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MULTILINEAR COMMUTATORS OF SINGULAR INTEGRAL OPERATORS WITH NON-SMOOTH KERNELS

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pace-0.68cm **Abstract.** The boundedness of multilinear commutators of singular integral operators with non-smooth kernels on spaces of homogeneous type is obtained.

1. Introduction

It is well-known that the multilinear commutator of singular integral operator and BMO functions on Euclidean spaces, which was introduced by Perez and Trujillo-Gonzalez in [10], is a generalization of commutators. Recently, Duong and Yan in [5] considered commutators of BMO functions and singular integral operators with non-smooth kernels on spaces of homogeneous type. In this paper, motivated by [10] and [5], we consider the boundedness of multilinear commutators of singular integral operators with non-smooth kernels on spaces of homogeneous type. Some ideas and methods of this paper directly come from those developed in both [10] and [5].

Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type, equipped with a metric d and a Borel measure μ. Let T be a bounded operator on $L^p(\mathcal{X})$ for some $p \in (1, \infty)$. A measurable function $K(x, y)$ on $\mathcal{X} \times \mathcal{X}$ is called to be an associated kernel of T, if

$$T(f)(x) = \int_{\mathcal{X}} K(x, y) f(y) d\mu(y) \tag{1.1}$$

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holds for each continuous function f with compact support, and for almost all x not in the support of f.

It is well-known (for example, see [6]) that T in (1.1) is bounded on $L^p(\mathcal{X})$ for $p \in (1, \infty)$ if the associated kernel K satisfies the following Hörmander conditions, that is,

(i) there exist $0 < r \leq 1$ and constants C and $C_1 \geq 1$ such that

$$|K(x, y) - K(x', y)| \leq C \frac{1}{\mu(B(x, d(x, y)))} \frac{d(x, x')^r}{d(x, y)^r} \tag{1.2}$$

when $d(x, y) > C_1 d(x, x')$, and $B(x, r_B)$ denotes the ball centered x with radius r_B.

(ii) there exist constants C and $C_2 > 1$ such that

$$\int_{d(x,y)>C_2 d(x,x')} |K(y, x) - K(y, x')| d\mu(y) \leq C \tag{1.3}$$

for all $x, x' \in \mathcal{X}$.

If $b \in BMO(\mathcal{X})$, and T is bounded on $L^q(\mathcal{X})$ for some $q \in (1, \infty)$, then the commutator, $[b, T]$, of b and T is defined by

$$[b, T](f) = bTf - T(bf) \tag{1.4}$$

for $f \in L^q(\mathcal{X})$.

It is well known that the Hörmander conditions (i) and (ii) above are sufficient to imply that the commutator $[b, T]f$ is bounded on $L^p(\mathcal{X})$ for all $p,\ 1 < p < \infty$, with norm

$$\|[b, T]f\|_p \leq C\|b\|_* \|f\|_p,$$

where $\|b\|_*$ the BMO semi-norm of b. See [2], [7] for $\mathcal{X} = \mathbb{R}^n$ Euclidean spaces, and [1] for spaces of homogeneous type.

Recently, Duong and Yan in [5] proved that the commutator $[b, T]$ in (1.4) is still bounded on $L^p(\mathcal{X})$ for all $p \in (1, \infty)$, if $\mathcal{X}$ has an infinite measure and the above Hörmander conditions (1.2) and (1.3) are replaced, respectively, by the following weak conditions (1.5) and (1.6), which firstly appeared in [4], that

(iii) There exists a class of operators A_t with kernels $a_t(x, y)$, which satisfy the condition (2.3) in Section 2, so that the kernel $k_t(x, y)$ of the operator $T - A_t T$ for $t > 0$ satisfies the condition

$$|k_t(x, y)| \leq C_1 \frac{1}{\mu(B(x, d(x, y)))} \frac{t^{\alpha/\beta}}{d(x, y)^\alpha}, \tag{1.5}$$

if $d(x, y) > C_2 t^{1/\beta}$ for some $\alpha, \beta > 0$; and

(iv) There exists a class of operators $\{B_t\}_{t>0}$ satisfy conditions (2.3) such that $(T - TB_t)$ for each $t > 0$ has an associated kernel $K_t(x, y)$ and there exist positive constants $C_3,\ C_4$ such that

$$\int_{d(x,y)>C_3 t^{1/\beta}} |K_t(x, y)| d\mu(x) \le C_4 \quad \text{for all } y \in \mathcal{X}. \tag{1.6}$$

Note that the classes of operators A_t and B_t play the role of generalized approximations to the identity. It is known that conditions (1.5) and (1.6) are consequences of conditions (1.2) and (1.3), respectively; see [4, Proposition 2].

