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A PRODUCT OF DOUBLING MEASURES ON THE REAL LINE

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Abstract. A product of doubling measures on the real line can be defined in such a way that another doubling measure on the line is obtained. It follows that doubling measures on the line form a semiring.

1. Introduction and Main Result

The main result of this note shows that suitably normalized quasisymmetric maps on the real line can be "multiplied" so that a new quasisymmetric map is obtained (by suitably normalized we mean that they are increasing and fix zero). In terms of doubling measures this means that they form a semiring. Before stating our main theorem precisely we need some definitions.

A measure on a metric space X is *doubling* if there exists a constant $K \geq 1$ such that for every $x \in X$ and every $t > 0$, $\mu(B(x, 2t)) \leq K\mu(B(x, t))$, where $B(x, t)$ denotes the open ball of radius t centered at x. Specializing this definition to the real line, one can easily check that for nontrivial measures this is equivalent to the following: μ is doubling if there exists a constant $K \geq 1$ such that for every $x \in \mathbb{R}$ and every $t > 0$,

$$\frac{1}{K} \leq \frac{\mu([x, x+t])}{\mu([x-t, x])} \leq K.$$

A homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is K-quasisymmetric if

$$\frac{1}{K} \leq \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \leq K,$$

with K, x and t as before. Additional background information on doubling measures and quasisymmetric maps can be obtained, for instance, from [2], as well as from several other sources.

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It is clear from the definitions that there is a close relationship between doubling measures and quasisymmetric maps on $\mathbb{R}$. Given f quasisymmetric, the measure μ_f defined on intervals by $\mu_f([a,b]) := |f(b) - f(a)|$ is doubling. If we assume that f is increasing, we can avoid the use of absolute value signs. Also, from the viewpoint of the defined measure it makes no difference if we add or substract a constant to f, so we may assume that $f(0) = 0$. Thus, with respect to measures it is enough to consider increasing quasisymmetric maps that fix the origin. Given μ, we shall say that f is the map associated to μ if f is increasing, $f(0) = 0$, and $\mu = \mu_f$. In the other direction, every nontrivial doubling measure μ on $\mathbb{R}$ defines an increasing quasisymmetric map f_μ that fixes 0, by setting $f_\mu(x) := \mu([0,x])$ if $x \geq 0$, and $f_\mu(x) := -\mu([x,0])$ if $x < 0$.

If $f, g : [0,\infty) \to [0,\infty)$ are homeomorfisms, their product fg is again a homeomorphism. Here the order structure of the line is crucial: Both f and g are nonnegative strictly increasing functions, and hence so is fg. But in general the product of two bijections need not be a bijection, so the possibility of defining a product via pointwise multiplication on collections of homeomorphisms defined on topological rings seems to be rather limited. To define such a product $\bullet$ on $\mathbb{R}$, we set, for increasing homeomorfisms $f, g : \mathbb{R} \to \mathbb{R}$ that fix the origin, $f \bullet g(x) := f(x)g(x)$ if $x \geq 0$, and $f \bullet g(x) := -f(x)g(x)$ if $x < 0$. If in addition f and g are quasisymmetric, then we call $f \bullet g$ their *quasisymmetric product*, the reason being that $f \bullet g$ is indeed quasisymmetric, as will be shown later. Therefore, this product induces a product of doubling measures via $\mu_f \bullet \mu_g := \mu_{f \bullet g}$. Note that the sum of two doubling measures μ and ν with doubling constants K_1 and K_2 respectively is again a doubling measure: $(\mu + \nu)(B(x, 2t)) = \mu(B(x, 2t)) + \nu(B(x, 2t)) \leq K_1\mu(B(x,t)) + K_2\nu(B(x,t)) \leq (K_1 + K_2)(\mu + \nu)(B(x,t))$. So we have two operations, addition and multiplication, defined on the set of doubling measures. Also, given $a < b$, it is immediate from the definitions that $(\mu_f + \mu_g)([a,b]) = \mu_{f+g}([a,b])$, so addition of measures corresponds to addition of the associated maps.

Definition 1.1. ([4], Def. 2.1 pp. 8-9) A nonempty set S with two binary operations $+, \cdot$ defined on it is called a *semiring* if

(1) $(S, +)$ is a commutative semigroup.
(2) $(S, \cdot)$ is a semigroup.
(3) The distributive laws $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ hold for all $a, b, c \in S$.

If in addition $(S, \cdot)$ is commutative, $(S, +, \cdot)$ is said to be a *commutative* semiring.

Theorem 1.2. *The set of doubling measures on the real line, with operations defined via sums and quasisymmetric products of the associated quasisymmetric functions, is a commutative semiring.*

A comment on terminology: Quite often a more restrictive notion of semiring is used (cf., for instance [1], p.1): Besides the above conditions, it is usually required that there exist an absorbing additive identity 0 (i.e. for every a, $0 = 0 \cdot a = a \cdot 0$) and a multiplicative identity 1. The existence of an absorbing additive identity poses no difficulties: Just consider the constant zero measure. But it is easy to check that no doubling measure can play the role of multiplicative identity, so if we used the terminology from [1], in our main theorem we would have to say that the set of doubling measures on the real line is a commutative *hemiring*, rather than semiring (the only difference between semirings and hemirings as defined in [1] is precisely whether or not of a multiplicative identity exists).

