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THE CONVERGENCE BALL OF NEWTON-LIKE METHODS IN BANACH SPACE AND APPLICATIONS

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Abstract. Under the hypothesis that the derivative satisfies some kind of weak Lipschitz condition, sharp estimates of the radii of convergence balls of Newton-like methods for operator equations are given in Banach space. New results can be used to analyze the convergence of other developed Newton iterative methods.

1. INTRODUCTION

We consider the operator equation:

$$f(x) = 0 \tag{1.1}$$

where $f(x)$ is an operator mapping from some domain D in a real or complex Banach space **X** to another **Y**. Let $f'(x)$ denote the Frechet derivative of f at x. Locally convergent iterative procedures commonly used to solve (1.1) have the general form:

For $n = 0$ *step 1 until convergence do*
Find the step Δ_n *which satisfies*

$$B_n \Delta_n = -f(x_n) \tag{1.2}$$

Set $x_{n+1} = x_n + \Delta_n$,

where x_0 is a given initial guess. The process is Newton method if $B_n = f'(x_n)$, and it represents Newton-like methods if $B_n = B(x_n)$ approximates $f'(x_n)$ (see[3]).

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Let x^* denote the solution of (1.1), $B(x,r)$ denote an open ball with radius r and center x, and let $\overline{B(x,r)}$ denote its closure. Traub & Wozniakowski (see[2]) and Wang (see[6]) independently gave an exact estimate for the convergence ball of Newton's method. Under the hypothesis that $f'(x)$ satisfies the some generalized Lipschitz conditions, Wang, Li et al (see[7,11]) also studied the convergence of the Newton's method. For more results on Newton iterative methods, we refer the reader to [1, 3] and the references therein.

In this paper we study the convergence of Newton-like methods under weak Lipschitz conditions and provide estimates of the radius of the convergence domain. The results obtained include, as a special case, the sharp estimate for the radius of convergence of Newton method given by Wang ([7, 11]). New results can also be used to analyze the convergence of other developed Newton iterative methods.

2. Preliminaries

The condition on the function f

$$\|f(x)-f(x^\tau)\| \le L\|x-x^\tau\|, \quad \forall x\in B(x^*,r), \tag{2.1}$$

where $x^\tau = x_* + \tau(x-x_*)$, $0\le\tau\le 1$, is usually called radius Lipschitz condition in the ball $B(x^*,r)$ with constant L. The function f satisfies the center Lipschitz condition in the ball $B(x^*,r)$ if it is only required to satisfy

$$\|f(x)-f(x^*)\| \le L\|x-x^*\|, \quad \forall x\in B(x^*,r). \tag{2.2}$$

The constant L in the Lipschitz condition needs not to be constant, but a positive integrable function, If this is the case, then (2.1) or (2.2) is replaced by

$$\|f(x)-f(x^\tau)\| \le \int_{\tau\rho(x)}^{\rho(x)} L(u)du, \quad \forall x\in B(x^*,r), 0\le\tau\le 1, \tag{2.3}$$

or

$$\|f(x)-f(x^*)\| \le \int_{0}^{\rho(x)} L(u)du, \quad \forall x\in B(x^*,r), \tag{2.4}$$

where $\rho(x)=\|x-x^*\|$. Here L is a positive integrable function in $(0,r)$ and conditions (2.3), (2.4) are denoted as Lipschitz conditions with the L average.

By Banach's theorem (see[3, 9]), the following result can be obtained directly.

