

## MULTIPLE POSITIVE SOLUTIONS OF CONJUGATE BOUNDARY VALUE PROBLEMS ON TIME SCALES

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**Abstract.** We consider the following differential equation on a time scale  $\mathbf{T}$

$$y^{\Delta\Delta}(t) + P(t, y(\sigma(t))) = 0, \quad t \in [a, b] \cap \mathbf{T}$$

subject to conjugate boundary conditions

$$y(a) = 0, \quad y(\sigma^2(b)) = 0$$

where  $a, b \in \mathbf{T}$  and  $a < \sigma(b)$ . By using different fixed point theorems, criteria are established for the existence of three positive solutions of the boundary value problem. Examples are also included to illustrate the results obtained.

### 1. INTRODUCTION

In this paper we shall consider the conjugate boundary value problem on a time scale  $\mathbf{T}$

$$(1.1) \quad \begin{aligned} y^{\Delta\Delta}(t) + P(t, y(\sigma(t))) &= 0, \quad t \in [a, b] \\ y(a) = 0, \quad y(\sigma^2(b)) &= 0 \end{aligned}$$

where  $a, b \in \mathbf{T}$  with  $a < \sigma(b)$ , and  $P : [a, \sigma(b)] \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous.

To understand the notations used in (1.1), we recall some standard definitions as follows. The reader may refer to [1.1] for an introduction to the subject.

- (a) Let  $\mathbf{T}$  be a time scale, i.e.,  $\mathbf{T}$  is a closed subset of  $\mathbb{R}$ . We assume that  $\mathbf{T}$  has the topology that it inherits from the standard topology on  $\mathbb{R}$ . Throughout, for any  $a, b (> a)$ , the interval  $[a, b]$  is defined as  $[a, b] = \{t \in \mathbf{T} \mid a \leq t \leq b\}$ . Analogous notations for open and half-open intervals will also be used in the paper.

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- (b) For  $t < \sup \mathbf{T}$  and  $s > \inf \mathbf{T}$ , the *forward jump operator*  $\sigma$  and the *backward jump operator*  $\rho$  are respectively defined by

$$\sigma(t) = \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T} \quad \text{and} \quad \rho(s) = \sup\{\tau \in \mathbf{T} \mid \tau < s\} \in \mathbf{T}.$$

- (c) Fix  $t \in \mathbf{T}$ . Let  $y : \mathbf{T} \rightarrow \mathbb{R}$ . We define  $y^\Delta(t)$  to be the number (if it exists) with the property that given  $\epsilon > 0$ , there is a neighborhood  $U$  of  $t$  such that for all  $s \in U$ ,

$$|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]| < \epsilon|\sigma(t) - s|.$$

We call  $y^\Delta(t)$  the *delta derivative* of  $y(t)$ . Define  $y^{\Delta\Delta}(t)$  to be the delta derivative of  $y^\Delta(t)$ , i.e.,  $y^{\Delta\Delta}(t) = (y^\Delta(t))^\Delta$ .

- (d) If  $F^\Delta(t) = f(t)$ , then we define the integral

$$\int_a^t f(\tau)\Delta\tau = F(t) - F(a).$$

A solution  $y$  of (1.1) will be sought in  $C[a, \sigma^2(b)]$ , the space of continuous functions  $\{y : [a, \sigma^2(b)] \rightarrow \mathbb{R}\}$ . We say that  $y$  is a *positive solution* if  $y(t) \geq 0$  for  $t \in [a, \sigma^2(b)]$ .

Boundary value problems have attracted a lot of attention in the recent literature, due mainly to the fact that they model many physical phenomena which include gas diffusion through porous media, nonlinear diffusion generated by nonlinear sources, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, adiabatic tubular reactor processes, as well as concentration in chemical or biological problems, just to name a few. In all these physical problems, only *positive* solutions are meaningful. Many papers have discussed the existence of single, double and triple positive solutions of boundary value problems on the real and discrete domains, we refer to [6,8,12,13,15,17-19] and the monographs [2,3] which give a good documentary of the literature. A recent trend is to consider boundary value problems on time scales, which include the real and the discrete as special cases, see [1,4,9-11,14].

In the present work, *both* fixed point theorems of Leggett and Williams [16] as well as of Avery [5] are used to derive criteria for the existence of *triple positive solutions* of (1.1). In addition, estimates on the norms of these solutions are also provided. Not only that *new* results are obtained, we also discuss the relationship between the results in terms of generality, and illustrate the importance of the results through some examples. Moreover, it is noted that the boundary value problem (1.1) considered has a nonlinear term  $P$  which is *more general* than those discussed in the literature.

The paper is outlined as follows. In Section 2 we state the necessary definitions and fixed point theorems. Our main results and discussion are presented in Section 3. Finally, some examples are included in Section 4 to illustrate the importance of the results obtained.

### 3. PRELIMINARIES

In this section we shall state some necessary definitions and the relevant fixed point theorems. Let  $B$  be a Banach space with norm  $\| \cdot \|$ .

**Definition 2.1.** Let  $C (\subset B)$  be a nonempty closed convex set. We say that  $C$  is a *cone* provided the following conditions are satisfied:

- (a) If  $u \in C$  and  $\alpha \geq 0$ , then  $\alpha u \in C$ ;
- (b) If  $u \in C$  and  $-u \in C$ , then  $u = 0$ .

**Definition 2.2.** Let  $C (\subset B)$  be a cone. A map  $\psi$  is a *nonnegative continuous concave functional* on  $C$  if the following conditions are satisfied:

- (a)  $\psi : C \rightarrow \mathbb{R}^+ \cup \{0\}$  is continuous;
- (b)  $\psi(ty + (1 - t)z) \geq t\psi(y) + (1 - t)\psi(z)$  for all  $y, z \in C$  and  $0 \leq t \leq 1$ .

**Definition 2.3.** Let  $C (\subset B)$  be a cone. A map  $\beta$  is a *nonnegative continuous convex functional* on  $C$  if the following conditions are satisfied:

- (a)  $\beta : C \rightarrow \mathbb{R}^+ \cup \{0\}$  is continuous;
- (b)  $\beta(ty + (1 - t)z) \leq t\beta(y) + (1 - t)\beta(z)$  for all  $y, z \in C$  and  $0 \leq t \leq 1$ .

Let  $\gamma, \beta, \Theta$  be nonnegative continuous convex functionals on  $C$  and  $\alpha, \psi$  be nonnegative continuous concave functionals on  $C$ . For nonnegative numbers  $w_i, 1 \leq i \leq 3$ , we shall introduce the following notations:

$$C(w_1) = \{u \in C \mid \|u\| < w_1\},$$

$$C(\psi, w_1, w_2) = \{u \in C \mid \psi(u) \geq w_1 \text{ and } \|u\| \leq w_2\},$$

$$P(\gamma, w_1) = \{u \in C \mid \gamma(u) < w_1\},$$

$$P(\gamma, \alpha, w_1, w_2) = \{u \in C \mid \alpha(u) \geq w_1 \text{ and } \gamma(u) \leq w_2\},$$

$$Q(\gamma, \beta, w_1, w_2) = \{u \in C \mid \beta(u) \leq w_1 \text{ and } \gamma(u) \leq w_2\},$$

$$P(\gamma, \Theta, \alpha, w_1, w_2, w_3) = \{u \in C \mid \alpha(u) \geq w_1, \Theta(u) \leq w_2 \text{ and } \gamma(u) \leq w_3\},$$

$$Q(\gamma, \beta, \psi, w_1, w_2, w_3) = \{u \in C \mid \psi(u) \geq w_1, \beta(u) \leq w_2 \text{ and } \gamma(u) \leq w_3\}.$$

The following fixed point theorems are needed later. The first is usually called *Leggett-Williams' fixed point theorem*, and the second is known as the *five-functional fixed point theorem*.

**Theorem 2.1.** [16]. *Let  $C (\subset B)$  be a cone, and  $w_4 > 0$  be given. Assume that  $\psi$  is a nonnegative continuous concave functional on  $C$  such that  $\psi(u) \leq \|u\|$  for all  $u \in \overline{C}(w_4)$ , and let  $S : \overline{C}(w_4) \rightarrow \overline{C}(w_4)$  be a continuous and completely continuous operator. Suppose that there exist numbers  $w_1, w_2, w_3$  where  $0 < w_1 < w_2 < w_3 \leq w_4$  such that*

- (a)  $\{u \in C(\psi, w_2, w_3) \mid \psi(u) > w_2\} \neq \emptyset$ , and  $\psi(Su) > w_2$  for all  $u \in C(\psi, w_2, w_3)$ ;
- (b)  $\|Su\| < w_1$  for all  $u \in \overline{C}(w_1)$ ;
- (c)  $\psi(Su) > w_2$  for all  $u \in C(\psi, w_2, w_4)$  with  $\|Su\| > w_3$ .