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2. Results and Proofs

Lemma 2.1. *Suppose that either* $0 \le x_1 < x_2 < x_3$ *and* $0 \le y_1 < y_2 < y_3$, *or* $x_1 < x_2 < x_3 \le 0$ *and* $y_1 < y_2 < y_3 \le 0$. *Let* $K_1, K_2 \ge 1$ *be such that*

$$\frac{1}{K_1} \le \frac{x_3 - x_2}{x_2 - x_1} \le K_1 \quad \textit{and} \quad \frac{1}{K_2} \le \frac{y_3 - y_2}{y_2 - y_1} \le K_2.$$

Then

$$\frac{1}{K_1 K_2 + K_1 + K_2} \le \frac{x_3 y_3 - x_2 y_2}{x_2 y_2 - x_1 y_1} \le K_1 K_2 + K_1 + K_2.$$

Proof. Assume first that $0 \le x_1 < x_2 < x_3$ and $0 \le y_1 < y_2 < y_3$. Note that for $i = 1, 2$,

$$x_{i+1}y_{i+1} - x_i y_i = (x_{i+1} - x_i) y_{i+1} + (y_{i+1} - y_i) x_i \ge (x_{i+1} - x_i) y_{i+1}, \tag{2.1.1}$$

$$x_{i+1}y_{i+1} - x_i y_i = (y_{i+1} - y_i) x_{i+1} + (x_{i+1} - x_i) y_i \ge (y_{i+1} - y_i) x_{i+1}, \quad \text{and} \tag{2.1.2}$$

$$\begin{aligned} x_{i+1}y_{i+1} - x_i y_i &= (x_{i+1} - x_i)(y_{i+1} - y_i) + (x_{i+1} - x_i) y_i + (y_{i+1} - y_i) x_i \\ &\ge (x_{i+1} - x_i)(y_{i+1} - y_i). \end{aligned} \tag{2.1.3}$$

To get the upper bound we use (2.1.3), (2.1.1) and (2.1.2) as follows:

$$\begin{aligned} \frac{x_3 y_3 - x_2 y_2}{x_2 y_2 - x_1 y_1} &= \frac{(x_3 - x_2)(y_3 - y_2) + (x_3 - x_2) y_2 + (y_3 - y_2) x_2}{x_2 y_2 - x_1 y_1} \\ &= \frac{(x_3 - x_2)(y_3 - y_2)}{(x_2 - x_1)(y_2 - y_1) + (x_2 - x_1) y_1 + (y_2 - y_1) x_1} \end{aligned}$$

$$
\begin{aligned}
&+\frac{(x_3-x_2)y_2}{(x_2-x_1)y_2+(y_2-y_1)x_1}+\frac{(y_3-y_2)x_2}{(y_2-y_1)x_2+(x_2-x_1)y_1}\\
&\leq \frac{(x_3-x_2)(y_3-y_2)}{(x_2-x_1)(y_2-y_1)}+\frac{(x_3-x_2)y_2}{(x_2-x_1)y_2}+\frac{(y_3-y_2)x_2}{(y_2-y_1)x_2}\\
&\leq K_1K_2+K_1+K_2.
\end{aligned}
$$

Regarding the lower bound, we have:

$$
\begin{aligned}
\frac{x_3y_3-x_2y_2}{x_2y_2-x_1y_1} &= \frac{(y_3-y_2)x_3+(x_3-x_2)y_2}{(y_2-y_1)x_2+(x_2-x_1)y_1}\\
\geq \frac{(y_3-y_2)x_2+(x_3-x_2)y_1}{(y_2-y_1)x_2+(x_2-x_1)y_1} &= \frac{1}{\dfrac{(y_2-y_1)x_2+(x_2-x_1)y_1}{(y_3-y_2)x_2+(x_3-x_2)y_1}}\\
\geq \frac{1}{\dfrac{(y_2-y_1)x_2}{(y_3-y_2)x_2}+\dfrac{(x_2-x_1)y_1}{(x_3-x_2)y_1}} &\geq \frac{1}{K_1+K_2}\geq\frac{1}{K_1K_2+K_1+K_2}.
\end{aligned}
$$

The case where $x_1 < x_2 < x_3 \leq 0$ and $y_1 < y_2 < y_3 \leq 0$ follows immediately by applying the previous argument to $-x_1 > -x_2 > -x_3 \geq 0$, $-y_1 > -y_2 > -y_3 \geq 0$, and simplifying. ■

The next theorem is essentially the same as Theorem 3.1 of [3], the difference being that we work on the whole real line, rather than the interval $[-1, M]$. The proof can be adapted without difficulty (in fact it is simpler in our case), and we include it here for the reader's convenience. I am indebted to Professor Juha Heinonen for pointing out this result to me.