Lemma 2.1. (see[7]) *Suppose that f has a continuous derivative in $B(x^*,r)$, $f'(x^*)^{-1}$ exists and $f'(x^*)^{-1}f'$ satisfies the center Lipschitz condition with the L average:*

$$\|f'(x^*)^{-1}f'(x)-I\| \le \int_{0}^{\rho(x)} L(u)du, \quad \forall x\in B(x^*,r), \tag{2.5}$$

where L is positive integrable function. Let r satisfy

$$\int_0^r L(u)du \leq 1, \tag{2.6}$$

then $f'(x)$ is invertible in this ball and

$$\|f'(x)^{-1}f'(x^*)\| \leq \left(1 - \int_0^{\rho(x)} L(u)du\right)^{-1}. \tag{2.7}$$

Lemma 2.2. (see [10, 11, 12]) *Let*

$$h(t) = \frac{1}{t^\alpha}\int_0^t L(u)u^{\alpha-1}du, \quad \alpha \geq 1, \ 0 \leq t \leq r, \tag{2.7}$$

where $L(u)$ is a positive integrable function nondecreasing in $[0, r]$. Then $h(t)$ is nondecreasing with respect to t.

Lemma 2.3. [12] *Let*

$$\varphi(t) = \frac{1}{t^2}\int_0^t L(u)(\alpha t - u)du, \quad \alpha \geq 1, \quad 0 \leq t \leq r, \tag{0.1}$$

where $L(u)$ is a positive integrable function and nondecreasing monotonically in $[0, r]$. Then $\varphi(t)$ is nondecreasing monotonically with respect to t.

3. Convergence Ball of Newton-like Methods

Theorem 3.1. *Suppose x^* satisfies (1.1), f has a continuous derivative in $B(x^*, r)$, $f'(x^*)^{-1}$ exists and $f'(x^*)^{-1}f'$ satisfies the radius Lipschitz condition with L average:*

$$\|f'(x^*)^{-1}\left(f'(x) - f'(x^\tau)\right)\| \leq \int_{\tau\rho(x)}^{\rho(x)} L(u)du, \quad \forall x \in B(x^*, r), 0 \leq \tau \leq 1, \tag{3.1}$$

where $x^\tau = x^* + \tau(x - x^*)$, $\rho(x) = \|x - x^*\|$, and L is nondecreasing. Let $B(x)$ be invertible and

$$\|B(x)^{-1}f'(x)\| \leq \upsilon_1, \quad \|B(x)^{-1}f'(x) - I\| \leq \upsilon_2, \qquad \forall x \in B(x^*, r), \tag{3.2}$$

where I is unit operator. Let $r > 0$ satisfy

$$\frac{\upsilon_1 \int_0^r L(u)udu}{r\left(1-\int_0^r L(u)du\right)} + \upsilon_2 \le 1. \tag{3.3}$$

Then Newton-like methods are convergent for all $x_0 \in B(x^*, r)$ *and*

$$\|x_{n+1} - x^*\| \le \frac{\upsilon_1 \int_0^{\rho(x_0)} L(u)udu}{\rho(x_0)^2\left(1-\int_0^{\rho(x_0)} L(u)du\right)}\|x_n - x^*\|^2 + \upsilon_2\|x_n - x^*\|, \quad n = 0, 1, \cdots, \tag{3.4}$$

where

$$q = \frac{\upsilon_1 \int_0^{\rho(x_0)} L(u)udu}{\rho(x_0)\left(1-\int_0^{\rho(x_0)} L(u)du\right)} + \upsilon_2 \tag{3.5}$$

is less than 1.

Proof. Choosing $x_0 \in B(x^*, r)$ where r satisfies (3.3), then q determined by (3.5) is less than 1. In fact, from the monotonicity of L and Lemma 2.2, we have

$$q = \frac{\upsilon_1 \int_0^{\rho(x_0)} L(u)udu}{\rho(x_0)^2\left(1-\int_0^{\rho(x_0)} L(u)du\right)}\rho(x_0) + \upsilon_2 < \frac{\upsilon_1 \int_0^r L(u)udu}{r^2\left(1-\int_0^r L(u)du\right)} r + \upsilon_2 \le 1.$$