Similar to [10], on a space of homogeneous type, we define the multilinear commutator $[\vec{b}, T]$ of BMO functions and a singular integral operator T as in (1.1) by

$$[\vec{b}, T]f(x) = \int_{\mathcal{X}} \prod_{i=1}^{m} (b_i(x) - b_i(y)) K(x, y) f(y) d\mu(y), \tag{1.7}$$

holds for each continuous function f with compact support, and for almost all x not in the support of f, where $\vec{b} = \{b_1, \cdots, b_m\}$, b_i's are BMO functions. The main purpose of this paper is to prove that multilinear commutators as in (1.7) under the weak conditions (1.5) and (1.6) are still bounded on $L^p(\mathcal{X})$ for $1 < p < \infty$.

To state our result, we need some definitions and preliminary results which will be given in Section 2. In Section 3 we will give the main result and its proof. A key of the method is to use the sharp maximal function $M_A^{\#} f$ which was previously introduced in [8].

Our result can be extended to the case that the underlying space being a subset of a space of homogeneous type of infinite measure, which can be applied to holomorphic functional calculi of Schrödinger operators, and divergence form operators on irregular domains. Then one can obtain an analogue to Section 4 and Section 5 in [5]. We omit the details.

Finally, in the sequel, we use C to denote a positive constant which is independent of the main parameters, but it may vary from line to line.

2. Definitions and Preliminary Results

Let $\mathcal{X}$ be a topological space equipped with a Borel measure μ and a metric d which is a measurable function on $\mathcal{X} \times \mathcal{X}$. We define $\mathcal{X}$ to be a space of homogeneous type if the balls $B(x; r) = \{y \in X : d(y, x) < r\}$ satisfy the doubling property

$$\mu(B(x; 2r)) \le C\mu(B(x; r)) < \infty$$

for some C uniformly for all $x \in \mathcal{X}$ and $r > 0$. For more about spaces of homogeneous type, one can see [3].

Note that the doubling property implies the following strong homogeneity property,

$$\mu(B(x;\lambda r)) \leq C\lambda^n \mu(B(x;r)) \tag{2.1}$$

for some constant $C, n > 0$ uniformly for all $\lambda \geq 1$. The parameter n is a measure of the dimension of the space. There also exist constants C and $N,\ 0 \leq N \leq n$ so that

$$\mu(B(y;r)) \leq C\left(1 + \frac{d(x,y)}{r}\right)^N \mu(B(x;r)), \tag{2.2}$$

uniformly for all $x, y \in \mathcal{X}$ and $r > 0$. Indeed, the property (2.2) with $N = n$ is a direct consequence of triangle inequality of the metric d and the strong homogeneity property. In the case of Euclidean spaces $\mathbb{R}^n$ and Lie groups of polynomial growth, N can be chosen to be 0.

The standard Hardy-Littlewood maximal function $M_s f,\ 1 \leq s < \infty$, is defined by

$$M_s f(x) = \sup_{x\in B}\left(\frac{1}{\mu(B)}\int_B |f(y)|^s d\mu(y)\right)^{1/s},$$

where the supremum is taken over all balls containing x. If $s = 1$, $M_s f$ will be denoted simply by Mf. The Fefferman-Stein sharp maximal function of f, $f^{\#}(x)$, is defined by

$$f^{\#}(x) = \sup_{x\in B}\frac{1}{\mu(B)}\int_B |f(y) - f_B| d\mu(y),$$

where $f_B = \frac{1}{\mu(B)}\int_B f(y)d\mu(y)$. We will say $f \in BMO(\mathcal{X})$ if $f \in L^1_{loc}(\mathcal{X})$ and $f^{\#}(x) \in L^\infty(\mathcal{X})$. If $f \in BMO(\mathcal{X})$, the BMO semi-norm of f is given by

$$\|f\|_* = \sup_x f^{\#}(x) = \sup_x \sup_{x\in B}\frac{1}{\mu(B)}\int_B |f(y) - f_B| d\mu(y).$$

