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EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTIONS FOR A KIND OF FIRST ORDER NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH A DEVIATING ARGUMENT

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Abstract. In this paper, we use the coincidence degree theory to establish new results on the existence and uniqueness of T-periodic solutions for the first order neutral functional differential equation with a deviating argument of the form

$$(x(t) + Bx(t-\delta))' = g_1(t, x(t)) + g_2(t, x(t-\tau(t))) + p(t).$$

1. INTRODUCTION

Consider the first order neutral functional differential equation(NFDE) with a deviating argument of the form

$$(x(t) + Bx(t-\delta))' = g_1(t, x(t)) + g_2(t, x(t-\tau(t))) + p(t), \tag{1.1}$$

where τ, $p : R \to R$ and $g_1, g_2 : R \times R \to R$ are continuous functions, B and δ are constants, τ and p are T-periodic, g_1 and g_2 are T-periodic in the first argument, $|B| \neq 1$ and $T > 0$.

Such kind of NFDE has been used for the study of distributed networks containing lossless transmission lines [6,7]. Hence, in recent years, the problem of the existence of periodic solutions for Eq. (1.1) has been extensively studied in the literature. For more details, we refer the reader to [1, 3-8,10] and the references cited therein. However, to the best of our knowledge, there exist no results for the existence and uniqueness of periodic solutions of Eq. (1.1).

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The main purpose of this paper is to establish sufficient conditions for the existence and uniqueness of T-periodic solutions of Eq. (1.1). The results of this paper are new and they compliment previously known results. An illustrative example is given in Section 4.

For ease of exposition, throughout this paper we will adopt the following notations:

$$|x|_k = (\int_0^T |x(t)|^k dt)^{1/k}, \quad |x|_\infty = \max_{t\in[0,T]} |x(t)|.$$

Let $X = \{x|x \in C(R,\ R),\ x(t+T) = x(t),\ \text{for all } t \in R\}$ be a Banach space with the norm $\|x\|_X = |x|_\infty$. Define linear operators A and L in the following form respectively

$$A : X \longrightarrow X, \quad (Ax)(t) = x(t) + Bx(t-\delta)$$

and

$$L : D(L) \subset X \longrightarrow X,\ Lx = (Ax)', \tag{1.2}$$

where $D(L) = \{x|x \in X, x' \in C(R,\ R)\}$.

We also define a nonlinear operator $N : X \longrightarrow X$ by setting

$$Nx = g_1(t, x(t)) + g_2(t, x(t-\tau(t))) + p(t). \tag{1.3}$$

By Hale's terminology [4], a solution $u(t)$ of Eq (1.1) is that $u \in C(R,\ R)$ such that $Au \in C^1(R,\ R)$ and Eq (1.1) is satisfied on R. In general, $u \notin C^1(R,\ R)$. But from Lemma 1 in [8], in view of $|B| \neq 1$, it is easy to see that $(Ax)' = Ax'$. So a $T-$periodic solution $u(t)$ of Eq (1.1) must be such that $u \in C^1(R,\ R)$. Meanwhile, according to Lemma 1 in [8], we can easily get that $\text{Ker}L = R$, and $\text{Im}L = \{x|x \in X, \int_0^T x(s)ds = 0\}$. Therefore, the operator L is a Fredholm operator with index zero. Define the continuous projectors $P : X \longrightarrow \text{Ker}L$ and $Q : X \longrightarrow X/ImL$ by setting

$$Px(t) = \frac{1}{T}\int_0^T x(s)ds$$

and

$$Qx(t) = \frac{1}{T}\int_0^T x(s)ds.$$

Hence, $\text{Im}P = \text{Ker}L$ and $\text{Ker}Q = \text{Im}L$. Set $L_P = L|_{D(L)\cap\text{KerP}}$, then L_P has continuous inverse L_P^{-1} defined by

$$L_P^{-1}y(t) = A^{-1}\left(\frac{1}{T}\int_0^T sy(s)ds + \int_0^t y(s)ds\right). \tag{1.4}$$

Therefore, it is easy to see from (1.3) and (1.4) that N is $L-$compact on $\overline{\Omega}$, where Ω is an open bounded set in X.