Now if $x_n \in B(x^*, r)$, we have by (1.2)

$$\begin{aligned} x_{n+1} - x^* &= x_n - x^* - B_n^{-1}(f(x_n) - f(x^*)) \\ &= x_n - x^* - \int_0^1 B_n^{-1} f'(x^\tau)d\tau(x_n - x^*) \\ &= B_n^{-1}f'(x_n)\int_0^1 f'(x_n)^{-1}f'(x^*)\left(f'(x^*)^{-1}(f'(x_n) - f'(x^\tau))\right)(x_n - x^*)d\tau \\ &\quad + B_n^{-1}(B_n - f'(x_n))(x_n - x^*), \end{aligned}$$

where $x^\tau = x^* + \tau(x_n - x^*)$. Hence, by Lemma 2.1 and condition (3.1) we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &\le \|B_n^{-1}f'(x_n)\| \int_0^1 \|f'(x_n)^{-1}f'(x^*)\|\|f'(x^*)^{-1}(f'(x_n) - f'(x^\tau))\| \\ &\qquad \cdot \|x_n - x^*\|d\tau + \|B_n^{-1}(B_n - f'(x_n)\| \cdot \|x_n - x^*\| \\ &\le \frac{\upsilon_1}{1 - \int_0^{\rho(x_n)} L(u)du} \int_0^1 \int_{\tau\rho(x_n)}^{\rho(x_n)} L(u)du\rho(x_n)d\tau + \upsilon_2\rho(x_n) \\ &= \frac{\upsilon_1 \int_0^{\rho(x_n)} L(u)udu}{1 - \int_0^{\rho(x_n)} L(u)du} + \upsilon_2\rho(x_n). \end{aligned}$$

Taking $n = 0$ above, we obtain $\|x_1 - x^*\| \leq q\|x_0 - x^*\| < \|x_0 - x^*\|$, i.e., $x_1 \in B(x^*, r)$. By mathematical induction, all x_n belong to $B(x^*, r)$ and $\rho(x_n) = \|x_n - x^*\|$ decreases monotonically. Therefore, for all $n = 0, 1, \cdots$, we have

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \frac{\upsilon_1 \displaystyle\int_0^{\rho(x_n)} L(u)udu}{\rho(x_n)^2 \left(1 - \displaystyle\int_0^{\rho(x_n)} L(u)du\right)} \rho(x_n)^2 + \upsilon_2 \rho(x_n) \\ &\leq \frac{\upsilon_1 \displaystyle\int_0^{\rho(x_0)} L(u)udu}{\rho(x_0)^2 \left(1 - \displaystyle\int_0^{\rho(x_0)} L(u)du\right)} \rho(x_n)^2 + \upsilon_2 \rho(x_n).\end{aligned}$$

Thus (3.4) follows. ■

Theorem 3.2. *Suppose x^* satisfies (1.1), f has a continuous derivative in $B(x^*, r)$, $f'(x^*)^{-1}$ exists and $f'(x^*)^{-1}f'$ satisfies the center Lipschitz condition with L average.*

$$\|f'(x^*)^{-1}f'(x) - I\| \leq \int_0^{\rho(x)} L(u)du, \quad \forall x \in B(x^*, r), \tag{3.6}$$

where $\rho(x) = \|x - x^*\|$, and L is nondecreasing. Let condition (3.2) hold. Let $r > 0$ satisfy

$$\frac{\upsilon_1 \displaystyle\int_0^r L(u)\,(2r - u)\,du}{r\left(1 - \displaystyle\int_0^r L(u)du\right)} + \upsilon_2 \leq 1. \tag{3.7}$$

Then Newton-like methods are convergent for all $x_0 \in B(x^*, r)$ and

$$\|x_{n+1} - x^*\| \leq \frac{\upsilon_1 \displaystyle\int_0^{\rho(x_n)} L(u)\,(2\rho(x_0) - u)\,du}{\rho(x_0)\left(1 - \displaystyle\int_0^{\rho(x_0)} L(u)du\right)} \|x_n - x^*\|^2 + \upsilon_2 \|x_n - x^*\|, \quad n = 0, 1, \cdots, \tag{3.8}$$

where

$$q = \frac{\upsilon_1 \displaystyle\int_0^{\rho(x_0)} L(u)\,(2\rho(x_0) - u)\,du}{\rho(x_0)\left(1 - \displaystyle\int_0^{\rho(x_0)} L(u)du\right)} + \upsilon_2 \tag{3.9}$$

is less than 1.

