}^{n} a_{nn}(\widetilde{X}_{k-1} - \widetilde{X}_k)| \\
&\le 2 \max_{1\le k\le n} |a_{nk} - a_{nn}| \max_{1\le m\le n} |\sum_{k=1}^{m}(\widetilde{X}_{k-1} - \widetilde{X}_k)| \\
&\quad + |a_{nn}||\widetilde{X}_0 - \widetilde{X}_n| \\
&\le C \max_{1\le k\le n} |a_{nk}(x)|(|\widetilde{X}_0| + \max_{1\le m\le n} |\widetilde{X}_m|) \\
&= O(n^{-1/2})(|\widetilde{X}_0| + \max_{1\le m\le n} |\widetilde{X}_m|).
\end{aligned}
\tag{3.6}
$$

Since $\sum_{j=1}^{\infty} j|c_j| < \infty \Rightarrow \sum_{j=0}^{\infty} |\widetilde{c}_j| < \infty$ by Lemma A3 in the Appendix,

$$E|\widetilde{X}_0| \le \sum_{j=0}^{\infty} |\widetilde{c}_j| E|\widetilde{X}_{-j}| < \infty. \tag{3.7}$$

On the other hand, observe that

$$
\begin{aligned}
|\widetilde{X}_m| \le & \sum_{i=0}^{m} |\widetilde{c}_i||X_{m-i}| + \sum_{i=1}^{\infty} |\widetilde{c}_{m+i}||X_{-i}| \\
\le & \max_{0\le i\le m} |X_i|(\sum_{i=0}^{m} |\widetilde{c}_i|) + \sum_{i=1}^{\infty} |\ \hat{c}_i\ |\ |X_{-i}|
\end{aligned} \tag{3.8}
$$

with $\hat{c}_i = \sum_{j=i+1}^{\infty} |c_j|$. Note that

$$
\sum_{j=1}^{\infty} |\widetilde{c}_j| \le \sum_{j=1}^{\infty} atc_j = \sum_{j=1}^{\infty} \sum_{i=j+1}^{\infty} |c_i| \le \sum_{j=1}^{\infty} j|c_j| < \infty, \tag{3.9}
$$

and $n^{-1/2} \max_{0\le m\le n} |X_m| \xrightarrow{P} 0$ is equivalent to

$$
n^{-1} \sum_{m=0}^{n} X_m^2 I(X_m| > n^{1/2}\epsilon) \xrightarrow{P} 0,\ \forall \epsilon > 0
$$

by Lemma A4 in the Appendix, which, together with (3.6)-(3.9), follows $J_n \xrightarrow{P} 0.$

(c) Note that

$$
\eta_i = D(1)X_i + \widetilde{X}_{i-1} - \widetilde{X}_i + \widetilde{\widetilde{X}}_{i+1} - \widetilde{\widetilde{X}}_i,
$$

where $\widetilde{X}_i = \sum_{j=0}^{\infty} \widetilde{c}_j X_{i-j}$, $\widetilde{\widetilde{X}}_i = \sum_{j=-\infty}^{0} \widetilde{\widetilde{c}}_j X_{i-j}$ and $\widetilde{c}_j = \sum_{k=j+1}^{\infty} c_k$, $\widetilde{\widetilde{c}}_j = \sum_{i=-\infty}^{j-1} c_i$. Similarly to the proof in (b), we need only prove that

$$
n^{-1}|\widetilde{X}_0|^2 \to 0 \text{ in probability},\ n^{-1} \max_{1\le m\le n} |\widetilde{\mathrm{X}}_{\mathrm{m}}|^2 \to 0 \text{ in probability},
$$

$$
n^{-1}|\widetilde{\widetilde{X}}_1|^2 \to 0 \text{ in probability},\ n^{-1} \max_{1\le m\le n} |\widetilde{\widetilde{\mathrm{X}}}_{\mathrm{m}}|^2 \to 0 \text{ in probability}.
$$