Then,  $S$  has (at least) three fixed points  $u^1, u^2$  and  $u^3$  in  $\overline{C}(w_4)$ . Furthermore, we have

$$(2.1) \quad \begin{aligned} u^1 \in C(w_1), \quad u^2 \in \left\{ u \in C(\psi, w_2, w_4) \mid \psi(u) > w_2 \right\} \\ \text{and } u^3 \in \overline{C}(w_4) \setminus (C(\psi, w_2, w_4) \cup \overline{C}(w_1)). \end{aligned}$$

**Theorem 2.2.** [5]. *Let  $C (\subset B)$  be a cone. Assume that there exist positive numbers  $w_5, M$ , nonnegative continuous convex functionals  $\gamma, \beta, \Theta$  on  $C$ , and nonnegative continuous concave functionals  $\alpha, \psi$  on  $C$ , with*

$$\alpha(u) \leq \beta(u) \quad \text{and} \quad \|u\| \leq M\gamma(u)$$

for all  $u \in \overline{P}(\gamma, w_5)$ . Let  $S : \overline{P}(\gamma, w_5) \rightarrow \overline{P}(\gamma, w_5)$  be a continuous and completely continuous operator. Suppose that there exist nonnegative numbers  $w_i, 1 \leq i \leq 4$  with  $0 < w_2 < w_3$  such that

- (a)  $\{u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(u) > w_3\} \neq \emptyset$ , and  $\alpha(Su) > w_3$  for all  $u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$ ;
- (b)  $\{u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(u) < w_2\} \neq \emptyset$ , and  $\beta(Su) < w_2$  for all  $u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$ ;
- (c)  $\alpha(Su) > w_3$  for all  $u \in P(\gamma, \alpha, w_3, w_5)$  with  $\Theta(Su) > w_4$ ;
- (d)  $\beta(Su) < w_2$  for all  $u \in Q(\gamma, \beta, w_2, w_5)$  with  $\psi(Su) < w_1$ .

Then,  $S$  has (at least) three fixed points  $u^1$ ,  $u^2$  and  $u^3$  in  $\overline{P}(\gamma, w_5)$ . Furthermore, we have

$$(2.2) \quad \beta(u^1) < w_2, \quad \alpha(u^2) > w_3, \quad \text{and} \quad \beta(u^3) > w_2 \quad \text{with} \quad \alpha(u^3) < w_3.$$

### 3. MAIN RESULTS

Let the Banach space

$$(3.1) \quad B = \left\{ y \mid y \in C[a, \sigma^2(b)] \right\}$$

be equipped with norm

$$(3.2) \quad \|y\| = \sup_{t \in [a, \sigma^2(b)]} |y(t)|.$$

To apply the fixed point theorems in Section 2, we need to define an operator  $S : B \rightarrow B$  so that a solution  $y$  of the boundary value problem (1.1) is a fixed point of  $S$ , i.e.,  $y = Sy$ . For this, let  $G(t, s)$  be the Green's function of the boundary value problem

$$(3.3) \quad \begin{aligned} -y^{\Delta\Delta}(t) &= 0, \quad t \in [a, b] \\ y(a) &= 0, \quad y(\sigma^2(b)) = 0. \end{aligned}$$

If  $y$  is a solution of (1.1), then it can be represented as

$$y(t) = \int_a^{\sigma(b)} G(t, s)P(s, y(\sigma(s)))\Delta s, \quad t \in [a, \sigma^2(b)].$$

Hence, we shall define the operator  $S : B \rightarrow B$  by

$$(3.4) \quad Sy(t) = \int_a^{\sigma(b)} G(t, s)P(s, y(\sigma(s)))\Delta s, \quad t \in [a, \sigma^2(b)].$$

It is clear that a fixed point of the operator  $S$  is a solution of (1.1).

Our first lemma gives the properties of the Green's function  $G(t, s)$  which will be used later.

**Lemma 3.1.** *We have the following:*

$$(a) \quad 0 \leq G(t, s) \leq G(\sigma(s), s), \quad (t, s) \in [a, \sigma^2(b)] \times [a, \sigma(b)];$$

(b) for fixed  $\delta$  such that  $0 < \delta < \frac{1}{2}$  and  $[a + \delta(\sigma^2(b) - a), \sigma^2(b) - \delta(\sigma^2(b) - a)] \neq \emptyset$ , we have

$$G(t, s) \geq kG(\sigma(s), s), \quad (t, s) \in [a + \delta(\sigma^2(b) - a), \sigma^2(b) - \delta(\sigma^2(b) - a)] \times [a, \sigma(b)]$$

where the constant  $0 < k < 1$  is given by

$$k = \min \left\{ \delta, \delta \frac{\sigma^2(b) - a}{\sigma^2(b) - \sigma(a)} \right\}.$$

*Proof.* It is known that [7]

$$(3.5) \quad G(t, s) = \begin{cases} \frac{(t-a)(\sigma^2(b) - \sigma(s))}{\sigma^2(b) - a}, & t \leq s \\ \frac{(\sigma(s) - a)(\sigma^2(b) - t)}{\sigma^2(b) - a}, & \sigma(s) \leq t. \end{cases}$$

Hence, (a) is obvious. To prove (b), first we consider the case  $t \leq s$ . Then, from (3.5) we find for  $(t, s) \in [a + \delta(\sigma^2(b) - a), \sigma^2(b) - \delta(\sigma^2(b) - a)] \times [a, \sigma(b)]$ ,

$$(3.6) \quad \frac{G(t, s)}{G(\sigma(s), s)} = \frac{t-a}{\sigma(s)-a} \geq \frac{a + \delta(\sigma^2(b) - a) - a}{\sigma^2(b) - a} = \delta.$$

Next, when  $\sigma(s) \leq t$ , it is clear that for  $(t, s) \in [a + \delta(\sigma^2(b) - a), \sigma^2(b) - \delta(\sigma^2(b) - a)] \times [a, \sigma(b)]$ ,

$$(3.7) \quad \begin{aligned} \frac{G(t, s)}{G(\sigma(s), s)} &= \frac{\sigma^2(b) - t}{\sigma^2(b) - \sigma(s)} \geq \frac{\sigma^2(b) - [\sigma^2(b) - \delta(\sigma^2(b) - a)]}{\sigma^2(b) - \sigma(a)} \\ &= \delta \frac{\sigma^2(b) - a}{\sigma^2(b) - \sigma(a)}. \end{aligned}$$

Combining (3.6) and (3.7) gives (b) immediately.  $\blacksquare$

**Lemma 3.2.** *The operator  $S$  defined in (3.4) is continuous and completely continuous.*

*Proof.* From (3.5), we have  $G(t, s) \in C[a, \sigma^2(b)]$ ,  $t \in [a, \sigma^2(b)]$  and the map  $t \rightarrow G(t, s)$  is continuous from  $[a, \sigma^2(b)]$  to  $C[a, \sigma^2(b)]$ . This together with  $P : [a, \sigma(b)] \times \rightarrow \mathbb{R}$  is continuous ensures (as in [1]) that  $S$  is continuous and completely continuous.  $\blacksquare$

For clarity, we shall list the conditions that are needed later. Note that in these conditions we use the notation  $y^\sigma = y \circ \sigma$ , and the sets  $\tilde{K}$  and  $K$  are given by

$$\tilde{K} = \left\{ y \in B \mid y(t) \geq 0 \text{ for } t \in [a, \sigma^2(b)] \right\}$$

and

$$K = \left\{ y \in \tilde{K} \mid y(t) > 0 \text{ for some } t \in [a, \sigma^2(b)] \right\} = \tilde{K} \setminus \{0\}.$$

(C1) Assume that

$$P(t, y^\sigma) \geq 0, y \in \tilde{K}, t \in [a, \sigma(b)] \quad \text{and} \quad P(t, y^\sigma) > 0, y \in K, t \in [a, \sigma(b)].$$

