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OPERATIONAL METHODS FOR INTEGRO-DIFFERENTIAL EQUATIONS AND APPLICATIONS TO PROBLEMS IN PARTICLE ACCELERATOR PHYSICS

G. Dattoli, L. Mezi and M. Migliorati

Abstract. We illustrate a general method, which is useful for the solution of integro-differential equations, and apply the technique to solve the betatron equations of motion with the inclusion of wake field effects.

1. Introduction

In a previous paper it has been shown that an extension of the evolution operator method is quite useful to study problems, involving higher order derivatives in the evolution variable [1].

We consider indeed the equation

$$\begin{aligned}\frac{\partial^2}{\partial t^2}y(x,t) &= \widehat{\Omega}_x^2 y(x,t),\\ y(x,0) &= f(x),\ \frac{\partial}{\partial t}y(x,t)|_{t=0} = g(x),\end{aligned} \tag{1}$$

where $\widehat{\Omega}_x$ is a not yet specified operator, which will be assumed to be independent of time. If we treat the operator on the r. h. s. of eq. (1) as a generic constant, we can write the formal solution of our problem as

$$\underline{y}(x,t) = \widehat{U}(x,t)\underline{y}(x,0), \tag{2}$$

where $\widehat{U}(x,t)$ is the evolution operator associated with the problem (1) written as [1]

$$\widehat{U}(x,t) = \begin{pmatrix} \cosh(\widehat{\Omega}_x t) & \frac{1}{\widehat{\Omega}_x}\sinh(\widehat{\Omega}_x t) \\ \widehat{\Omega}_x \sinh(\widehat{\Omega}_x t) & \cosh(\widehat{\Omega}_x t) \end{pmatrix}. \tag{3}$$

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Furthermore $\underline{y}(x,t)$ and $\underline{y}(x,0)$ are two column vectors specified by

(4) $$\underline{y}(x,t) = \begin{pmatrix} y(x,t) \\ \dfrac{\partial}{\partial t} y(x,t) \end{pmatrix}, \ \underline{y}(x,0) = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}.$$

If for example $\widehat{\Omega}_x = x\frac{\partial}{\partial x}$, we use the following operational rules

(5) $$\exp(\alpha x \frac{\partial}{\partial x}) f(x) = f(x \exp(\alpha)), \quad \frac{1}{\widehat{\Omega}_x} = \int_0^\infty \exp(-s\widehat{\Omega}_x) ds,$$

to get from eqs. (2-4) the solution of eq. (1) in the form

(6) $$\begin{aligned} y(x,t) = & \ \frac{1}{2}\left[f(x\exp(t)) + f(x\exp(-t))\right] \\ & + \frac{1}{2}\int_0^\infty \left[g(x\exp(-(s-t))) - g(x\exp(-(s+t)))\right] ds. \end{aligned}$$

It has also been shown that it is not mandatory that the operator $\widehat{\Omega}_x$ be of differential nature.

By considering indeed the integro differential equation

(7) $$\frac{\partial^2}{\partial t^2} y(x,t) = \int_a^x K(x-x') y(x',t) dx',$$

with the same initial conditions as in eq. (1), we can recast the above problem in a form equivalent to (1), by introducing the operator $\widehat{V}$, defined in such a way that

(8) $$\widehat{V}^n f(x) = f_n(x) = \int_a^x K(x-x') f_{n-1}(x',t) dx'.$$

Therefore, by just applying eq. (2), we obtain the solution of eq. (7) in the form of an infinite series

(9) $$y(x,t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}\left[f_{2n}(x) + \frac{t}{(2n+1)} g_{2n}(x)\right],$$

which can be viewed as a generalization of the ordinary Volterra series (see [2] for further comments).

In the forthcoming section we will see how the method can be applied to a well defined physical problem.