A family of operators $A_t,\ t > 0$ is said to be generalized approximations to the identity, if, for every $t > 0$, A_t can be represented by kernels $a_t(x,y)$ in the following sense: for every function $f \in L^p(\mathcal{X}),\ p \geq 1$,

$$A_t f(x) = \int_{\mathcal{X}} a_t(x,y) f(y) d\mu(y),$$

and the following condition holds:

$$|a_t(x,y)| \leq h_t(x,y) = \left(\mu(B(x;t^{1/\beta}))\right)^{-1} s(d(x,y)^\beta t^{-1}), \tag{2.3}$$

in which β is a positive constant and s is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+N+\epsilon}s(r^\beta)=0 \tag{2.4}$$

for some $\epsilon>0$, where n and N are the same as in (2.1) and (2.2). Note that (2.2) and (2.3) imply that

$$|a_t(x,y)|\le C\left(\mu(B(y;t^{1/\beta}))\right)^{-1}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(n+\epsilon)}.$$

In [8], the sharp maximal function $M_A^{\#}f$ associated with the generalized approximation to the identity $\{A_t\}_{t>0}$ is defined by

$$M_A^{\#}f(x)=\sup_{x\in B}\frac{1}{\mu(B)}\int_B|f(y)-A_{t_B}f(y)|d\mu(y), \tag{2.5}$$

where $t_B=r_B^\beta$, and $f\in L^p(\mathcal{X})$ for some $p\ge 1$. This sharp maximal function $M_A^{\#}f$ is a variant of Fefferman-Stein's sharp maximal function. For the latter, one can see [12].

In the following, we recall some results, which have been proved in the context of spaces of homogeneous type in [3], [4] and [8].

Lemma 2.1. *(i) For every $p\in[1,\infty]$, there exists a constant C such that for every $f\in L^p(\mathcal{X})$,*

$$A_tf(x)\le CMf(x);$$

(ii) Assume $b\in BMO(\mathcal{X})$ and $M>1$. Then for every ball $B(x;r)$, we have

$$|b_B-b_{MB}|\le C\|b\|_*\log M;$$

(iii) (John-Nirenberg Lemma) Let $1\le p<\infty$ and $B\subset\mathcal{X}$, then $b\in BMO(\mathcal{X})$ if and only if

$$\frac{1}{\mu(B)}\int_B|b(x)-b_B|^pd\mu(x)\le\|b\|_*^p.$$

Lemma 2.2. *Let $\lambda>0$ and $f\in L^p(\mathcal{X})$ for some $1<p<\infty$. Then for every $0<\eta<1$, there exists $\gamma>0$ independent of λ and f such that*

$$\mu(\{x\in\mathcal{X}:Mf(x)>D\lambda,\ M_A^{\#}f(x)\le\gamma\lambda\})\le\eta\mu(\{x\in\mathcal{X}:Mf(x)>\lambda\}),$$

where $D > 1$ is a fixed constant which depends only on the space $\mathcal{X}$ and the generalized approximations of the identity $\{A_t\}_{t>0}$. For the proof of this lemma, see [8, Proposition 4.1]. As a consequence, we have the following estimate:

$$\|f\|_p \leq \|M(f)\|_p \leq C\|M_A^{\#} f\|_p \tag{2.6}$$

for every $f \in L^p(\mathcal{X}),\ 1 < p < \infty$.

Lemma 2.3. *Let $\{A_t\}_{t>0}$ be a generalized approximations of the identity and let $b_i \in BMO(\mathcal{X})$ for $i = 1, 2, \cdots, m$. Then, for every function $f \in L^p(\mathcal{X}),\ p > 1,\ x \in \mathcal{X}$ and $1 < r < \infty$, we have*

$$\sup_{x\in B} \frac{1}{\mu(B)} \int_B \left| A_{t_B}\left(\prod_{i=1}^m (b_i - b_{iB})f\right)(y)\right| d\mu(y) \leq C \prod_{i=1}^m \|b_i\|_* M_q f(x), \tag{2.7}$$

where $t_B = r_B^m$.