(C2) There exist continuous functions  $f, \mu, \nu$  with  $f : \mathbb{R}^+ \cup \{0\} \rightarrow \mathbb{R}^+ \cup \{0\}$  and  $\mu, \nu : [a, \sigma(b)] \rightarrow \mathbb{R}^+ \cup \{0\}$  such that

$$\mu(t)f(y^\sigma) \leq P(t, y^\sigma) \leq \nu(t)f(y^\sigma), y \in \tilde{K}, t \in [a, \sigma(b)].$$

(C3) There exists a number  $0 < c \leq 1$  such that

$$\mu(t) \geq c\nu(t), t \in [a, \sigma(b)].$$

Now, let  $\delta$  be fixed, where  $0 < \delta < \frac{1}{2}$  and  $[a + \delta(\sigma^2(b) - a), \sigma^2(b) - \delta(\sigma^2(b) - a)] \neq \emptyset$ . Let  $[a, \sigma^2(b)]$  be such that

$$(3.8) \quad \begin{aligned} \xi &\equiv \min\{t \in \mathbf{T} \mid t \geq a + \delta[\sigma^2(b) - a]\} \quad \text{and} \\ w &\equiv \max\{t \in \mathbf{T} \mid t \leq \sigma^2(b) - \delta[\sigma^2(b) - a]\} \end{aligned}$$

exist and satisfy

$$(3.9) \quad a < a + \delta[\sigma^2(b) - a] \leq \xi < \rho(w) \leq w \leq \sigma^2(b) - \delta[\sigma^2(b) - a] < \sigma^2(b).$$

Next, we define a cone in  $B$  as

$$(3.10) \quad C = \left\{ y \in B \mid \begin{aligned} &y^{\Delta\Delta}(t) \leq 0 \text{ for } t \in [a, b], y(a) = 0 = y(\sigma^2(b)), \text{ and} \\ &y(t) \geq 0 \text{ for } t \in [a, \sigma^2(b)] \end{aligned} \right\}.$$

**Remark 3.1.**

- (a) Note that  $C \subseteq \tilde{K}$ . Moreover, a fixed point of  $S$  obtained in  $C$  will be a *positive solution* of the boundary value problem (1.1).
- (b) Let  $y \in C$ . Then, since  $y^{\Delta\Delta}(t) \leq 0$  for  $t \in [a, b]$ ,  $y^\Delta(t)$  is nonincreasing for  $t \in [a, \sigma(b)]$ , and so  $y^\Delta(a) \geq y^\Delta(\sigma(b))$ . Noting that  $y$  is nonnegative on  $[a, \sigma^2(b)]$  and also  $y(a) = 0 = y(\sigma^2(b))$ , it follows that  $y^\Delta(a) \geq 0 \geq y^\Delta(\sigma(b))$ , and the maximum of  $y$  over the interval  $[a, \sigma^2(b)]$  occurs at some  $t \in (a, \sigma^2(b))$ . Indeed, if we define

$$(3.11) \quad \begin{aligned} \eta_1 &= \sup\{t \in [a, \sigma^2(b)] \mid y^\Delta(t) \geq 0\} \quad \text{and} \\ \eta_2 &= \inf\{t \in [a, \sigma^2(b)] \mid y^\Delta(t) < 0\}, \end{aligned}$$

then

$$\left\{ \begin{array}{l} y(t) \text{ is nondecreasing for } t \in [a, \eta_1], \text{ and is nonincreasing for } t \in [\eta_2, \sigma^2(b)], \\ \eta_1 \leq \eta_2, \\ \text{if } \eta_1 \neq \eta_2, \text{ then } \sigma(\eta_1) = \eta_2, \\ \|y\| = \max\{y(\eta_1), y(\eta_2)\}. \end{array} \right.$$

Taking into account all these and (3.8), (3.9), we see that

$$(3.12) \quad \min_{t \in [\xi, w]} y(t) = \begin{cases} y(\xi), & \xi < w \leq \eta_1 \leq \eta_2 \\ \min\{y(\xi), y(w)\}, & \xi \leq \eta_1 \leq \eta_2 \leq w \\ y(w), & \eta_1 \leq \eta_2 \leq \xi < w \end{cases} \\ = \min\{y(\xi), y(w)\}.$$

**Lemma 3.3.** *Let (C1) hold. Then, the operator  $S$  maps  $C$  into itself.*

*Proof.* Let  $y \in C$ . From (3.4), Lemma 3.1(a) and (C1) we have

$$(3.13) \quad Sy(t) = \int_a^{\sigma(b)} G(t, s)P(s, y(\sigma(s)))\Delta s \geq 0, \quad t \in [a, \sigma^2(b)].$$

Next, it is clear from the property of Green's function that

$$(3.14) \quad Sy(a) = 0 = Sy(\sigma^2(b)).$$

Finally, from (3.4) and (C1) it follows that

$$(3.15) \quad (Sy)^{\Delta\Delta}(t) = -P(t, y(\sigma(t))) \leq 0, \quad t \in [a, b].$$

Hence, (3.13)-(3.15) give  $Sy \in C$ . ■

**Remark 3.2.** If (C1) and (C2) hold, then from (3.4) and Lemma 3.1(a) it follows for  $y \in \tilde{K}$  and  $t \in [a, \sigma^2(b)]$  that

$$(3.16) \quad \int_a^{\sigma(b)} G(t, s)\mu(s)f(y(\sigma(s)))\Delta s \leq Sy(t) \leq \int_a^{\sigma(b)} G(t, s)\nu(s)f(y(\sigma(s)))\Delta s.$$



Moreover, using (3.13), (3.16) and Lemma 3.1(a), we obtain for  $t \in [a, \sigma^2(b)]$ ,

$$\begin{aligned} |Sy(t)| = Sy(t) &\leq \int_a^{\sigma(b)} G(t, s)\nu(s)f(y(\sigma(s)))\Delta s \\ &\leq \int_a^{\sigma(b)} G(\sigma(s), s)\nu(s)f(y(\sigma(s)))\Delta s. \end{aligned}$$

Therefore, we have

$$(3.17) \quad \|Sy\| \leq \int_a^{\sigma(b)} G(\sigma(s), s)\nu(s)f(y(\sigma(s)))\Delta s.$$

For subsequent results, we define the following constants for fixed numbers  $\tau_1, \tau_2, \tau_3, \tau_4 \in [a, \sigma^2(b)]$  where  $\tau_3 > \tau_2$  and  $\tau_4 > \tau_1$  :

$$\begin{aligned} q &= \sup_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} G(t, s)\nu(s)\Delta s, \\ r &= \min\left\{ \int_{\xi}^{\rho(w)} \frac{(\xi - a)(\sigma^2(b) - \sigma(s))\mu(s)}{\sigma^2(b) - a} \Delta s, \right. \\ &\quad \left. \int_{\xi}^{\rho(w)} \frac{(\sigma(s) - a)(\sigma^2(b) - w)\mu(s)}{\sigma^2(b) - a} \Delta s \right\}, \\ (3.18) \quad d_1 &= \min\left\{ \int_{\tau_2}^{\rho(\tau_3)} \frac{(\tau_2 - a)(\sigma^2(b) - \sigma(s))\mu(s)}{\sigma^2(b) - a} \Delta s, \right. \\ &\quad \left. \int_{\tau_2}^{\rho(\tau_3)} \frac{(\sigma(s) - a)(\sigma^2(b) - \tau_3)\mu(s)}{\sigma^2(b) - a} \Delta s \right\}, \\ d_2 &= \max_{t \in [\tau_1, \tau_4]} \int_{\tau_1}^{\rho(\tau_4)} G(t, s)\nu(s)\Delta s, \\ d_3 &= \max_{t \in [\tau_1, \tau_4]} \left[ \int_a^{\tau_1} G(t, s)\nu(s)\Delta s + \int_{\rho(\tau_4)}^{\sigma(b)} G(t, s)\nu(s)\Delta s \right]. \end{aligned}$$

**Lemma 3.4.** *Let (C1) and (C2) hold, and assume*

(C4) *for each  $t \in [a, \sigma^2(b)]$ , the function  $G(t, s)\nu(s)$  is nonzero for some  $s \in [a, \sigma(b)]$ .*

Suppose that there exists a number  $d > 0$  such that for  $0 \leq x \leq d$ ,

$$(3.19) \quad f(x) < \frac{d}{q}.$$

Then,

$$(3.20) \quad S(\overline{C}(d)) \subseteq C(d) \subset \overline{C}(d).$$

*Proof.* Let  $y \in \overline{C}(d)$ . Then, it follows that  $0 \leq y(s) \leq d$  for  $s \in [a, \sigma^2(b)]$ . This implies

$$(3.21) \quad 0 \leq y(\sigma(s)) \leq d, \quad s \in [a, \sigma(b)].$$

Noting (3.21), we apply (3.16), (C4), (3.19), (3.18) and obtain for  $t \in [a, \sigma^2(b)]$ ,

$$\begin{aligned} Sy(t) &\leq \int_a^{\sigma(b)} G(t, s) \nu(s) f(y(\sigma(s))) \Delta s \\ &< \int_a^{\sigma(b)} G(t, s) \nu(s) \frac{d}{q} \Delta s \\ &\leq q \frac{d}{q} = d. \end{aligned}$$

Hence,  $\|Sy\| < d$ . Coupling with the fact that  $Sy \in C$  (Lemma 3.3), we get  $Sy \in C(d)$ . The conclusion (3.20) is now immediate. ■

The next lemma is similar to Lemma 3.4 and its proof is omitted.