Proof. Arbitrarily choosing $x_0 \in B(x^*, r)$, where r satisfies (3.7), then q determined by (3.9) is less than 1. In fact, by the monotonicity of L and Lemma 2.3, we have

$$q = \frac{\upsilon_1 \displaystyle\int_0^{\rho(x_0)} L(u)\,(2\rho(x_0) - u)\,du}{\rho(x_0)^2 \left(1 - \displaystyle\int_0^{\rho(x_0)} L(u)du\right)} \rho(x_0) + \upsilon_2 < \frac{\upsilon_1 \int_0^r L(u)\,(2r - u)\,du}{r^2 \left(1 - \displaystyle\int_0^r L(u)du\right)} r + \upsilon_2 \leq 1.$$

Now if $x_n \in B(x^*, r)$, we have by (1.2)

$$\begin{aligned} x_{n+1} - x^* &= x_n - x^* - B_n^{-1}(f(x_n) - f(x^*)) \\ &= x_n - x^* - \int_0^1 B_n^{-1} f'(x^\tau) d\tau (x_n - x^*) \\ &= B_n^{-1} f'(x_n) \int_0^1 f'(x_n)^{-1} f'(x^*) \left(f'(x^*)^{-1}(f'(x_n) - f'(x^\tau))\right)(x_n - x^*) d\tau \\ &\quad - B_n^{-1}(f'(x_n) - B_n)(x_n - x^*), \end{aligned}$$

where $x^\tau = x + \tau(x_n - x^*)$. Hence, by Lemma 2.1 and condition (3.6) we obtain

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \|B_n^{-1} f'(x_n)\| \int_0^1 \|f'(x_n)^{-1} f'(x^*)\| \|f'(x^*)^{-1}(f'(x_n) - f'(x^\tau))\| \\ &\quad \cdot \|x_n - x^*\| d\tau \\ &\leq \|B_n^{-1} f'(x_n)\| \cdot \|f'(x_n)^{-1} f'(x^*)\| \left(\int_0^1 \|f'(x^*)^{-1}(f'(x_n) \right. \\ &\quad \left. - f'(x^*))\| d\tau + \int_0^1 \|f'(x^*)^{-1}(f'(x^\tau) - f'(x^*))\| d\tau\right) \cdot \|x_n - x^*\| \\ &\quad + \|B_n^{-1}(f'(x_n) - B_n)\| \cdot \|x_n - x^*\| \\ &\leq \frac{\upsilon_1}{1 - \displaystyle\int_0^{\rho(x_n)} L(u)du} \int_0^1 \left(\int_0^{\rho(x_n)} + \int_0^{\tau\rho(x_n)}\right) L(u) du \rho(x_n) d\tau + \upsilon_2 \rho(x_n) \\ &= \frac{\upsilon_1 \displaystyle\int_0^{\rho(x_n)} L(u)\,(2\rho(x_n) - u)\,du}{1 - \displaystyle\int_0^{\rho(x_n)} L(u)du} + \upsilon_2 \rho(x_n). \end{aligned}$$

Taking $n = 0$ above, we obtain $\|x_1 - x^*\| \leq q\|x_0 - x^*\| < \|x_0 - x^*\|$. Hence, i.e. $x_1 \in B(x^*, r)$, this shows that (1.2) can be continued an infinite number of times.