2. Betatron Motion and Wake Field Forces

The betatron motion distortions induced by wake field effects [3] can be treated using the integro-differential equation

$$\begin{aligned}\frac{\partial^2}{\partial s^2}y(z,s)+k^2y(z,s) &= -\alpha\int_z^{\infty}y(z',s)W\left(z'-z\right)\rho(z')dz',\\ y(z,0) = p(z),\ \frac{\partial}{\partial s}y(z,s)|_{s=0} &= d(z)\end{aligned} \tag{10}$$

in which the transverse wake function per unit of displacement is denoted by $W(z)$ and the e-beam charge density by $\rho(z)$, with z being the coordinate in the e-bunch. Both W and ρ are assumed to be independent of the propagation coordinate s. The constant α depends on the number of particles in the bunch and on other specific parameters.

We introduce the operator $\widehat{W}$, defined in such a way that

$$\widehat{W}f(z)=\int_z^{\infty}f(z')W\left(z'-z\right)\rho(z')dz' \tag{11}$$

and recast eq. (10) in the form

$$\begin{aligned}\frac{\partial^2}{\partial s^2}y(z,s)+\widehat{K}^2y(z,s) &= 0\\ \widehat{K}^2 &= k^2+\alpha\widehat{W}.\end{aligned} \tag{12}$$

According to the introductory remarks we can write the formal solution of the previous equation as

$$y(z,s)=\cos(\widehat{K}s)p(z)+\frac{1}{\widehat{K}}\sin(\widehat{K}s)d(z). \tag{13}$$

The initial conditions $p(z)$ and $d(z)$ are not necessarily function of z, we have made this assumption to discuss a more general problem; in the concluding section, devoted to the evaluation of the wake induced beam dimension increase, we will use constant values.

Let us now assume, for simplicity, that $d(z)=0$. At the first order in $\alpha\widehat{W}$ we find

$$\begin{aligned}y_1(z,s) &\simeq \left[\cos(ks)-\frac{s}{2k}\alpha\sin(ks)\widehat{W}\right]p(z)\\ &= \cos(ks)p(z)-\frac{s}{2k}\alpha\sin(ks)p_1(z),\end{aligned} \tag{14}$$

where

$$p_1(z) = \int_z^\infty p(z')W\left(z' - z\right)\rho(z')dz'.$$

The procedure for the inclusion of higher order corrections is only more cumbersome. By taking, indeed, into account that

$$\begin{aligned} \widehat{K} &= k\sum_{r=0}^{\infty}\begin{pmatrix} \frac{1}{2} \\ r \end{pmatrix}\left(\frac{\alpha}{k^2}\widehat{W}\right)^r, \\ \begin{pmatrix} \frac{1}{2} \\ r \end{pmatrix} &= \frac{\Gamma(\frac{3}{2})}{r!\Gamma(\frac{3}{2}-r)}, \end{aligned} \tag{15}$$

we find at the second order in $\alpha\widehat{W}$

$$\begin{aligned} y_2(z,s) &\simeq \cos(ks)\left[p(z) - \frac{s^2}{8k^2}\alpha^2 p_2(z)\right] - \frac{s}{2k}\alpha\sin(ks)\left[p_1(z) - \frac{\alpha}{4k^2}p_2(z)\right] \\ &= y_1(z,s) - \frac{\alpha^2 s}{8k^2}\left[s\cos(ks) - \frac{1}{k}\sin(ks)\right]p_2(z). \end{aligned} \tag{16}$$

The inclusion of the contribution with $d(z) \neq 0$ can be obtained quite straightforwardly too, by noting that the last term in eq. (13) can be written as

$$\frac{1}{\widehat{K}}\sin(\widehat{K}s)d(z) = \int_0^\infty d\xi\exp(-\widehat{K}\xi)\sin(\widehat{K}s)d(z) \tag{17}$$

and that, once expanded, yields, at the lowest order in $\alpha\widehat{W}$

$$\frac{1}{\widehat{K}}\sin(\widehat{K}s)d(z) \simeq \frac{\sin(ks)}{k}\left[d(z) - \frac{1}{2}\frac{\alpha}{k^2}d_1(z)\right] + \frac{1}{2}\frac{\alpha s}{k^2}\cos(ks)d_1(z) \tag{18}$$

The physical applications of the previous results will be discussed in the forthcoming section.