**Lemma 3.5.** *Let (C1) and (C2) hold. Suppose that there exists a number  $d > 0$  such that for  $0 \leq x \leq d$ ,*

$$f(x) \leq \frac{d}{q}.$$

Then,

$$S(\overline{C}(d)) \subseteq \overline{C}(d).$$

We are now ready to establish existence criteria for three positive solutions. Our first result employs Theorem 2.1.

**Theorem 3.1.** *Let (C1)-(C4) hold, and assume*

(C5) *the functions  $[\sigma^2(b) - \sigma(s)]\mu(s)$  and  $[\sigma(x) - a]\mu(x)$  are nonzero for some  $s, x \in [\xi, \rho(w)]$ .*

*Suppose that there exist numbers  $w_1, w_2, w_3$  with*

$$0 < w_1 < w_2 < \frac{w_2}{kc} \leq w_3$$

*such that the following hold:*

(P)  $f(x) < \frac{w_1}{q}$  for  $0 \leq x \leq w_1$ ;

(Q) one of the following holds:

(Q1)  $\limsup_{x \rightarrow \infty} \frac{f(x)}{x} < \frac{1}{q}$ ;

(Q2) there exists a number  $d$  ( $\geq w_3$ ) such that  $f(x) \leq \frac{d}{q}$  for  $0 \leq x \leq d$ ;

(R)  $f(x) > \frac{w_2}{r}$  for  $w_2 \leq x \leq w_3$ .

Then, the boundary value problem (1.1) has (at least) three positive solutions  $y^1, y^2, y^3 \in C$  such that

$$(3.22) \quad \begin{aligned} & \|y^1\| < w_1; \quad y^2(t) > w_2, \quad t \in [\xi, w]; \\ & \|y^3\| > w_1 \quad \text{and} \quad \min_{t \in [\xi, w]} y^3(t) = \min \{y^3(\xi), y^3(w)\} < w_2. \end{aligned}$$

*Proof.* We shall employ Theorem 2.1. First, we shall prove that condition (Q) implies the existence of a number  $w_4$  where  $w_4 \geq w_3$  such that

$$(3.23) \quad S(\overline{C}(w_4)) \subseteq \overline{C}(w_4).$$

Suppose that (Q2) holds. Then, by Lemma 3.5 we immediately have (3.23) where we pick  $w_4 = d$ . Suppose now that (Q1) is satisfied. Then, there exist  $N > 0$  and  $\epsilon < \frac{1}{q}$  such that

$$(3.24) \quad \frac{f(x)}{x} < \epsilon, \quad x > N.$$

Define  $M_0 = \max_{0 \leq x \leq N} f(x)$ . In view of (3.24), it is clear that the following holds for all  $x \in \mathbb{R}$ ,

$$(3.25) \quad f(x) \leq M_0 + \epsilon x.$$

Now, pick the number  $w_4$  so that

$$(3.26) \quad w_4 > \max \left\{ w_3, M_0 \left( \frac{1}{q} - \epsilon \right)^{-1} \right\}.$$

Let  $y \in \overline{C}(w_4)$ . Then,  $0 \leq y(s) \leq w_4$  for  $s \in [a, \sigma^2(b)]$ . This implies

$$(3.27) \quad 0 \leq y(\sigma(s)) \leq w_4, \quad s \in [a, \sigma(b)].$$

Then, using (3.16), (3.25), (3.27) and (3.26) we find for  $t \in [a, \sigma^2(b)]$ ,

$$\begin{aligned}
Sy(t) &\leq \int_a^{\sigma(b)} G(t, s)\nu(s)f(y(\sigma(s)))\Delta s \\
&\leq \int_a^{\sigma(b)} G(t, s)\nu(s)[M_0 + \epsilon y(\sigma(s))]\Delta s \\
&\leq \int_a^{\sigma(b)} G(t, s)\nu(s)[M_0 + \epsilon w_4]\Delta s \\
&\leq q(M_0 + \epsilon w_4) \\
&< q[w_4(\frac{1}{q} - \epsilon) + \epsilon w_4] = w_4.
\end{aligned}$$

This leads to  $\|Sy\| < w_4$  and so  $Sy \in C(w_4) \subset \overline{C}(w_4)$ . Thus, (3.23) follows immediately.

Let  $\psi : C \rightarrow \mathbb{R}^+ \cup \{0\}$  be defined by

$$(3.28) \quad \psi(y) = \min_{t \in [\xi, w]} y(t) = \min\{y(\xi), y(w)\}$$

where we have also used (3.12) in the second equality. Clearly,  $\psi$  is a nonnegative continuous concave functional on  $C$  and  $\psi(y) \leq \|y\|$  for all  $y \in C$ .

We shall verify that condition (a) of Theorem 2.1 is satisfied. First, we note that

$$y^*(t) = \frac{w_2 + w_3}{2} \in \left\{ y \in C(\psi, w_2, w_3) \mid \psi(y) > w_2 \right\}.$$

Thus,  $\{y \in C(\psi, w_2, w_3) \mid \psi(y) > w_2\} \neq \emptyset$ . Next, let  $y \in C(\psi, w_2, w_3)$ . Then,  $w_2 \leq \psi(y) \leq \|y\| \leq w_3$  provides  $w_2 \leq y(s) \leq w_3$  for  $s \in [\xi, w]$ , which leads to

$$(3.29) \quad w_2 \leq y(\sigma(s)) \leq w_3, \quad s \in [\xi, \rho(w)].$$

Noting (3.29), we apply (3.16), (3.29), (C5), (R) and (3.18) to get

$$\begin{aligned}
\psi(Sy) &= \min\{Sy(\xi), Sy(w)\} \\
&\geq \min \left\{ \int_a^{\sigma(b)} G(\xi, s)\mu(s)f(y(\sigma(s)))\Delta s, \int_a^{\sigma(b)} G(w, s)\mu(s)f(y(\sigma(s)))\Delta s \right\} \\
&\geq \min \left\{ \int_\xi^{\rho(w)} G(\xi, s)\mu(s)f(y(\sigma(s)))\Delta s, \int_\xi^{\rho(w)} G(w, s)\mu(s)f(y(\sigma(s)))\Delta s \right\}
\end{aligned}$$

$$\begin{aligned}
 &= \min \left\{ \int_{\xi}^{\rho(w)} \frac{(\xi - a)(\sigma^2(b) - \sigma(s))\mu(s)}{\sigma^2(b) - a} f(y(\sigma(s)))\Delta s, \right. \\
 &\quad \left. \int_{\xi}^{\rho(w)} \frac{(\sigma(s) - a)(\sigma^2(b) - w)\mu(s)}{\sigma^2(b) - a} f(y(\sigma(s)))\Delta s \right\} \\
 &> \min \left\{ \int_{\xi}^{\rho(w)} \frac{(\xi - a)(\sigma^2(b) - \sigma(s))\mu(s)}{\sigma^2(b) - a} \frac{w_2}{r} \Delta s, \right. \\
 &\quad \left. \int_{\xi}^{\rho(w)} \frac{(\sigma(s) - a)(\sigma^2(b) - w)\mu(s)}{\sigma^2(b) - a} \frac{w_2}{r} \Delta s \right\} \\
 &= r \frac{w_2}{r} = w_2.
 \end{aligned}$$

Therefore, we have shown that  $\psi(Sy) > w_2$  for all  $y \in C(\psi, w_2, w_3)$ .