Proof. Let $f \in L^p(\mathcal{X}),\ p > 1$, and $x \in \mathcal{X}$ and $x \in B$ for some ball B. Then,

$$\begin{aligned}
\frac{1}{\mu(B)} \int_B & \left| A_{t_B}\left(\prod_{i=1}^m (b_i - b_{iB})f\right)(y)\right| d\mu(y) \\
&\leq \frac{1}{\mu(B)} \int_B \int_{\mathcal{X}} \left| h_{t_B}(y,z) \prod_{i=1}^m (b_i(z) - b_{iB}) f(z)\right| d\mu(z) d\mu(y) \\
&\leq \frac{1}{\mu(B)} \int_B \int_{2B} \left| h_{t_B}(y,z) \prod_{i=1}^m (b_i(z) - b_{iB}) f(z)\right| d\mu(z) d\mu(y) \\
&\quad + \sum_{k=1}^{\infty} \frac{1}{\mu(B)} \int_B \int_{2^{k+1}B\setminus 2^k B} \left| h_{t_B}(y,z) \prod_{i=1}^m (b_i(z) - b_{iB}) f(z)\right| d\mu(z) d\mu(y) \\
&= I + II
\end{aligned}$$

To estimate I, by (2.2), we have $\mu(B) \leq 2^N \mu(B(x, r_B))$ since $x \in B$. From this, it follows that for $z \in 2B$, we have

$$h_{t_B}(y,z) = \frac{s(d(y,z)^\beta t_B^{-1})}{\mu(B(y, t_B^{1/\beta}))} \leq \frac{s(0)}{\mu(B(y, r_B))} \leq \frac{C}{\mu(B)} \leq \frac{C}{\mu(2B)}.$$

Let $1/q + 1/q' = 1$. An iterated application of Lemma 2.1 (iii) yields

$$\begin{aligned}
I &\le C\frac{1}{\mu(B)\mu(2B)}\int_B\int_{2B}\left|\prod_{i=1}^m(b_i(z)-b_{iB})f(z)\right|d\mu(z)d\mu(y)\\
&\le C\frac{1}{\mu(2B)}\int_{2B}\left|\prod_{i=1}^m(b_i(z)-b_{iB})f(z)\right|d\mu(z)\\
&\le C\left[\frac{1}{\mu(2B)}\int_{2B}\prod_{i=1}^m|b_i(z)-b_{iB}|^{q'}d\mu(z)\right]^{1/q'}\left[\frac{1}{\mu(2B)}\int_{2B}|f(z)|^q d\mu(z)\right]^{1/q}\\
&\le C\prod_{i=1}^m\|b_i\|_* M_q f(x).
\end{aligned}$$

Now we turn to estimate II. For $y\in B$ and $z\in 2^{k+1}B\setminus 2^kB$, we have $d(y,z)\ge 2^{k-1}r_B$, and from the doubling volume property, it follows that

$$h_{t_B}(y,z)=\frac{s(d(y,z)^\beta t_B^{-1})}{\mu(B(y,t_B^{1/\beta}))}\le\frac{s(2^{(k-1)\beta})}{\mu(B(y,r_B))}\le\frac{C2^{(k+1)n}s(2^{(k-1)\beta})}{\mu(2^{k+1}B)}.$$

By (2.4) and Lemma 2.1, we obtain

$$\begin{aligned}
II &\le C\sum_{k=1}^\infty 2^{kn}s(2^{(k-1)\beta})\frac{1}{\mu(B)\mu(2^{k+1}B)}\int_B\int_{2^{k+1}B}\\
&\qquad\left|\prod_{i=1}^m(b_i(z)-b_{iB})\right|\ |f(z)|d\mu(z)d\mu(y)\\
&\le C\sum_{k=1}^\infty 2^{kn}s(2^{(k-1)\beta})\frac{1}{\mu(2^{k+1}B)}\int_{2^{k+1}B}\left|\prod_{i=1}^m(b_i(z)-b_{iB})\right|\ |f(z)|d\mu(z)\\
&\le C\sum_{k=1}^\infty 2^{kn}s(2^{(k-1)\beta})\\
&\quad\times\left[\frac{1}{\mu(2^{k+1}B)}\int_{2^{k+1}B}\left|\prod_{i=1}^m(b_i(z)-b_{i,2^{k+1}B}+b_{i,2^{k+1}}-b_{iB})\right|^{q'}d\mu(z)\right]^{1/q'}\\
&\quad\times\left[\frac{1}{\mu(2^{k+1}B)}\int_{2^{k+1}B}|f(z)|^q d\mu(z)\right]^{1/q}\\
&\le C\prod_{i=1}^m\|b_i\|_*\sum_{k=1}^\infty(k+1)^m 2^{kn}s(2^{(k-1)\beta})M_q f(x)\\
&\le C\prod_{i=1}^m\|b_i\|_* M_q f(x),
\end{aligned}$$

where $b_{i,2^{k+1}B} = \frac{1}{\mu(2^{k+1}B)} \int_{2^{k+1}B} b_i(z)dz$, and we used (ii) and (iii) of Lemma 2.1. This finishes the proof of Lemma 2.3.