3. Concluding Remarks

A fairly immediate consequence of the previously discussed results is the derivation of the increase of the e-beam transverse dimensions, induced by wake field effects.

To this aim we suppose $p(z)$ and $d(z)$ indipendent of z and referring to the ith particle in the bunch, so that we can write

$$\begin{aligned} p(z) &\Rightarrow y_{i0} \\ d(z) &\Rightarrow y'_{i0}. \end{aligned} \tag{19}$$

At first order in $\alpha\widehat{W}$ we rewrite the solution of eq. (10) in the form

$$\begin{aligned} y_i(z,s) = \; & y_{i0}(s) + \alpha y_{i\alpha}(s) W_p(z) = \cos(ks) y_{i0} + \frac{\sin(ks)}{k} y'_{i0} \\ & + \alpha \left[-\frac{s}{2k} \sin(ks) y_{i0} + \frac{1}{2k^2} \left(s\cos(ks) - \frac{\sin(ks)}{k} \right) y'_{i0} \right] W_p(z), \end{aligned} \tag{20}$$

with

$$W_p(z) = \int_z^\infty W\left(z' - z\right) \rho(z') dz' \tag{21}$$

the transverse wake potential at the position z inside the bunch, and $y_{i0}(s)$ the transverse unperturbed displacement at the location s in the machine. For the following we define the beam section, divergence and covariance as

$$\sigma_y^2 = \left\langle y_i^2 \right\rangle - \left\langle y_i \right\rangle^2 \;,\; \sigma_{y'}^2 = \left\langle y_i'^2 \right\rangle - \left\langle y_i' \right\rangle^2 \;,\; \mathrm{cov}(y,y') = \left\langle y_i y_i' \right\rangle - \left\langle y_i \right\rangle \left\langle y_i' \right\rangle, \tag{22}$$

where the average is taken over all the particles.

If we are interested to first order perturbations, and we assume that the wake potential $W_p(z)$ has zero average, we get, at the lowest order in $\alpha\widehat{W}$,

$$\sigma_y^2(s) = \sigma_{y0}^2(s) + \sigma_\alpha^2(s) = \sigma_{y0}^2(s) + \alpha^2 \left\langle W_p^2(z) \right\rangle \left\langle y_{i\alpha}^2(s) \right\rangle, \tag{23}$$

where $\sigma_{y0}^2(s)$ is the unperturbed beam section and $\sigma_\alpha^2(s)$ the increase due to the wake forces. An analogous expression can be obtained for the divergence $\sigma_{y'}^2(s)$ and the covariance $\mathrm{cov}[y(s), y'(s)]$.

If we make the simplifying assumption that the bunch has initially zero average in the transverse space phase, that is $\langle y_i(z,0)\rangle = \langle y_i'(z,0)\rangle = 0$, the use of eqs. (20-23) allows the derivation of the explicit form of $\sigma_\alpha(s)$, $\sigma'_\alpha(s)$ and the covariance $\mathrm{cov}_\alpha(s)$ as reported below