Next, condition (b) of Theorem 2.1 is fulfilled since by Lemma 3.4 and condition (P), we have  $S(\overline{C}(w_1)) \subseteq C(w_1)$ .

Finally, we shall show that condition (c) of Theorem 2.1 holds. Let  $y \in C(\psi, w_2, w_4)$  with  $\|Sy\| > w_3$ . Using (3.16), Lemma 3.1(b), (C3) and (3.17), we get

$$\begin{aligned}
 \psi(Sy) &= \min\{Sy(\xi), Sy(w)\} \\
 &\geq \min\left\{ \int_a^{\sigma(b)} G(\xi, s)\mu(s)f(y(\sigma(s)))\Delta s, \right. \\
 &\quad \left. \int_a^{\sigma(b)} G(w, s)\mu(s)f(y(\sigma(s)))\Delta s \right\} \\
 &\geq \min\left\{ \int_a^{\sigma(b)} kG(\sigma(s), s)c\nu(s)f(y(\sigma(s)))\Delta s, \right. \\
 &\quad \left. \int_a^{\sigma(b)} kG(\sigma(s), s)c\nu(s)f(y(\sigma(s)))\Delta s \right\} \\
 &= kc \int_a^{\sigma(b)} G(\sigma(s), s)\nu(s)f(y(\sigma(s)))\Delta s \\
 &\geq kc\|Sy\|kcw_3 \geq kc \frac{w_2}{kc} = w_2.
 \end{aligned}$$

Hence, we have proved that  $\psi(Sy) > w_2$  for all  $y \in C(\psi, w_2, w_4)$  with  $\|Sy\| > w_3$ .

It now follows from Theorem 2.1 that the boundary value problem (1.1) has (at least) three positive solutions  $y^1, y^2, y^3 \in \overline{C}(w_4)$  satisfying (2.1). It is easy to see that here (2.1) reduces to (3.22). ■

We shall now employ Theorem 2.2 to give other existence criteria. In applying Theorem 2.2 it is possible to choose the functionals and constants in many different ways. We shall present two results to show the arguments involved. In particular the first result is a generalization of Theorem 3.1.

**Theorem 3.2.** *Let (C1)-(C3) hold. Assume there exist  $\tau_1, \tau_2, \tau_3, \tau_4 \in [a, \sigma^2(b)]$  with*

$$(3.30) \quad \begin{aligned} a \leq \tau_1 \leq \xi \leq \tau_2 < \rho(\tau_3) \leq \rho(w) \leq \tau_4 \leq \sigma^2(b), \\ \tau_3 \leq \tau_4, \quad \tau_1 < \rho(\tau_4) \leq \sigma(b) \end{aligned}$$

such that

(C6) *the functions  $[\sigma^2(b) - \sigma(s)]\mu(s)$  and  $[\sigma(x) - a]\mu(x)$  are nonzero for some  $s, x \in [\tau_2, \rho(\tau_3)]$ ;*

(C7) *for each  $t \in [\tau_1, \tau_4]$ , the function  $G(t, s)\nu(s)$  is nonzero for some  $s \in [\tau_1, \rho(\tau_4)]$ .*

Suppose that there exist numbers  $w_i$ ,  $2 \leq i \leq 5$  with

$$0 < w_2 < w_3 < \frac{w_3}{kc} \leq w_4 \leq w_5 \quad \text{and} \quad w_2 > \frac{d_3}{q} w_5$$

such that the following hold:

$$(P) \quad f(x) < \frac{1}{d_2}(w_2 - \frac{w_5 d_3}{q}) \text{ for } 0 \leq x \leq w_2;$$

$$(Q) \quad f(x) \leq \frac{w_5}{q} \text{ for } 0 \leq x \leq w_5;$$

$$(R) \quad f(x) > \frac{w_3}{d_1} \text{ for } w_3 \leq x \leq w_4.$$

Then, the boundary value problem (1.1) has (at least) three positive solutions  $y^1, y^2, y^3 \in \overline{C}(w_5)$  such that

$$(3.31) \quad \begin{aligned} \max_{t \in [\tau_1, \tau_4]} y^1(t) < w_2; \quad y^2(t) > w_3, \quad t \in [\tau_2, \tau_3]; \\ \max_{t \in [\tau_1, \tau_4]} y^3(t) > w_2 \quad \text{and} \quad \min_{t \in [\tau_2, \tau_3]} y^3(t) = \min\{y^3(\tau_2), y^3(\tau_3)\} < w_3. \end{aligned}$$

*Proof.* In the context of Theorem 2.2, we define the following functionals on  $C$ :

$$(3.32) \quad \begin{aligned} \gamma(y) &= \|y\|, \\ \psi(y) &= \min_{t \in [\xi, w]} y(t) = \min\{y(\xi), y(w)\}, \\ \beta(y) &= \Theta(y) = \max_{t \in [\tau_1, \tau_4]} y(t), \\ \alpha(y) &= \min_{t \in [\tau_2, \tau_3]} y(t) = \min\{y(\tau_2), y(\tau_3)\} \end{aligned}$$

where we have used the idea of (3.12) in the definitions of  $\psi$  and  $\alpha$ .

First, we shall show that the operator  $S$  maps  $\overline{P}(\gamma, w_5)$  into  $\overline{P}(\gamma, w_5)$ . Let  $x \in \overline{P}(\gamma, w_5) = \overline{C}(w_5)$ . Then, we have  $0 \leq x \leq w_5$ . Together with (Q) and Lemma 3.5, we get  $S(\overline{C}(w_5)) \subseteq \overline{C}(w_5)$ , or equivalently  $S : \overline{P}(\gamma, w_5) \rightarrow \overline{P}(\gamma, w_5)$ .

Next, we shall prove that condition (a) of Theorem 2.2 is fulfilled. It is noted that

$$y^*(t) = \frac{w_3 + w_4}{2} \in \left\{ y \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(y) > w_3 \right\}$$

and hence  $\{y \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(y) > w_3\} \neq \emptyset$ . Now, let  $y \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$ . Then, by definition we have  $\alpha(y) \geq w_3$  and  $\Theta(y) \leq w_4$  which imply  $w_3 \leq y(s) \leq w_4$  for  $s \in [\tau_2, \tau_3]$ . Thus, we have

$$(3.33) \quad w_3 \leq y(\sigma(s)) \leq w_4, \quad s \in [\tau_2, \rho(\tau_3)].$$

Noting (3.33), we apply (3.16), (C6), (R) and (3.18) to obtain

$$\begin{aligned} \alpha(Sy) &= \min\{Sy(\tau_2), Sy(\tau_3)\} \\ &\geq \min \left\{ \int_a^{\sigma(b)} G(\tau_2, s)\mu(s)f(y(\sigma(s)))\Delta s, \int_a^{\sigma(b)} G(\tau_3, s)\mu(s)f(y(\sigma(s)))\Delta s \right\} \\ &\geq \min \left\{ \int_{\tau_2}^{\rho(\tau_3)} G(\tau_2, s)\mu(s)f(y(\sigma(s)))\Delta s, \int_{\tau_2}^{\rho(\tau_3)} G(\tau_3, s)\mu(s)f(y(\sigma(s)))\Delta s \right\} \\ &= \min \left\{ \int_{\tau_2}^{\rho(\tau_3)} \frac{(\tau_2 - a)(\sigma^2(b) - \sigma(s))\mu(s)}{\sigma^2(b) - a} f(y(\sigma(s)))\Delta s, \right. \\ &\quad \left. \int_{\tau_2}^{\rho(\tau_3)} \frac{(\sigma(s) - a)(\sigma^2(b) - \tau_3)\mu(s)}{\sigma^2(b) - a} f(y(\sigma(s)))\Delta s \right\} \\ &> \min \left\{ \int_{\tau_2}^{\rho(\tau_3)} \frac{(\tau_2 - a)(\sigma^2(b) - \sigma(s))\mu(s)}{\sigma^2(b) - a} \frac{w_3}{d_1} \Delta s, \right. \\ &\quad \left. \int_{\tau_2}^{\rho(\tau_3)} \frac{(\sigma(s) - a)(\sigma^2(b) - \tau_3)\mu(s)}{\sigma^2(b) - a} \frac{w_3}{d_1} \Delta s \right\} \\ &= d_1 \frac{w_3}{d_1} = w_3. \end{aligned}$$

Hence,  $\alpha(Sy) > w_3$  for all  $y \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$ .