By $\sum_{j=-\infty}^{\infty} j^2 c_j^2 < \infty$, we can get $E|\widetilde{X}_0|^2 < \infty$ and $E|\widetilde{\widetilde{X}}_1|^2 < \infty$, which follow $n^{-1}|\widetilde{X}_0|^2 \to 0$ in probability and $n^{-1}|\widetilde{\widetilde{X}}_1|^2 \to 0$ in probability, respectively. On the other hand, note that $n^{-1} \max_{1\le m\le n} |\widetilde{X}_m|^2 \to 0$ in probability if and only if

$$
\begin{aligned}
& \frac{1}{n} \sum_{i=1}^{n} \widetilde{X}_i^2 I(\widetilde{X}_i^2 > nc) \to 0 \text{ in probability for any c} > 0, \\
& n^{-1} \max_{1\le m\le n} |\widetilde{\widetilde{X}}_m|^2 \to 0 \text{ in probability if and only if}
\end{aligned} \tag{3.10}
$$

$$(3.11) \qquad \frac{1}{n}\sum_{i=1}^{n} \widetilde{\widetilde{X}}_i^2 I(\widetilde{\widetilde{X}}_i^2 > nc) \to 0 \text{ in probability for any c} > 0$$

by Lemma A4 in the Appendix. Since $\{\widetilde{X}_i^2\}$ and $\{\widetilde{\widetilde{X}}_i^2\}$ are uniformly integrable by $\sum_{j=-\infty}^{\infty} j^2 c_j^2 < \infty$ and $\sup_i E|X_i|^{2+\delta} < \infty$, from (3.10) and (3.11) we get

$$n^{-1} \max_{1\le m\le n} |\widetilde{X}_m|^2 \to 0 \text{ in probability,} \;\; n^{-1} \max_{1\le m\le n} |\widetilde{\widetilde{\mathrm{X}}}_{\mathrm{m}}|^2 \to 0 \text{ in probability.}$$

APPENDIX

The following result is Lemma 2 of Matula ([9]):

Lemma A1. *If $\{X_i, i \ge 1\}$ is a sequence of pairwise NQD random variables and $\{f_i, i \ge 1\}$ a sequence of nondecreasing functions $f_i : \mathbb{R} \to \mathbb{R}$, then $\{f_i(X_i)\}$ are also pairwise NQD.*

Lemma A2. *Suppose that, for each u, $X_{un} \to^{\mathcal{D}} X_u$ as $n \to \infty$ and $X_u \to^{\mathcal{D}} X$ as $u \to \infty$. Suppose further that*

$$\lim_{u\to\infty} \lim_{n\to\infty} \sup P\{\rho(X_{un}, Y_n) \ge \varepsilon\} = 0 \textit{ for each } \varepsilon > 0.$$

Then, $Y_n \to^{\mathcal{D}} X$ as $n \to \infty$, where $\mathcal{D}$ means convergence in distribution.

Proof. See the proof of Theorem 4.2 in Billingsley ([1]).

The following result is Lemma 2.1 of Phillips and Solo([12]):

Lemma A3. *Let $C(L) = \sum_{j=0}^{\infty} c_j L^j$. Then, we have*

$$C(L) = C(1) - (1 - L)\widetilde{C}(L),$$

where $\widetilde{C}(L) = \sum_{j=0}^{\infty} \widetilde{c}_j L^j$, $\widetilde{c}_j = \sum_{j+1}^{\infty} c_k$. If $p \ge 1$, then

$$\sum_{j=1}^{\infty} j^p |c_j|^p < \infty \Rightarrow \sum_{j=0}^{\infty} |\widetilde{c}_j|^p < \infty \text{ and } |C(1)| < \infty.$$

From the fact $P(\max_i |X_{ni}| > \varepsilon) = P(\sum_i X_{ni}^2 I(|X_{ni}| > \varepsilon) > \varepsilon^2)$ we have the following result (See Hall and Heyde [4] p. 53):

Lemma A4. $\max_i |X_{ni}| \to^p 0$ *is equivalent to the weak Lindeberg condition*

$$\sum_i X_{ni}^2 I(|X_{ni}|) > \varepsilon) \to^p 0 \text{ for all } \varepsilon > 0.$$

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