2. Preliminary Results

In view of (1.2) and (1.3), the operator equation

$$Lx = \lambda Nx$$

is equivalent to the following equation

$$x'(t) + Bx'(t-\delta) = \lambda[g_1(t, x(t)) + g_2(t, x(t-\tau(t))) + p(t)], \tag{2.1}$$

where $\lambda \in (0, 1)$.

For convenience, we introduce the Continuation Theorem [3] as follows.

Lemma 2.1. *Let X be a Banach space. Suppose that $L : D(L) \subset X \longrightarrow X$ is a Fredholm operator with index zero and $N : \overline{\Omega} \longrightarrow X$ is $L-$compact on $\overline{\Omega}$, where Ω is an open bounded subset of X. Moreover, assume that all the following conditions are satisfied:*

(1) $Lx \neq \lambda Nx$, *for all* $x \in \partial\Omega \cap D(L), \lambda \in (0,\ 1)$;

(2) $Nx \notin ImL$, *for all* $x \in \partial\Omega \cap KerL$;

(3) *The Brower degree*

$$deg\{QN,\ \Omega \cap KerL,\ 0\} \neq 0.$$

Then equation $Lx = Nx$ has at least one solution on $\overline{\Omega} \cap D(L)$.

The following lemmas will be useful to prove our main results in Section 3.

Lemma 2.2. *Let $x(t) \in X \bigcap C^1(R, R)$. Suppose that there exists a constant $D \geq 0$ such that*

$$|x(\tau_0)| \leq D,\ \ \tau_0 \in [0, T]. \tag{2.2}$$

Then

$$|x|_2 \leq \frac{T}{\pi}|x'|_2 + \sqrt{T}D. \tag{2.3}$$

Proof. Let

$$y(t) = \begin{cases} x(t+\tau_0 - T) - x(\tau_0), & T - \tau_0 \leq t \leq T, \\ x(t+\tau_0) - x(\tau_0), & 0 \leq t < T - \tau_0. \end{cases}$$

Then

$$y(0) = y(T) = 0, \quad \text{and} \quad y'(t) = x'(t+\tau_0) \quad \text{for all} \quad t \in [0, T]. \tag{2.4}$$

Thus, by Theorem 225 in [2], (2.4) implies that

$$|y|_2 \leq \frac{T}{\pi}|y'|_2. \tag{2.5}$$

In view of the inequality of Minkowski, we have

$$\begin{aligned}
|x|_2^2 &= \int_{\tau_0}^{T} |x(t)|^2 dt + \int_0^{\tau_0} |x(t)|^2 dt \\
&= \int_0^{T-\tau_0} |x(t+\tau_0)|^2 dt + \int_{T-\tau_0}^{T} |x(t+\tau_0-T)|^2 dt \\
&= |y(t) + x(\tau_0)|_2^2 \\
&\leq (|y|_2 + |x(\tau_0)|_2)^2 \\
&\leq (|y|_2 + \sqrt{T}D)^2.
\end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we obtain

$$|x|_2 \leq |y|_2 + \sqrt{T}D \leq \frac{T}{\pi}|y'|_2 + \sqrt{T}D = \frac{T}{\pi}|x'|_2 + \sqrt{T}D.$$

This completes the proof of Lemma 2.2.

Lemma 2.3. *Assume that the following conditions are satisfied.*

(A_0) *one of the following conditions holds:*

(1) $(g_i(t,u_1) - g_i(t,u_2))(u_1 - u_2) > 0$, *for* $i = 1,2, u_i \in R$, $\forall t \in R$ *and* $u_1 \neq u_2$,

(2) $(g_i(t,u_1) - g_i(t,u_2))(u_1 - u_2) < 0$, *for* $i = 1,2, u_i \in R$, $\forall t \in R$ *and* $u_1 \neq u_2$;