Theorem 2.1. (Heinonen and Hinkkanen) *Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing homeomorphism with $f(0) = 0$. If the restrictions of f to $(-\infty, 0]$ and $[0, \infty)$ are K-quasisymmetric maps, and for every $t > 0$*

$$\frac{1}{K} \leq \frac{f(t)}{-f(-t)} \leq K,$$

then f is $(K + 1)^3$-quasisymmetric on $\mathbb{R}$.

Proof. By hypothesis, it is enough to consider the case where $x-t < 0 < x+t$ (so $x < t$), and we may also assume that $x > 0$ (the argument for $x < 0$ is similar). Since $f(0) = 0$, given $y > 0$, from

$$\text{(2.2.1)}\qquad \frac{1}{K} \leq \frac{f(2y)-f(y)}{f(y)-f(0)} \leq K \quad\text{and}\quad \frac{1}{K} \leq \frac{-f(-y)}{f(y)} \leq K,$$

we obtain

$$\left(\frac{1}{K}+1\right) f(y) \leq f(2y) \leq (K+1) f(y), \quad \text{so} \quad \frac{K+1}{K} \leq \frac{f(2y)}{f(y)} \leq K+1, \tag{2.2.2}$$

and

$$\left(\frac{1}{K}+1\right) f(y) \leq f(y)-f(-y) \leq (K+1) f(y). \tag{2.2.3}$$

We consider separately the cases $2x \leq t$ and $2x > t$. If $2x \leq t$, then replacing y with $t/2$ in (2.2.1), with $t/2$ and t in (2.2.2), and with t in (2.2.3), we get

$$\begin{aligned} \frac{1}{K(K+1)^2} \leq \frac{f(t/2)}{K(K+1)f(t)} &\leq \frac{f(t)-f(t/2)}{f(t)-f(-t)} \leq \frac{f(x+t)-f(x)}{f(x)-f(x-t)} \\ &\leq \frac{f(2t)}{-f(-t/2)} = \frac{f(2t)}{f(t)} \frac{f(t)}{f(t/2)} \frac{f(t/2)}{(-f(-t/2))} \leq (K+1)^2 K. \end{aligned}$$

And if $2x > t$, again by (2.2.1), (2.2.2), and (2.2.3), we have

$$\begin{aligned} \frac{1}{K(K+1)^2} \leq \frac{f(t/2)}{K(K+1)f(t)} \leq \frac{f(x)}{K(K+1)f(t)} \leq \frac{f(2x)-f(x)}{f(t)-f(-t)} \\ \leq \frac{f(x+t)-f(x)}{f(x)-f(x-t)} \leq \frac{f(2t)}{f(t/2)} = \frac{f(2t)}{f(t)} \frac{f(t)}{f(t/2)} \leq (K+1)^2. \end{aligned}$$ ■

We recall from the introduction the notion of quasisymmetric product.

Definition 2.3. Let $f, g : \mathbb{R} \to \mathbb{R}$ be increasing homeomorphisms with $f(0) = g(0) = 0$. The *quasisymmetric product* $f \bullet g$ of f and g is defined via $f \bullet g(x) := f(x)g(x)$ if $x \geq 0$ and $f \bullet g(x) := -f(x)g(x)$ if $x < 0$.

Corollary 2.4. *If $f, g : \mathbb{R} \to \mathbb{R}$ are increasing homeomorphisms with $f(0) = g(0) = 0$, then so is $f \bullet g$. If in addition f and g are K_1 and K_2-quasisymmetric maps respectively, then $f \bullet g$ is $(K_1K_2 + K_1 + K_2 + 1)^3$-quasisymmetric.*

Proof. The first assertion is obvious, so we only need to verify that the hypotheses of Theorem 2.2 are satisfied. Let $t > 0$. Since

$$\frac{f \bullet g(t)}{-f \bullet g(-t)} = \frac{f(t)}{(-f(-t))} \frac{g(t)}{(-g(-t))},$$

it follows that

$$\frac{1}{K_1K_2} \leq \frac{f \bullet g(t)}{-f \bullet g(-t)} \leq K_1K_2.$$

To see that the restrictions of $f \bullet g$ to $[0, \infty)$ and to $(-\infty, 0]$ are $(K_1K_2+K_1+K_2)$-quasisymmetric maps, set $x_1 = f(x-t), x_2 = f(x), x_3 = f(x+t)$, $y_1 = g(x-t), y_2 = g(x), y_3 = g(x+t)$ and apply Lemma 2.1. ■

Proof of Theorem 1.2. Denote by $\mathcal{D}$ the set of doubling measures on $\mathbb{R}$. Clearly addition and multiplication are both associative and commutative on $\mathcal{D}$, so $(\mathcal{D}, +)$ and $(\mathcal{D}, \bullet)$ are commutative semigroups. And distributivity follows from the corresponding fact for functions: $\mu_f \bullet (\mu_g + \mu_h) = \mu_f \bullet \mu_{g+h} = \mu_{f\bullet(g+h)} = \mu_{f\bullet g+f\bullet h} = \mu_{f\bullet g} + \mu_{f\bullet h} = \mu_f \bullet \mu_g + \mu_f \bullet \mu_h$. ■

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