We shall now verify that condition (b) of Theorem 2.2 is satisfied. Let  $w_1$  be such that  $0 < w_1 < w_2$ . We note that

$$y^*(t) = \frac{w_1 + w_2}{2} \in \left\{ y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(y) < w_2 \right\}.$$

Hence,  $\{y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(y) < w_2\} \neq \emptyset$ . Next, let  $y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$ . Then, we have  $\beta(y) \leq w_2$  and  $\gamma(y) \leq w_5$  which provide

$$(3.34) \quad \begin{aligned} 0 \leq y(s) \leq w_2, \quad s \in [\tau_1, \tau_4] \quad \text{or} \quad 0 \leq y(\sigma(s)) \leq w_2, \quad s \in [\tau_1, \rho(\tau_4)]; \\ 0 \leq y(s) \leq w_5, \quad s \in [a, \sigma^2(b)] \quad \text{or} \quad 0 \leq y(\sigma(s)) \leq w_5, \quad s \in [a, \sigma(b)]. \end{aligned}$$

Noting (3.16), (3.34), (C7), (P), (Q) and (3.18), we find

$$\begin{aligned} \beta(Sy) &= \max_{t \in [\tau_1, \tau_4]} Sy(t) \\ &\leq \max_{t \in [\tau_1, \tau_4]} \int_a^{\sigma(b)} G(t, s) \nu(s) f(y(\sigma(s))) \Delta s \\ &= \max_{t \in [\tau_1, \tau_4]} \left[ \int_a^{\tau_1} G(t, s) \nu(s) f(y(\sigma(s))) \Delta s + \int_{\rho(\tau_4)}^{\sigma(b)} G(t, s) \nu(s) f(y(\sigma(s))) \Delta s \right. \\ &\quad \left. + \int_{\tau_1}^{\rho(\tau_4)} G(t, s) \nu(s) f(y(\sigma(s))) \Delta s \right] \\ &< \max_{t \in [\tau_1, \tau_4]} \left[ \int_a^{\tau_1} G(t, s) \nu(s) f(y(\sigma(s))) \Delta s + \int_{\rho(\tau_4)}^{\sigma(b)} G(t, s) \nu(s) \Delta s \right] \frac{w_5}{q} \\ &\quad + \max_{t \in [\tau_1, \tau_4]} \left[ \int_{\tau_1}^{\rho(\tau_4)} G(t, s) \nu(s) \Delta s \right] \frac{1}{d_2} \left( w_2 - \frac{w_5 d_3}{q} \right) \\ &= d_3 \frac{w_5}{q} + d_2 \frac{1}{d_2} \left( w_2 - \frac{w_5 d_3}{q} \right) = w_2. \end{aligned}$$

Therefore,  $\beta(Sy) < w_2$  for all  $y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$ .

Next, we shall show that condition (c) of Theorem 2.2 is met. Using Lemma 3.1(a), we observe that for  $y \in C$ ,

$$(3.35) \quad \begin{aligned} \Theta(Sy) &= \max_{t \in [\tau_1, \tau_4]} Sy(t) \\ &\leq \max_{t \in [\tau_1, \tau_4]} \int_a^{\sigma(b)} G(t, s) \nu(s) f(y(\sigma(s))) \Delta s \\ &\leq \int_a^{\sigma(b)} G(\sigma(s), s) \nu(s) f(y(\sigma(s))) \Delta s. \end{aligned}$$

Moreover, (C3) and Lemma 3.1(b) yield for  $y \in C$ ,

$$(3.36) \quad \begin{aligned} \alpha(Sy) &= \min_{t \in [\tau_2, \tau_3]} Sy(t) \\ &\geq \min_{t \in [\tau_2, \tau_3]} \int_a^{\sigma(b)} G(t, s) \mu(s) f(y(\sigma(s))) \Delta s \\ &\geq \int_a^{\sigma(b)} kG(\sigma(s), s) c\nu(s) f(y(\sigma(s))) \Delta s. \end{aligned}$$



A combination of (3.35) and (3.36) gives

$$(3.37) \quad \alpha(Sy) \geq kc \Theta(Sy), \quad y \in C.$$

Let  $y \in P(\gamma, \alpha, w_3, w_5)$  with  $\Theta(Sy) > w_4$ . Then, it follows from (3.37) that

$$\alpha(Sy) \geq kc \Theta(Sy) > kc w_4 \geq kc \frac{w_3}{kc} = w_3.$$

Thus,  $\alpha(Sy) > w_3$  for all  $y \in P(\gamma, \alpha, w_3, w_5)$  with  $\Theta(Sy) > w_4$ .

Finally, we shall prove that condition (d) of Theorem 2.2 is fulfilled. Let  $y \in Q(\gamma, \beta, w_2, w_5)$  with  $\psi(Sy) < w_1$ . Then, we have  $\beta(y) \leq w_2$  and  $\gamma(y) \leq w_5$  which give (3.34). Using (3.16), (3.34), (C7), (P), (Q) and (3.18), we get as in an earlier part  $\beta(Sy) < w_2$  for all  $y \in Q(\gamma, \beta, w_2, w_5)$  with  $\psi(Sy) < w_1$ .

It now follows from Theorem 2.2 that the boundary value problem (1.1) has (at least) three *positive* solutions  $y^1, y^2, y^3 \in \overline{P}(\gamma, w_5) = \overline{C}(w_5)$  satisfying (2.2). It is clear that (2.2) reduces to (3.31) immediately. ■

Consider the special case when

$$\tau_1 = a, \quad \tau_2 = \xi, \quad \tau_3 = w \quad \text{and} \quad \tau_4 = \sigma^2(b).$$

Then, it is clear that

$$(3.38) \quad d_1 = r, \quad d_2 = q, \quad d_3 = 0.$$

In this case Theorem 3.2 yields the following corollary.

**Corollary 3.1.** *Let (C1) – (C3) hold, and assume*

(C6)' *the functions  $[\sigma^2(b) - \sigma(s)]\mu(s)$  and  $[\sigma(x) - a]\mu(x)$  are nonzero for some  $s, x \in [\xi, \rho(w)]$ ;*

(C7)' *for each  $t \in [a, \sigma^2(b)]$ , the function  $G(t, s)\nu(s)$  is nonzero for some  $s \in [a, \sigma(b)]$ .*

*Suppose that there exist numbers  $w_i, 2 \leq i \leq 5$  with*

$$0 < w_2 < w_3 < \frac{w_3}{kc} \leq w_4 \leq w_5$$

*such that the following hold:*

(P)  $f(x) < \frac{w_2}{q}$  for  $0 \leq x \leq w_2$ ;

(Q)  $f(x) \leq \frac{w_5}{q}$  for  $0 \leq x \leq w_5$ ;

(R)  $f(x) > \frac{w_3}{r}$  for  $w_3 \leq x \leq w_4$ .

Then, the boundary value problem (1.1) has (at least) three positive solutions  $y^1, y^2, y^3 \in \overline{C}(w_5)$  such that

$$\|y^1\| < w_2; \quad y^2(t) > w_3, \quad t \in [\xi, w];$$

$$\|y^3\| > w_2 \quad \text{and} \quad \min_{t \in [\xi, w]} y^3(t) = \min\{y^3(\xi), y^3(w)\} < w_3.$$

**Remark 3.2.** Corollary 3.1 is actually Theorem 3.1. Hence, Theorem 3.2 is more general than Theorem 3.1.

Another application of Theorem 2.2 yields the next result.

**Theorem 3.3.** Let (C1)-(C3) hold. Assume there exist numbers  $\tau_1, \tau_2, \tau_3, \tau_4 \in [a, \sigma^2(b)]$  with

$$(3.39) \quad \xi \leq \tau_1 \leq \tau_2 < \rho(\tau_3) \leq \tau_4 \leq w \quad \text{and} \quad \tau_1 < \rho(\tau_4) \leq \sigma(b)$$

such that (C6) and (C7) hold. Suppose that there exist numbers  $w_i$ ,  $1 \leq i \leq 5$  with

$$0 < w_1 \leq w_2 \cdot kc < w_2 < w_3 < \frac{w_3}{kc} \leq w_4 \leq w_5 \quad \text{and} \quad w_2 > \frac{d_3}{q} w_5$$

such that the following hold:

(P)  $f(x) < \frac{1}{d_2} \left( w_2 - \frac{w_5 d_3}{q} \right)$  for  $w_1 \leq x \leq w_2$ ;

(Q)  $f(x) \leq \frac{w_5}{q}$  for  $0 \leq x \leq w_5$ ;

(R)  $f(x) > \frac{w_3}{d_1}$  for  $w_3 \leq x \leq w_4$ .