$(\overline{A}_0)$ *one of the following conditions holds:*

(1) *there exists constants* b_1 *and* b_2 *such that* $b_1\frac{T}{\pi} + \frac{1}{2}b_2T < 1 - |B|$, *and*
$$|g_i(t,u_1) - g_i(t,u_2)| \leq b_i|u_1 - u_2|, \quad \textit{for } i = 1,2,\ u_i \in R, \forall t \in R,$$

(2) *there exists constants* b_1 *and* b_2 *such that* $b_1\frac{T}{\pi} + \frac{1}{2}b_2T < |B| - 1$, *and*
$$|g_i(t,u_1) - g_i(t,u_2)| \leq b_i|u_1 - u_2|, \quad \textit{for } i = 1,2,\ u_i \in R, \forall t \in R.$$

Then Eq. (1.1) has at most one T-periodic solution.

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of Eq. (1.1). Then, we have

$$(x_1(t) + Bx_1(t-\delta))' - g_1(t, x_1(t)) - g_2(t, x_1(t-\tau(t))) = p(t)$$

and

$$(x_2(t) + Bx_2(t-\delta))' - g_1(t, x_2(t)) - g_2(t, x_2(t-\tau(t))) = p(t).$$

This implies that

$$[(x_1(t) - x_2(t)) + B(x_1(t-\delta) - x_2(t-\delta))]' - (g_1(t, x_1(t)) - g_1(t, x_2(t)))$$

$$-(g_2(t, x_1(t-\tau(t))) - g_2(t, x_2(t-\tau(t)))) = 0. \tag{2.7}$$

Set $Z(t) = x_1(t) - x_2(t)$. Then, from (2.7), we obtain

$$\begin{aligned} &Z'(t) + BZ'(t-\delta) - (g_1(t, x_1(t)) - g_1(t, x_2(t))) \\ &\quad -(g_2(t, x_1(t-\tau(t))) - g_2(t, x_2(t-\tau(t)))) = 0. \end{aligned} \tag{2.8}$$

Thus, integrating (2.8) from 0 to T, we have

$$\int_0^T [(g_1(t, x_1(t)) - g_1(t, x_2(t))) + (g_2(t, x_1(t-\tau(t))) - g_2(t, x_2(t-\tau(t))))]dt = 0.$$

Therefore, in view of integral mean value theorem, it follows that there exists a constant $\gamma \in [0,\ T]$ such that

$$\begin{aligned} &g_1(\gamma, x_1(\gamma)) - g_2(\gamma, x_2(\gamma))) + g_2(\gamma, x_1(\gamma - \tau(\gamma))) \\ &\quad -g_2(\gamma, x_2(\gamma - \tau(\gamma)))) = 0. \end{aligned} \tag{2.9}$$

From (A_0), (2.9) implies that

$$(x_1(\gamma) - x_2(\gamma))(x_1(\gamma - \tau(\gamma)) - x_2(\gamma - \tau(\gamma))) \le 0.$$

Since $Z(t) = x_1(t) - x_2(t)$ is a continuous function on R, it follows that there exists a constant $\xi \in R$ such that

$$Z(\xi) = 0. \tag{2.10}$$

Let $\xi = nT + \widetilde{\gamma}$, where $\widetilde{\gamma} \in [0,\ T]$ and n is an integer. Then, (2.10) implies that there exists a constant $\widetilde{\gamma} \in [0,\ T]$ such that

$$Z(\widetilde{\gamma}) = Z(\xi) = 0, \tag{2.11}$$

which implies that

$$|Z(t)| = |Z(\widetilde{\gamma}) + \int_{\widetilde{\gamma}}^t Z'(s)ds| \le \int_{\widetilde{\gamma}}^t |Z'(s)|ds,\ t \in [\widetilde{\gamma},\ \widetilde{\gamma} + T],$$

and

$$|Z(t)| = |Z(t-T)| = |Z(\widetilde{\gamma}) - \int_{t-T}^{\widetilde{\gamma}} Z'(s)ds| \leq \int_{t-T}^{\widetilde{\gamma}} |Z'(s)|ds, t \in [\widetilde{\gamma},\ \widetilde{\gamma}+T].$$

Therefore,

$$|Z(t)| \leq \frac{1}{2}\int_{t-T}^{t} |Z'(s)|ds = \frac{1}{2}\int_{0}^{T} |Z'(s)|ds,\ t \in [\widetilde{\gamma},\ \widetilde{\gamma}+T],$$

which, together with Lemma 2.2 and Schwarz inequality, implies that

$$|Z|_\infty \leq \frac{1}{2}\sqrt{T}|Z'|_2, \quad \text{and } |Z|_2 \leq \frac{T}{\pi}|Z'|_2. \tag{2.12}$$

Now, we consider two cases.