Remark . Lemma 2.3 is a generalization of Lemma 2.3 in [5], which is just the case $m = 1$.

3. Main Result and Its Proof

Assume that $\mathcal{X}$ is a space of homogeneous type of infinite measure, equipped with a metric d and a Borel measure μ. In this section, we assume the following:

(a) T is a bounded linear operator on $L^2(\mathcal{X})$;

(b) There exists a generalized approximations of the identity $\{B_t\}_{t>0}$ such that $(T - TB_t)$ for $t > 0$ has an associated kernel $K_t(x, y)$ and there exist positive constants $C_1,\ C_2$ such that

$$\int_{d(x,y)>C_1 t^{1/\beta}} |K_t(x, y)| d\mu(x) \le C_2 \quad \text{for all } y \in \mathcal{X};$$

(c) There exists a generalized approximations of the identity $\{A_t\}_{t>0}$ such that the kernel $k_t(x, y)$ of the operator $(T - A_t T)$ for $t > 0$ satisfies

$$|k_t(x, y)| \le C_4 \frac{1}{\mu(B(x; d(x, y))} t^{\alpha/\beta} d(x, y)^{\alpha}, \tag{3.1}$$

if $d(x, y) \ge C_3 t^{1/\beta}$ for some $C_3,\ C_4,\ \alpha > 0$.

It is proved in [4] that if T is an operator satisfying (a) and (b) above, then T is of weak type (1,1) and of strong type (p, p) for $1 < p \le 2$. In addition, if (c) is also satisfied, the operator T is bounded on $L^p(\mathcal{X})$ for all $1 < p < \infty$.

Our main result is the following.

Theorem 3.1. *Let $1 < p < \infty$, and $b_i \in BMO(\mathcal{X}),\ i = 1, \cdots, m$. Then there exists a positive constant C such that*

$$\|[\vec{b}, T]f\|_p \le C \prod_{i=1}^{m} \|b_i\|_* \|f\|_p$$

for any function $f \in L^p(\mathcal{X})$.

The idea of the proof comes from [10] and [5]. We need some basic pointwise estimates of sharp functions for the multilinear commutators of singular integral.

For convenience, we use the following notation. Given any positive integer m, for any $i \in \{1, \cdots, m\}$, we denote by C_i^m the family of all finite subsets

$\sigma = \{\sigma(1), \ldots, \sigma(i)\}$ of i different elements of $\{1, 2, \ldots, m\}$. For any $\sigma \in C_i^m$, we associate the complementary sequence σ' given by $\sigma' = \{1, 2, \ldots, m\} \backslash \sigma$. For any $\sigma \in C_i^m$, we define

$$[\vec{b}_\sigma, T]f(x) = \int_{\mathcal{X}} (b_{\sigma(1)}(x) - b_{\sigma(1)}(y)) \ldots (b_{\sigma(i)}(x) - b_{\sigma(i)}(y))K(x, y)f(y)d\mu(y)$$

for each continuous function f with compact support, and for almost all x not in the support of f. In the case that $\sigma = \{1, 2, \ldots, m\}$, we denote $[\vec{b}_\sigma, T]$ simply by $[\vec{b}, T]$.

To prove Theorem 3.1, we only need to prove the following lemma. In fact, when $m = 1$, Theorem 3.1 was proved in [5, Theorem 3.1]. For $m > 1$, using (2.6), we can deduce Theorem 3.1 from the following Lemma 3.1 by induction on m. We omit the details.