$$\begin{aligned} \sigma_\alpha^2(s) \simeq \; & \varepsilon_0(0)\alpha^2 \left\langle W_p^2(z) \right\rangle \left[\pi_1^2(s)\beta_y + \delta_1^2(s)\gamma_y - 2\delta_1(s)\pi_1(s)\alpha_y \right] \\ \sigma_\alpha'^2(s) \simeq \; & \varepsilon_0(0)\alpha^2 \left\langle W_p^2(z) \right\rangle \left[\pi_1'^2(s)\beta_y + \delta_1'^2(s)\gamma_y - 2\pi_1'(s)\delta_1'(s)\alpha_y \right] \\ \mathrm{cov}_\alpha(s) \simeq \; & \varepsilon_0(0)\alpha^2 \left\langle W_p^2(z) \right\rangle \\ & \left[\pi_1(s)\pi_1'(s)\beta_y + \delta_1(s)\delta_1'(s)\gamma_y - \left(\pi_1(s)\delta_1'(s) + \delta_1(s)\pi_1'(s) \right) \alpha_y \right], \end{aligned} \tag{24}$$

where $\varepsilon_0(0)$ is the e-beam emittance in the vertical direction at $s = 0$, the quantities $(\alpha_y, \beta_y, \gamma_y)$ are the initial Twiss parameters defined as

$$\sigma_{y0}^2(0) = \beta_y \varepsilon_0(0) \;,\; \sigma_{y'0}^2(0) = \gamma_y \varepsilon_0(0) \;,\; \mathrm{cov}[y_0(0), y_0'(0)] = -\alpha_y \varepsilon_0(0), \tag{25}$$

and furthermore

$$\begin{aligned} \pi_1(s) &= -\frac{s}{2k}\sin(ks), \\ \delta_1(s) &= \frac{1}{2k^2}\left[s\cos(ks) - \frac{\sin(ks)}{k}\right]. \end{aligned} \tag{26}$$

It is to be underlined that in the thin lens approximation the above functions write

$$\begin{aligned} \pi_1(s) &\simeq -\frac{1}{2}s^2, \\ \delta_1(s) &\simeq -\frac{1}{6}s^3. \end{aligned} \tag{27}$$

which show that, at first order, the wake corrections are independent of the quadrupole focusing conditions.

One of the most significant consequences of the wake field contribution is an increase of the beam emittance, which can be evaluated quite straightforwardly along the lines we have just illustrated.

The standard definition of emittance

$$\varepsilon = \sqrt{\sigma_y^2\sigma_{y'}^2 - \mathrm{cov}^2(y, y')}, \tag{28}$$

can be exploited to derive the corrections at the lowest order in $\alpha\widehat{W}$, giving

$$\varepsilon \simeq \varepsilon_0(s) + \frac{1}{2}\left[\beta_y(s)\sigma_\alpha'^2(s) + \gamma_y(s)\sigma_\alpha^2(s) + 2\alpha_y(s)\mathrm{cov}_\alpha(s)\right]. \tag{29}$$

It is then easy to verify that

$$\Delta\varepsilon \propto \alpha^2 \left\langle W_p^2(z)\right\rangle. \tag{30}$$

The method we have developed in the paper can be extended to equations of the type (10), containing an extra non homogeneous term. In the case of accelerator physics this corresponds to the inclusion of contributions to the betatron motion due to off-energy effects.

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References

1. G. Dattoli, P. L. Ottaviani, A. Torre and L. Vazquez, Evolution operator equations: Integration with finite-difference methods. Application to physical problems in classical and quantum mechanics and quantum field theory, *Riv. Nuovo Cimento*, **20** (1997), 1-133.

2. J. Kondo, *Integral Equations*, Oxford Applied Mathematics and Computing Science Series, Clarendon (Oxford University) Press, Oxford, 1991.

3. A. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators*, John Wiley and Sons, New York, 1993.

G. Dattoli and L. Mezi
ENEA, Divisione Fisica Applicata,
Centro Ricerche Frascati,
CP 65, I-00044 Frascati,
Rome, Italy
E-mail: dattoli@frascati.enea.it
E-mail: mezi@frascati.enea.it

M. Migliorati
Università di Roma “La Sapienza”,
Dipartimento Di Energetica,
Via A. Scarpa 14, I-00161 Rome,
Italy
E-mail: mauro.migliorati@uniroma1.it