Case (i). If $(\overline{A}_0)(1)$ holds, multiplying both sides of (2.8) by $Z'(t)$ and then integrating them from 0 to T, using (2.12) and Schwarz inequality, we have

$$\begin{aligned}
|Z'|_2^2 &= \int_0^T |Z'(t)|^2 dt \\
&= -B\int_0^T Z'(t)Z'(t-\delta)dt + \int_0^T (g_1(t,x_1(t)) - g_1(t,x_2(t)))Z'(t)dt \\
&\quad + \int_0^T (g_2(t,x_1(t-\tau(t))) - g_2(t,x_2(t-\tau(t))))Z'(t)dt \\
&\leq |B||Z'|_2^2 + b_1\int_0^T |x_1(t)-x_2(t)||Z'(t)|dt \\
&\quad + b_2\textstyle\int_0^T |x_1(t-\tau(t))-x_2(t-\tau(t))||Z'(t)|dt \\
&\leq |B||Z'|_2^2 + b_1|Z|_2||Z'|_2 + b_2|Z|_\infty\sqrt{T}|Z'|_2 \\
&\leq |B||Z'|_2^2 + b_1\tfrac{T}{\pi}|Z'|_2|Z'|_2 + \tfrac{1}{2}b_2T|Z'|_2^2 \\
&\leq [|B| + (b_1\tfrac{T}{\pi} + \tfrac{1}{2}b_2T)]|Z'|_2^2.
\end{aligned} \tag{2.13}$$

From (2.11) and $(\overline{A}_0)(1)$, (2.13) implies that

$$Z(t) \equiv Z'(t) \equiv 0, \quad \text{for all } \ t \in R.$$

Hence, $x_1(t) \equiv x_2(t)$, for all $t \in R$. Therefore, Eq. (1.1) has at most one T-periodic solution.

Case (ii). If $(\overline{A}_0)(2)$ holds, multiplying both sides of (2.8) by $Z'(t-\delta)$ and then integrating them from 0 to T, using (2.12) and Schwarz inequality, we have

$$
\begin{aligned}
|B||Z'|_2^2 &= |\int_0^T B|Z'(t-\delta)|^2 dt| \\
&= |-\int_0^T Z'(t)Z'(t-\delta)dt + \int_0^T (g_1(t,x_1(t)) - g_2(t,x_2(t)))Z'(t-\delta)dt \\
&\quad + \int_0^T (g_2(t,x_1(t-\tau(t))) - g_2(t,x_2(t-\tau(t))))Z'(t-\delta)dt| \\
&\le |Z'|_2^2 + b_1 \int_0^T |x_1(t)-x_2(t)||Z'(t-\delta)|dt \\
&\quad + b_2 \int_0^T |x_1(t-\tau(t)) - x_2(t-\tau(t))||Z'(t-\delta)|dt \\
&\le |Z'|_2^2 + b_1|Z|_2|Z'|_2 + b_2|Z|_\infty \sqrt{T}|Z'|_2 \\
&\le |Z'|_2^2 + b_1 \frac{T}{\pi}|Z'|_2|Z'|_2 + \frac{1}{2} b_2 T |Z'|_2^2 \\
&\le [1 + (b_1 \frac{T}{\pi} + \frac{1}{2} b_2 T)]|Z'|_2^2
\end{aligned} \tag{2.14}
$$

Then using the methods similar to those used in Case (i), from (2.11), (2.14) and $(\overline{A}_0)(2)$, we can conclude that Eq. (1.1) has at most one T-periodic solution. The proof of Lemma 2.3 is now complete.