Lemma 3.1. *Let $[\vec{b}, T]$ be as in (1.7), and $q, s > 1$. Then there exists a constant $C > 0$, depending only on q, s, T, and $\mathcal{X}$, such that*

$$([\vec{b}, T]f)^{\#}_A(x) \leq C \left[\prod_{j=1}^m \|b_j\|_* M_{sq}f(x) + \sum_{i=1}^m \sum_{\sigma \in C_i^m} \prod_{j \in \sigma} \|b_j\|_* M_q([\vec{b}_{\sigma'}, T]f)(x) \right] \tag{3.2}$$

for any function $f \in L_c^\infty(\mathcal{X})$ and for every $x \in \mathcal{X}$.

Proof. For $m = 1$, Lemma 3.1 was proved in [5]. So we only need to prove the lemma for $m > 1$. To this end we make use of induction on m. For any $\vec{\lambda} = (\lambda_1, \cdots, \lambda_m) \in \mathbb{R}^m$, we have

$$\begin{aligned}
[\vec{b}, T]f(x) &= \int_{\mathcal{X}} (b_1(x) - b_1(y)) \cdots (b_m(x) - b_m(y))K(x, y)f(y)d\mu(y) \\
&= \int_{\mathcal{X}} \prod_{i=1}^m ((b_i(x) - \lambda_i) - (b_i(y) - \lambda_i))K(x, y)f(y)d\mu(y) \\
&= \sum_{i=0}^m \sum_{\sigma \in C_i^m} (-1)^{m-i}(b(x) - \vec{\lambda})_\sigma \int_{\mathcal{X}} (b(y) - \vec{\lambda})_{\sigma'} K(x, y)f(y)d\mu(y) \\
&= \prod_{i=1}^m (b_i(x) - \lambda_i)Tf(x) + (-1)^m T \left(\prod_{i=1}^m (b_i - \lambda_i)f \right)(x) \\
&\quad + \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} (-1)^{m-i}(b(x) - \vec{\lambda})_\sigma \int_{\mathcal{X}} (b(y) - \vec{\lambda})_{\sigma'} K(x, y)f(y)d\mu(y).
\end{aligned}$$

By expanding $(b(y) - \vec{\lambda})_{\sigma'} = [(b(y) - b(x)) + (b(x) - \vec{\lambda}]_{\sigma'}$, we obtain

$$\begin{aligned}
[\vec{b},T]f(x) &= \int_{\mathcal{X}} (b_1(x)-b_1(y))\cdots(b_m(x)-b_m(y))K(x,y)f(y)d\mu(y) \\
&= \int_{\mathcal{X}} \prod_{i=1}^m ((b_i(x)-\lambda_i)-(b_i(y)-\lambda_i))K(x,y)f(y)d\mu(y) \\
&= \sum_{i=0}^m \sum_{\sigma\in C_i^m} (-1)^{m-i}(b(x)-\vec{\lambda})_\sigma \int_{\mathcal{X}} (b(y)-\vec{\lambda})_{\sigma'}K(x,y)f(y)d\mu(y) \\
&= \prod_{i=1}^m (b_i(x)-\lambda_i)Tf(x) + (-1)^m T\left(\prod_{i=1}^m (b_i-\lambda_i)f\right)(x) \\
&\quad + \sum_{i=1}^{m-1} \sum_{\sigma\in C_i^m} C_{m,i}(b(x)-\vec{\lambda})_\sigma[\vec{b}_{\sigma'},T]f(x),
\end{aligned}$$

where $C_{m,i}$ are constants depending only on m and i.

For fixed $x\in\mathcal{X}$, B denotes a ball containing x center at x_0 with radius r_B, and $2B$ denotes the ball concentric with B and radius two times the radius of B. Split $f=f_1+f_2$, where $f_1=f\chi_{2B}$. Then we can write that

$$\begin{aligned}
[\vec{b},T]f(y) &= \prod_{i=1}^m (b_i(y)-\lambda_i)Tf(y) + (-1)^m T\left(\prod_{i=1}^m (b_i-\lambda_i)f_1\right)(y) \\
&\quad +(-1)^m T\left(\prod_{i=1}^m (b_i-\lambda_i)f_2\right)(y) \\
&\quad + \sum_{i=1}^{m-1} \sum_{\sigma\in C_i^m} C_{m,i}(b(y)-\vec{\lambda})_\sigma[\vec{b}_{\sigma'},T]f(y).
\end{aligned}$$