Then, the boundary value problem (1.1) has (at least) three positive solutions  $y^1, y^2, y^3 \in \overline{C}(w_5)$  such that

$$(3.40) \quad \max_{t \in [\tau_1, \tau_4]} y^1(t) < w_2; \quad y^2(t) > w_3, \quad t \in [\tau_2, \tau_3];$$

$$\max_{t \in [\tau_1, \tau_4]} y^3(t) > w_2 \quad \text{and} \quad \min_{t \in [\tau_2, \tau_3]} y^3(t) = \min\{y^3(\tau_2), y^3(\tau_3)\} < w_3.$$

*Proof.* In the context of Theorem 2.2, we define the following functionals on  $C$ :

$$\begin{aligned}
 \gamma(y) &= \|y\|, \\
 \psi(y) &= \min_{t \in [\tau_1, \tau_4]} y(t) = \min\{y(\tau_1), y(\tau_4)\}, \\
 \beta(y) &= \max_{t \in [\tau_1, \tau_4]} y(t), \\
 \alpha(y) &= \min_{t \in [\tau_2, \tau_3]} y(t) = \min\{y(\tau_2), y(\tau_3)\}, \\
 \Theta(y) &= \max_{t \in [\tau_2, \tau_3]} y(t).
 \end{aligned}
 \tag{3.41}$$

First, using (Q) it can be shown (as in the proof of Theorem 3.2) that  $S : \overline{P}(\gamma, w_5) \rightarrow \overline{P}(\gamma, w_5)$ .

Next, using (R), (C6) and a similar argument as in the proof of Theorem 3.2, we can verify that condition (a) of Theorem 2.2 is fulfilled.

Now, we shall check that condition (b) of Theorem 2.2 is satisfied. It is clear that

$$y^*(t) = \frac{w_1 + w_2}{2} \in \left\{ y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(y) < w_2 \right\} \neq \emptyset.$$

For  $y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$ , we have  $\psi(y) \geq w_1$ ,  $\beta(y) \leq w_2$  and  $\gamma(y) \leq w_5$  which imply

$$\begin{aligned}
 (3.42) \quad & w_1 \leq y(s) \leq w_2, \quad s \in [\tau_1, \tau_4] \quad \text{or} \quad w_1 \leq y(\sigma(s)) \leq w_2, \quad s \in [\tau_1, \rho(\tau_4)]; \\
 & 0 \leq y(s) \leq w_5, s \in [a, \sigma^2(b)] \quad \text{or} \quad 0 \leq y(\sigma(s)) \leq w_5, \quad s \in [a, \sigma(b)].
 \end{aligned}$$

Using (3.16), (3.42), (C7), (P), (Q) and (3.18), we find, as in the proof of Theorem 3.2, that  $\beta(Sy) < w_2$  for all  $y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$ .

Next, we shall show that condition (c) of Theorem 2.2 is met. We observe that, by (3.16) and Lemma 3.1(a), for  $y \in C$ ,

$$\begin{aligned}
 (3.43) \quad \Theta(Sy) &\leq \max_{t \in [\tau_2, \tau_3]} \int_a^{\sigma(b)} G(t, s) \nu(s) f(y(\sigma(s))) \Delta s \\
 &\leq \int_a^{\sigma(b)} G(\sigma(s), s) \nu(s) f(y(\sigma(s))) \Delta s.
 \end{aligned}$$

Moreover, using (3.16), (C3) and Lemma 3.1(c), we obtain (3.36) for  $y \in C$ . A combination of (3.36) and (3.43) yields (3.37). Following similar argument as in the proof of Theorem 3.2, we get  $\alpha(Sy) > w_3$  for all  $y \in P(\gamma, \alpha, w_3, w_5)$  with  $\Theta(Sy) > w_4$ .

Finally, we shall prove that condition (d) of Theorem 2.2 is fulfilled. By (3.16) and Lemma 3.1(a), we see that for  $y \in C$ ,

$$(3.44) \quad \beta(Sy) = \max_{t \in [\tau_1, \tau_4]} Sy(t) \leq \int_a^{\sigma(b)} G(\sigma(s), s) \nu(s) f(y(\sigma(s))) \Delta s.$$

On the other hand, it follows from (3.16), (C3) and Lemma 3.1(b) that for  $y \in C$ ,

$$(3.45) \quad \begin{aligned} \psi(Sy) &\geq \min_{t \in [\tau_1, \tau_4]} \int_a^{\sigma(b)} G(t, s) \mu(s) f(y(\sigma(s))) \Delta s \\ &\geq \int_a^{\sigma(b)} kG(\sigma(s), s) c \nu(s) f(y(\sigma(s))) \Delta s. \end{aligned}$$

A combination of (3.44) and (3.45) gives

$$(3.46) \quad \psi(Sy) \geq kc \beta(Sy), \quad y \in C.$$

Let  $y \in Q(\gamma, \beta, w_2, w_5)$  with  $\psi(Sy) < w_1$ . Then, (3.46) leads to

$$\beta(Sy) \leq \frac{1}{kc} \psi(Sy) < \frac{1}{kc} w_1 \leq \frac{1}{kc} w_2 \cdot kc = w_2.$$

Thus,  $\beta(Sy) < w_2$  for all  $y \in Q(\gamma, \beta, w_2, w_5)$  with  $\psi(Sy) < w_1$ .

It now follows from Theorem 2.2 that the boundary value problem (1.1) has (at least) three *positive* solutions  $y^1, y^2, y^3 \in \overline{P}(\gamma, w_5) = \overline{C}(w_5)$  satisfying (2.2). Furthermore, (2.2) reduces to (3.40) immediately. ■

#### 4. EXAMPLES

In this section we shall present some examples to illustrate the usefulness of the results obtained. Throughout, we consider the time scale

$$\mathbf{T} = \{2^k \mid k \in \mathbf{Z}\} \cup \{0\}.$$

**Example 4.1.** Consider the boundary value problem (1.1) with  $a = 2$ ,  $b = 2^9 = 512$  and the nonlinear term

$$(4.1) \quad P(t, x) = f(x) = \begin{cases} \frac{w_1}{2q}, & 0 \leq x \leq w_1 \\ l(x), & w_1 \leq x \leq w_2 \\ \frac{1}{2} \left( \frac{d}{q} + \frac{w_2}{r} \right), & x \geq w_2 \end{cases}$$

where  $l(x)$  satisfies

$$(4.2) \quad l''(x) = 0, \quad l(w_1) = \frac{w_1}{2q}, \quad l(w_2) = \frac{1}{2} \left( \frac{d}{q} + \frac{w_2}{r} \right)$$

and  $w_i$ 's,  $d$ ,  $q$  and  $r$  are as in the context of Theorem 3.1.

Let  $\delta = \frac{1}{16}$ . Then, it is easy to see that  $\xi = 256$ ,  $w = 1024$  and  $k = \frac{1}{16}$ . Taking the functions  $\mu = \nu \equiv 1$  (this implies  $c = 1$ ), by direct computation we get

$$(4.3) \quad q = \frac{1046528}{3} = 348842.67 \quad \text{and} \quad r = \frac{16646144}{341} = 48815.67.$$

We assume the  $w_i$ 's and  $d$  are numbers satisfying the relation

$$(4.4) \quad 0 < w_1 < w_2 < \frac{w_2}{kc} = 16w_2 \leq w_3 \leq d.$$

We shall check the conditions of Theorem 3.1. First, it is clear that (C1)–(C5) are fulfilled. Next, condition (P) is obviously satisfied. In view of (4.3) and (4.4), we have

$$(4.5) \quad \frac{w_2}{r} < \frac{d}{q} \quad (\text{or equivalently } d > \frac{q}{r} w_2 = 7.15w_2).$$

Hence, we find for  $0 \leq x \leq d$ ,

$$f(x) \leq \max\{l(w_1), l(w_2)\} = l(w_2) = \frac{1}{2} \left( \frac{d}{q} + \frac{w_2}{r} \right) < \frac{1}{2} \left( \frac{d}{q} + \frac{d}{q} \right) = \frac{d}{q}.$$