Lemma 2.4. *Assume that (A_0) holds, and there exists a constant $d > 0$ such that one of the following conditions holds:*

(A_1) $x(g_1(t,x) + g_2(t,x) + p(t)) > 0$, *for all* $t \in R, |x| \ge d$;

(A_2) $x(g_1(t,x) + g_2(t,x) + p(t)) < 0$, *for all* $t \in R, |x| \ge d$.

If $x(t)$ is a T-periodic solution of $(2.1)_\lambda$, then

$$|x|_\infty \le d + \frac{1}{2}\sqrt{T}|x'|_2. \tag{2.15}$$

Proof. Let $x(t)$ be a T-periodic solution of $(2.1)_\lambda$. Then, integrating $(2.1)_\lambda$ from 0 to T, we have

$$\int_0^T [g_1(t,x(t)) + g_2(t,x(t-\tau(t))) + p(t)]dt = 0.$$

This implies that there exists a constant $t_1 \in R$ such that

$$g_1(t_1,x(t_1)) + g_2(t_1,x(t_1-\tau(t_1))) + p(t_1) = 0. \tag{2.16}$$

We show next that the following claim is true.

Claim. If $x(t)$ is a $T-$periodic solution of $(2.1)_\lambda$, then there exists a constant $t_2 \in R$ such that

$$|x(t_2)| \leq d. \tag{2.17}$$

Assume, by way of contradiction, that (2.17) does not hold. Then

$$|x(t)| > d, \quad \text{for all } t \in R, \tag{2.18}$$

which, together with (A_1), (A_2) and (2.16), implies that one of the following relations holds:

$$x(t_1) > x(t_1 - \tau(t_1)) > d; \tag{2.19}$$

$$x(t_1 - \tau(t_1)) > x(t_1) > d; \tag{2.20}$$

$$x(t_1) < x(t_1 - \tau(t_1)) < -d; \tag{2.21}$$

$$x(t_1 - \tau(t_1)) < x(t_1) < -d. \tag{2.22}$$

If (2.19) holds, in view of $(A_0)(1)$, $(A_0)(2)$, (A_1) and (A_2) , we shall consider four cases as follows.

Case (i). If (A_1) and $(A_0)(1)$ hold, according to (2.19), we obtain

$$\begin{aligned} 0 &< g_1(t_1, x(t_1 - \tau(t_1))) + g_2(t_1, x(t_1 - \tau(t_1))) + p(t_1) \\ &< g_1(t_1, x(t_1)) + g_2(t_1, x(t_1 - \tau(t_1))) + p(t_1), \end{aligned}$$

which contradicts (2.16). This contradiction implies that (2.17) holds.

Case (ii). If (A_1) and $(A_0)(2)$ hold, according to (2.19), we obtain

$$0 < g_1(t_1, x(t_1)) + g_2(t_1, x(t_1)) + p(t_1) < g_1(t_1, x(t_1)) + g_2(t_1, x(t_1 - \tau(t_1))) + p(t_1),$$

which contradicts (2.16). This contradiction implies that (2.17) holds.

Case (iii). If (A_2) and $(A_0)(1)$ hold, according to (2.19), we obtain

$$g_1(t_1, x(t_1)) + g_2(t_1, x(t_1 - \tau(t_1))) + p(t_1) < g_1(t_1, x(t_1)) + g_2(t_1, x(t_1)) + p(t_1) < 0,$$

which contradicts (2.16). This contradiction implies that (2.17) holds.

Case (iv). If (A_2) and $(A_0)(2)$ hold, according to (2.19), we obtain

$$\begin{aligned} &g_1(t_1, x(t_1)) + g_2(t_1, x(t_1 - \tau(t_1))) + p(t_1) \\ &< g_1(t_1, x(t_1 - \tau(t_1))) + g_2(t_1, x(t_1 - \tau(t_1))) + p(t_1) < 0, \end{aligned}$$

which contradicts (2.16). This contradiction implies that (2.17) holds.