From this, it follows that

$$\begin{aligned}
A_{t_B}([\vec{b},T]f)(y) &= A_{t_B}\left(\prod_{i=1}^m (b_i-\lambda_i)Tf\right)(y) + (-1)^m A_{t_B} \\
&\qquad \left(T\left(\textstyle\prod_{i=1}^m (b_i-\lambda_i)f_1\right)\right)(y) \\
&\quad +(-1)^m A_{t_B}\left(T\left(\prod_{i=1}^m (b_i-\lambda_i)f_2\right)\right)(y) \\
&\quad + \sum_{i=1}^{m-1} \sum_{\sigma\in C_i^m} C_{m,i}A_{t_B}((b-\vec{\lambda})_\sigma[\vec{b}_{\sigma'},T]f)(y)
\end{aligned}$$

Let $y\in B$. Now we estimate $|[\vec{b},T]f(y)-A_{t_B}([\vec{b},T]f)(y)|$ by

$$\begin{aligned}
&|[\vec{b},T]f(y)-A_{t_B}([\vec{b},T]f)(y)| \\
&\le \left|\prod_{i=1}^m (b_i(y)-\lambda_i)Tf(y)\right| + \left|T\left(\prod_{i=1}^m (b_i-\lambda_i)f_1\right)(y)\right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|A_{t_B}\left(\prod_{i=1}^{m}(b_i-\lambda_i)Tf\right)(y)\right| + \left|A_{t_B}\left(T\left(\prod_{i=1}^{m}(b_i-\lambda_i)f_1\right)\right)(y)\right| \\
&+\left|T\left(\prod_{i=1}^{m}(b_i-\lambda_i)f_2\right)(y) - A_{t_B}\left(T\left(\prod_{i=1}^{m}(b_i-\lambda_i)f_2\right)\right)(y)\right| \\
&+C\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}|(b(y)-\vec{\lambda})_\sigma[\vec{b}_{\sigma'},T]f(y)| \\
&+C\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}|A_{t_B}((b-\vec{\lambda})_\sigma[\vec{b}_{\sigma'},T]f)(y)| \\
=\;& F_1(y)+F_2(y)+F_3(y)+F_4(y)+F_5(y)+F_6(y)+F_7(y).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{\mu(B)}\int_B |[\vec{b},T]f(y)-A_{t_B}([\vec{b},T]f)(y)|d\mu(y) \\
&\quad\leq \sum_{j=1}^{7}\frac{1}{\mu(B)}\int_B F_j(y)d\mu(y) = \sum_{j=1}^{7} I_j(x).
\end{aligned}
\tag{3.3}
$$

Let q' be the dual exponent of q such that $1/q+1/q'=1$. We first estimate I_1. The Hölder inequality and Lemma 2.1 tell us that

$$
\begin{aligned}
I_1(x) =\;& \frac{1}{\mu(B)}\int_B \prod_{i=1}^{m}|b_i(y)-\lambda_i||T(f)(y)|\,d\mu(y) \\
\leq\;& \left[\frac{1}{\mu(B)}\int_B \prod_{i=1}^{m}|b_i(y)-\lambda_i|^{q'}d\mu(y)\right]^{1/q'}\left[\frac{1}{\mu(B)}\int_B |T(f)(y)|^q\,d\mu(y)\right]^{1/q} \\
\leq\;& C\prod_{i=1}^{m}\|b_i\|_* M_q(Tf)(x),
\end{aligned}
$$

where $\lambda_i=(b_i)_{2B}, i=1,\cdots,m$. For the term I_2, Since T is bounded on $L^p(\mathcal{X})$ for $1<p<\infty$, by the Hölder inequality and Lemma 2.1 again, we have

$$
\begin{aligned}
I_2(x) \;\leq\;& \left[\frac{1}{\mu(B)}\int_B \left|T\left(\prod_{i=1}^{m}(b_i(y)-\lambda_i)f_1\right)(y)\right|^s d\mu(y)\right]^{1/s} \\
\leq\;& C\left[\frac{1}{\mu(B)}\int_{2B} \left|\prod_{i=1}^{m}(b_i(y)-\lambda_i)f(y)\right|^s d\mu(y)\right]^{1/s} \\
\leq\;& C\left[\frac{1}{\mu(B)}\int_B \prod_{i=1}^{m}|b_i(y)-\lambda_i|^{sq'}d\mu(y)\right]^{1/sq'}\left[\frac{1}{\mu(B)}\int_{2B} |f(y)|^{sq}\,d\mu(y)\right]^{1/sq} \\
\leq\;& C\prod_{i=1}^{m}\|b_i\|_* M_{sq}(f)(x).
\end{aligned}
$$