Thus, condition (Q2) is met. Finally, (R) is satisfied since for  $w_2 \leq x \leq w_3$ , we have, noting (4.5),

$$f(x) = \frac{1}{2} \left( \frac{d}{q} + \frac{w_2}{r} \right) > \frac{1}{2} \left( \frac{w_2}{r} + \frac{w_2}{r} \right) = \frac{w_2}{r}.$$

By Theorem 3.1, we conclude that the boundary value problem (1.1) with  $a = 2$ ,  $b = 512$  and (4.1)–(4.4) has (at least) three positive solutions  $y^1, y^2, y^3 \in C$  such that

$$(4.6) \quad \|y^1\| < w_1; \quad y^2(t) > w_2, \quad t \in [256, 1024]; \\ \|y^3\| > w_1 \quad \text{and} \quad \min_{t \in [256, 1024]} y^3(t) = \min\{y^3(256), y^3(1024)\} < w_2.$$

To illustrate further, let us fix

$$(4.7) \quad w_1 = 1, \quad w_2 = 2 \quad \text{and} \quad w_3 = d = 32.$$

Clearly, (4.4) is fulfilled. We find that the boundary value problem (1.1) with  $a = 2$ ,  $b = 512$ , (4.7), (4.1)–(4.3) indeed has three positive solutions  $y^1, y^2, y^3 \in C$  that satisfy (4.6). In fact,

$$(4.8) \quad \begin{aligned} \|y^1\| &= y^1(1024) = 0.5 < w_1 = 1; & y^2(t) &> w_2 = 2, \quad t \in [256, 1024]; \\ \|y^3\| &= y^3(512) = 1.1158 > w_1 = 1 \\ \text{and } \min_{t \in [256, 1024]} y^3(t) &= y^3(256) = 0.5868 < w_2 = 2. \end{aligned}$$

The solutions are tabulated as follows:

$t$	4	8	16	32	64	128	256	512	1024
$y^1(t)$	0.0020	0.0058	0.0136	0.0289	0.0588	0.1156	0.2175	0.3743	0.5000
$y^2(t)$	0.0872	0.2616	0.6102	1.3074	2.6908	5.3218	10.0401	17.3026	23.1308
$y^3(t)$	0.0049	0.0146	0.0339	0.0725	0.1489	0.2988	0.5868	1.1158	1.0005

**Example 4.2.** Consider the boundary value problem (1.1) with  $a = 2$ ,  $b = 2^9 = 512$  and the nonlinear term

$$(4.9) \quad P(t, x) = f(x) = \begin{cases} \frac{1}{2d_2} \left( w_2 - \frac{w_5 d_3}{q} \right), & 0 \leq x \leq w_2 \\ l(x), & w_2 \leq x \leq w_3 \\ \frac{1}{2} \left( \frac{w_3}{r} + \frac{w_3}{d_1} \right), & x \geq w_3 \end{cases}$$

where  $l(x)$  satisfies

$$(4.10) \quad l''(x) = 0, \quad l(w_2) = \frac{1}{2d_2} \left( w_2 - \frac{w_2 d_3}{q} \right), \quad l(w_3) = \frac{1}{2} \left( \frac{w_3}{r} + \frac{w_3}{d_1} \right)$$

and  $w_i$ 's,  $d_i$ 's,  $q$  and  $r$  are as in the context of Theorem 3.2.

Fix  $\delta = \frac{1}{17}$ . Then, it follows that  $\xi = 128$ ,  $w = 1024$  and  $k = \frac{1}{17}$ . Taking the functions  $\mu = \nu \equiv 1$  (this implies  $c = 1$ ), and  $\tau_1 = 64$ ,  $\tau_2 = 256$ ,  $\tau_3 = 1024$  and  $\tau_4 = 2048$ , by direct computation we have

$$(4.11) \quad \begin{aligned} q &= 348842.67, & r &= 38341.44, & d_1 &= 48815.67, \\ d_2 &= 78196.36, & d_3 &= 2525.09. \end{aligned}$$

We assume the  $w_i$ 's are numbers satisfying the relation

$$(4.12) \quad 0 < w_2 < w_3 < \frac{w_3}{kc} = 17w_3 \leq w_4 \leq w_5 < \frac{q}{d_3} w_2 = 138.15w_2.$$

We shall check the conditions of Theorem 3.2. Clearly, (C1)–(C3), (C6) and (C7) are fulfilled. Next, condition (P) is obviously satisfied. From (4.11) and (4.12), we note that

$$(4.13) \quad r < d_1 < d_2 \quad \text{and} \quad \frac{w_3}{r} < \frac{w_5}{q} \quad (\text{equivalent to } w_5 > \frac{q}{r} w_3 = 9.1w_3).$$

Thus, we find for  $0 \leq x \leq w_5$ ,

$$f(x) \leq \max\{l(w_2), l(w_3)\} = l(w_3) = \frac{1}{2} \left( \frac{w_3}{r} + \frac{w_3}{d_1} \right) < \frac{1}{2} \left( \frac{w_3}{r} + \frac{w_3}{r} \right) = \frac{w_3}{r} < \frac{w_5}{q}.$$

Hence, condition (Q) is met. Finally, (R) is satisfied since for  $w_3 \leq x \leq w_4$  we have, in view of (4.13),

$$f(x) = \frac{1}{2} \left( \frac{w_3}{r} + \frac{w_3}{d_1} \right) > \frac{1}{2} \left( \frac{w_3}{d_1} + \frac{w_3}{d_1} \right) = \frac{w_3}{d_1}.$$

It follows from Theorem 3.2 that the boundary value problem (1.1) with  $a = 2$ ,  $b = 512$  and (4.9)-(4.12) has (at least) three positive solutions  $y^1, y^2, y^3 \in \overline{C}(w_5)$  such that

$$(4.14) \quad \begin{aligned} & \max_{t \in [64, 2048]} y^1(t) < w_2; \quad y^2(t) > w_3, \quad t \in [256, 1024]; \\ & \max_{t \in [64, 2048]} y^3(t) > w_2 \quad \text{and} \quad \min_{t \in [256, 1024]} y^3(t) = \min\{y^3(256), y^3(1024)\} < w_3. \end{aligned}$$

As an example, fix

$$(4.15) \quad w_2 = 1, \quad w_3 = 2, \quad w_5 = 128 \quad \text{and any } w_4 \text{ such that } 17w_3 \leq w_4 \leq w_5$$

so that (4.12) holds. We find that the boundary value problem (1.1) with  $a = 2$ ,  $b = 512$ , (4.9)-(4.11), (4.15) in fact has three positive solutions  $y^1, y^2, y^3 \in \overline{C}(128)$  satisfying (4.14). Indeed,

$$(4.16) \quad \begin{aligned} & \max_{t \in [64, 2048]} y^1(t) = y^1(1024) = 0.1639 < w_2 = 1; \\ & y^2(t) > w_3 = 2, \quad t \in [256, 1024]; \\ & \max_{t \in [64, 2048]} y^3(t) = y^3(1024) = 1.0755 > w_2 = 1 \quad \text{and} \\ & \min_{t \in [256, 1024]} y^3(t) = y^3(256) = 0.2978 < w_3 = 2. \end{aligned}$$

The solutions are tabulated as follows:

$t$	4	8	16	32	64	128	256	512	1024
$y^1(t)$	0.0006	0.0019	0.0045	0.0095	0.0193	0.0379	0.0713	0.1227	0.1639
$y^2(t)$	0.0603	0.1810	0.4224	0.9050	1.8701	3.7172	7.0298	12.1291	16.2242
$y^3(t)$	0.0024	0.0073	0.0169	0.0362	0.0746	0.1503	0.2978	0.5776	1.0755

**Remark 4.1.** In Example 4.2, noting (4.13) we find for  $w_3 \leq x \leq w_4$ ,

$$f(x) = \frac{1}{2} \left( \frac{w_3}{r} + \frac{w_3}{d_1} \right) < \frac{1}{2} \left( \frac{w_3}{r} + \frac{w_3}{r} \right) = \frac{w_3}{r}.$$

Thus, condition (R) of Corollary 3.1 is *not* satisfied. Recalling that Corollary 3.1 is actually Theorem 3.1, Example 4.2 illustrates the case when Theorem 3.2 is applicable but not Theorem 3.1. Hence, this example shows that Theorem 3.2 is indeed *more general* than Theorem 3.1.

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