If (2.20)(or (2.21), or (2.22)) holds, using the methods similar to those used in Case (i)-Case (iv), we can show that (2.17) holds. This completes the proof of the Claim.

Let $t_1 = mT + t_0$, where $t_0 \in [0, T]$ and m is an integer. Then, using an argument similar to that in proof of (2.12), we obtain

$$|x|_\infty = \max_{t\in[0,T]} |x(t)| \le d + \frac{1}{2}\sqrt{T}|x'|_2.$$

This completes the proof of Lemma 2.4.

3. Main Results

Theorem 3.1. *Let (A_0) and $(\overline{A}_0)$ hold. Assume that either the condition (A_1) or the condition (A_2) is satisfied. Then Eq. (1.1) has a unique T-periodic solution.*

Proof. From Lemma 2.3, together with (A_0) and $(\overline{A}_0)$, it is easy to see that Eq. (1.1) has at most one T-periodic solution. Thus, to prove Theorem 3.1, it suffices to show that Eq. (1.1) has at least one T-periodic solution. To do this, we shall apply Lemma 2.1. Firstly, we will claim that the set of all possible T-periodic solutions of Eq. $(2.1)_\lambda$ are bounded.

Let $x(t)$ be a T-periodic solution of equation $(2.1)_\lambda$. In view of $(\overline{A}_0)(1)$ and $(\overline{A}_0)(2)$, we shall consider two cases as follows.

Case (i). If $(\overline{A}_0)(1)$ holds, multiplying both sides of $(2.1)_\lambda$ by $x'(t)$ and then integrating them from 0 to T, from (2.3), (2.15), $(\overline{A}_0)(1)$ and the inequality of Schwarz, we have

$$\begin{aligned}
|x'|_2^2 &= \int_0^T |x'(t)|^2 dt \\
&= -\int_0^T Bx'(t-\delta)x'(t)dt + \lambda\int_0^T g_1(t,x(t))x'(t)dtt \\
&\quad +\lambda\int_0^T g_2(t,x(t-\tau(t)))x'(t)d + \lambda\int_0^T p(t)x'(t)dt \\
&\le |B||x'|_2^2 + |p|_2|x'|_2 + \lambda\int_0^T (g_1(t,x(t)) \\
&\quad -g_1(t,0))x'(t)dt + \lambda\int_0^T g_1(t,0)x'(t)dt \\
&\quad +\lambda\int_0^T (g_2(t,x(t-\tau(t))) - g_2(t,0))x'(t)dt + \lambda\int_0^T g_2(t,0)x'(t)dt
\end{aligned} \tag{3.1}$$

$$
\begin{aligned}
&\leq |B||x'|_2^2 + |p|_2|x'|_2 + b_1\int_0^T |x(t)||x'(t)|dt \\
&\quad + \max_{t\in[0,\ T]} |g_1(t,0)| \int_0^T |x'(t)|dt \\
&\quad + b_2\int_0^T |x(t-\tau(t))|\cdot|x'(t)|dt + \max_{t\in[0,\ T]} |g_2(t,0)| \int_0^T |x'(t)|dt \\
&\leq |B||x'|_2^2 + |p|_2|x'|_2 + b_1|x|_2|x'|_2 + b_2|x|_\infty\sqrt{T}|x'|_2 + (\max_{t\in[0,\ T]} |g_1(t,0)| \\
&\quad + \max_{t\in[0,\ T]} |g_2(t,0)|)\sqrt{T}|x'(t)|_2 \\
&\leq (|B| + b_1\frac{T}{\pi} + \frac{1}{2}b_2T)|x'|_2^2 + [|p|_2 \\
&\quad + (db_2 + \max_{t\in[0,\ T]} |g_1(t,0)| + \max_{t\in[0,\ T]} |g_2(t,0)|)\sqrt{T}]|x'|_2.
\end{aligned} \tag{3.1}
$$

Now, let

$$
D_1 = \frac{|p|_2 + (db_2 + \max_{t\in[0,\ T]} |g_1(t,0)| + \max_{t\in[0,\ T]} |g_2(t,0)|)\sqrt{T}}{1 - |B| - (b_1\frac{T}{\pi} + \frac{1}{2}b_2T)}.
$$