Similarly, by Lemma 2.1, Lemma 2.3 and the $L^p(\mathcal{X})$ boundedness of T, we obtain

$$I_3(x)+I_4(x)\le C\prod_{i=1}^m\|b_i\|_*\left[M_q(Tf)(x)+M_{sq}(f)(x)\right].$$

Now we turn to estimate the term $I_5(x)$. By the assumption (c), we have

$$\begin{aligned}
I_5(x) &\le \frac{1}{\mu(B)}\int_B\int_{\mathcal{X}\setminus 2B}\left|k_{t_B}(y,z)\prod_{i=1}^m(b_i(z)-\lambda_i)f(z)\right|d\mu(z)d\mu(y)\\
&\le C\sum_{k=1}^\infty\int_{2^kr_B\le d(x_0,z)<2^{k+1}r_B}\frac{1}{\mu(B(x_0,d(x_0,z)))}\frac{r_B^\alpha}{d(x_0,z)^\alpha}\\
&\quad\left|\prod_{i=1}^m(b_i(z)-\lambda_i)f(z)\right|d\mu(z)
\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{k=1}^\infty 2^{-k\alpha}\frac{1}{\mu(B(x_0,2^kr_B))}\int_{d(x_0,z)<2^{k+1}r_B}\left|\prod_{i=1}^m(b_i(z)-\lambda_i)f(z)\right|d\mu(z)\\
&\le C\sum_{k=1}^\infty 2^{-k\alpha}\frac{1}{\mu(B(x_0,2^kr_B))}\\
&\quad\times\int_{d(x_0,z)<2^{k+1}r_B}\left|\prod_{i=1}^m(b_i(z)-b_{i,2^{k+1}B}+b_{i,2^{k+1}B}-b_{i,B})f(z)\right|d\mu(z)\\
&\le C\sum_{k=1}^\infty 2^{-k\alpha}\frac{1}{\mu(B(x_0,2^kr_B))}\\
&\quad\int_{d(x_0,z)<2^{k+1}r_B}\sum_{i=1}^m\sum_{\sigma\in C_i^m}|(b_i(z)-b_{i,2^{k+1}B})_\sigma(b_{i,2^{k+1}B}-b_{i,B})_{\sigma'}f(z)|d\mu(z)\\
&\le C\sum_{i=1}^m\sum_{\sigma\in C_i^m}\sum_{k=1}^\infty 2^{-k\alpha}\frac{1}{\mu(B(x_0,2^kr_B))}|(b_{j,2^{k+1}B}-b_{j,B})_{\sigma'}|\\
&\quad\times\int_{d(x_0,z)<2^{k+1}r_B}|(b_j(z)-b_{j,2^{k+1}B})_\sigma f(z)|d\mu(z)\\
&\le C\sum_{i=1}^m\sum_{\sigma\in C_i^m}\sum_{k=1}^\infty 2^{-k\alpha}(k+1)^{m-i}\prod_{j\in\sigma'}\|b_j\|_*\\
&\quad\times\frac{1}{\mu(B(x_0,2^kr_B))}\int_{d(x_0,z)<2^{k+1}r_B}|(b_j(z)-b_{j,2^{k+1}B})_\sigma f(z)|d\mu(z)\\
&\le C\sum_{i=1}^m\sum_{\sigma\in C_i^m}\sum_{k=1}^\infty 2^{-k\alpha}(k+1)^{m-i}\prod_{j=1}^m\|b_j\|_*M_qf(x)\\
&\le C\prod_{j=1}^m\|b_j\|_*M_qf(x),
\end{aligned}$$

where $b_{i,2^{k+1}B} = \frac{1}{\mu(2^{k+1}B)} \int_{2^{k+1}B} b_i(z)dz$. Finally, by an argument similar to above, we can obtain

$$I_6(x) + I_7(x) \leq C \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} \prod_{j \in \sigma} \|b_j\|_* M_q([\vec{b}_{\sigma'}, T]f(x).$$

Combining the estimates for $I_1(x)$ to $I_7(x)$ with (3.3) and then taking supremum over all balls containing x in (3.3) gives us (3.2), which completes the proof of Lemma 3.1.

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