In view of (2.15) and (3.1), we obtain

$$
|x'|_2 \leq D_1, |x|_\infty \leq d + \sqrt{T}D_1. \tag{3.2}
$$

Case (ii). If $(\overline{A}_0)(2)$ holds, multiplying both sides of $(2.1)_\lambda$ by $x'(t-\delta)$ and then integrating them from 0 to T, from (2.3), (2.15), $(\overline{A}_0)(2)$ and the inequality of Schwarz, we have

$$
\begin{aligned}
|B||x'|_2^2 &= |\int_0^T B|x'(t-\delta)|^2 dt| \\
&= |-\int_0^T x'(t-\delta)x'(t)dt + \lambda\int_0^T g_1(t,x(t))x'(t-\delta)dt \\
&\quad + \lambda\int_0^T g_2(t,x(t-\tau(t)))x'(t-\delta)dt + \lambda\int_0^T p(t)x'(t-\delta)dt| \\
&\leq |x'|_2^2 + |p|_2|x'|_2 + |\lambda\int_0^T (g_1(t,x(t)) \\
&\quad - g_1(t,0))x'(t-\delta)dt + \lambda\int_0^T g_1(t,0)x'(t-\delta)dt \\
&\quad + \lambda\int_0^T (g_2(t,x(t-\tau(t))) \\
&\quad - g_2(t,0))x'(t-\delta)dt + \lambda\int_0^T g_2(t,0)x'(t-\delta)dt| \\
&\leq |x'|_2^2 + |p|_2|x'|_2 + b_1\int_0^T |x(t)||x'(t-\delta)|dt
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
& +\max_{t\in[0,\ T]}|g_1(t,0)|\int_0^T|x'(t-\delta)|dt \\
& +b_2\int_0^T|x(t-\tau(t))|\cdot|x'(t-\delta)|dt \\
& +\max_{t\in[0,\ T]}|g_2(t,0)|\int_0^T|x'(t-\delta)|dt \\
(3.3) \qquad & \le |x'|_2^2+|p|_2|x'|_2+b_1|x|_2|x'|_2+b_2|x|_\infty\sqrt{T}|x'|_2+(\max_{t\in[0,\ T]}|g_1(t,0)| \\
& +\max_{t\in[0,\ T]}|g_2(t,0)|)\sqrt{T}|x'(t)|_2 \\
& \le (1+b_1\frac{T}{\pi}+\frac{1}{2}b_2T)|x'|_2^2+[|p|_2 \\
& +(db_2+\max_{t\in[0,\ T]}|g_1(t,0)|+\max_{t\in[0,\ T]}|g_2(t,0)|)\sqrt{T}]|x'|_2.
\end{aligned}
$$

Now, let

$$\overline{D}_1=\frac{|p|_2+(db_2+\max_{t\in[0,\ T]}|g_1(t,0)|+\max_{t\in[0,\ T]}|g_2(t,0)|)\sqrt{T}}{|B|-1-(b_1\frac{T}{\pi}+\frac{1}{2}b_2T)}.$$

In view of (2.15) and (3.3), we obtain

$$|x'|_2\le\overline{D}_1,|x|_\infty\le d+\sqrt{T}\overline{D}_1. \tag{3.4}$$

If $x\in\Omega_1=\{x|x\in KerL\cap X \text{ and } Nx\in ImL\}$, then there exists a constant M_1 such that

$$x(t)\equiv M_1 \text{ and } \int_0^T[g_1(t,M_1)+g_2(t,M_1)+p(t)]dt=0. \tag{3.5}$$

Thus,

$$|x(t)|\equiv|M_1|<d, \text{ for all } x(t)\in\Omega_1. \tag{3.6}$$

Let $M=(D_1+\overline{D}_1)\sqrt{T}+d+1$. Set

$$\Omega=\{x|x\in X,\ |x|_\infty<M\}.$$

It is easy to see from (1.3) and (1.4) that N is $L-$compact on $\overline{\Omega}$. We have from (3.5), (3.6) and the fact $M>\max\{D_1\sqrt{T}+d,\ \overline{D}_1\sqrt{T}+d,\ d\}$ that the conditions (1) and (2) in Lemma 2.1 hold.

Furthermore, define continuous functions $H_1(x,\mu)$ and $H_2(x,\mu)$ by setting

$$H_1(x,\mu)=(1-\mu)x+\mu\cdot\frac{1}{T}\int_0^T[g_1(t,x)+g_2(t,x)+p(t)]dt;\ \mu\in[0\ \ 1],$$

$$H_2(x,\mu) = -(1-\mu)x + \mu \cdot \frac{1}{T}\int_0^T [g_1(t,x) + g_2(t,x) + p(t)]dt;\ \mu \in [0\ \ 1].$$

If (A_1) holds, then

$$xH_1(x,\mu) \neq 0 \text{ for all } x \in \partial\Omega \cap KerL.$$

Hence, using the homotopy invariance theorem, we have

$$deg\{QN,\ \Omega\cap KerL,\ 0\} = deg\{\frac{1}{T}\int_0^T [g_1(t,x)+g_2(t,x)+p(t)]dt,\ \Omega\cap KerL,\ 0\}$$

$$= deg\{x,\ \Omega\cap KerL,\ 0\} \neq 0.$$

If (A_2) holds, then

$$xH_2(x,\mu) \neq 0 \text{ for all } x \in \partial\Omega \cap KerL.$$

Hence, using the homotopy invariance theorem, we obtain

$$deg\{QN,\ \Omega\cap KerL,\ 0\} = deg\{\frac{1}{T}\int_0^T [g_1(t,x)+g_2(t,x)+p(t)]dt,\ \Omega\cap KerL,\ 0\}$$

$$= deg\{-x,\ \Omega\cap KerL,\ 0\} \neq 0.$$

In view of all the discussions above, we conclude from Lemma 2.1 that Theorem 3.1 is proved.

4. Example and Remark

Example 4.1. The first order NFDE

$$(x(t) + \frac{1}{8}x(t-\delta))' = -\frac{1}{8\pi}x + \frac{1}{2\pi}[1 - x(t - \frac{3}{2}\sin t)] + e^{\cos^2 t} \tag{4.1}$$

has a unique 2π-periodic solution.

Proof. From (4.1), we have $B = \frac{1}{8}$, $g_1(x) = -\frac{1}{8\pi}x$, $g_2(x(t-\tau(t))) = \frac{1}{2\pi}[1 - x(t - \frac{3}{2}\sin t)]$ and $p(t) = e^{\cos^2 t}$. Then, $b_1 = \frac{1}{8\pi}, b_2 = \frac{1}{2\pi}$. It is straight forward to check that all the conditions needed in Theorem 3.1 are satisfied. Therefore, Eq. (4.1) has a unique 2π-periodic solution.

Remark 4.1. Eq. (4.1) is a very simple version of first order NFDE. Since $B \neq 0$, all the results in [1-12] and the references therein can not be applicable to Eq. (4.1) to obtain the existence and uniqueness of 2π-periodic solutions. Moreover, one can easily see that all the results in Ref. [12] are special ones of this paper. This implies that the results of this paper are essentially new.

Remark 4.2. By using the methods similarly to those used for Eq. (1.1), we can deal with the NFDE with multiple deviating arguments of the following type:

$$(x(t)+Bx(t-\delta))'=\sum_{i=1}^{n}g_i(t,x(t-\tau_i(t)))+p(t), \tag{4.2}$$

where $\tau_i(i=1,2,\cdots,n)$, $p:R\to R$ and $g_i(i=1,2,\cdots,n):R\times R\to R$ are continuous functions, $\tau_i(i=1,2,\cdots,n)$ and p are T-periodic, $g_i, i=1,2,\cdots,n,$ are T-periodic in the first argument, and $T>0$. One may also establish the results similar to those in Theorem 3.1 under some minor additional assumptions on $g_i(t,x)(i=1,2,\cdots,n)$.

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