By mathematical induction, all x_n belong to $B(x^*, r)$ and $\rho(x_n) = \|x_n - x^*\|$ decreases monotonically. Therefore, for all $n = 0, 1, \cdots$, we have

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \frac{\upsilon_1 \int_0^{\rho(x_n)} L(u)\,(2\rho(x_n) - u)\,du}{\rho(x_n)^2 \left(1 - \int_0^{\rho(x_n)} L(u)du\right)}\rho(x_n)^2 + \upsilon_2 \rho(x_n) \\ &\leq \frac{\upsilon_1 \int_0^{\rho(x_0)} L(u)\,(2\rho(x_0) - u)\,du}{\rho(x_0)^2 \left(1 - \int_0^{\rho(x_0)} L(u)du\right)}\rho(x_n)^2 + \upsilon_2 \rho(x_n).\end{aligned}$$

Thus (3.8) follows. ∎

Remark. Suppose that $B(x) = f'(x)$ and the equality sign holds in the inequality (3.3) in Theorem 3.1. Then the given value r of the convergence ball is the best possible, see Theorem 5.1 [7]. Furthermore, note that r depends on L, υ_1, υ_2, but it is independent of f.

4. Corollaries of the Main Results

In the study of the Newton's method, the assumption that the derivative is Lipschitz continuous is considered traditional. Combining Theorems $3.1 \sim 3.2$, and taking L as a constant, the following corollaries are obtained directly.

Corollary 4.1. *Suppose x^* satisfies (1.1), f has a continuous derivative in $B(x^*, r)$, $f'(x^*)^{-1}$ exists and $f'(x^*)^{-1}f'$ satisfies the radius Lipschitz condition with L average:*

(4.1) $$\|f'(x^*)^{-1}(f'(x) - f'(x^\tau))\| \leq (1-\tau)L\|x - x^*\|, \;\; \forall x \in B(x^*, r), 0 \leq \tau \leq 1,$$

where $x^\tau = x^* + \tau(x - x^*)$, L is positive number. Let condition (3.2) hold and

(4.2) $$r = \frac{2(1-\upsilon_2)}{L(\upsilon_1 + 2 - 2\upsilon_2)}.$$

Then, Newton-like methods are convergent for all $x_0 \in B(x^*, r)$,

(4.3) $$q = \frac{\upsilon_1 L\|x_0 - x^*\|}{2\,(1 - L\|x_0 - x^*\|)} + \upsilon_2 < 1$$

and the inequality (3.4) holds.

Corollary 4.2. *Suppose x^* satisfies (1.1), f has a continuous derivative in $B(x^*, r)$, $f'(x^*)^{-1}$ exists and $f'(x^*)^{-1}f'$ satisfies the center Lipschitz condition with L average:*

(4.4) $$\|f'(x^*)^{-1}f'(x) - I\| \leq L\|x - x^*\|, \;\; \forall x \in B(x^*, r).$$

Let condition (3.2) hold and

$$r = \frac{2\alpha}{(3+2\alpha)L}, \tag{4.5}$$

where $\alpha = \frac{1-\upsilon_2}{\upsilon_1}$*. Then Newton-like methods are convergent for all* $x_0 \in B(x^*, r)$*,*

$$q = \frac{3\upsilon_1 L\|x_0 - x^*\|}{2\,(1 - L\|x_0 - x^*\|)} + \upsilon_2 < 1 \tag{4.6}$$

and the inequality (3.9) holds.

By Theorems 3.1 and 3.2, and taking

$$L(u) = \frac{2\gamma}{(1-\gamma u)^3}, \tag{4.7}$$

we obtain the following corollaries.

Corollary 4.3. *Suppose* x^* *satisfies (1.1),* f *has a continuous derivative in* $B(x^*, r)$*,* $f'(x^*)^{-1}$ *exists and* $f'(x^*)^{-1}f'$ *satisfies the radius Lipschitz condition with* L *average:*

$$\|f'(x^*)^{-1}(f'(x) - f'(x^\tau))\| \le \frac{1}{(1-\gamma\|x-x^*\|)^2} - \frac{1}{(1-\tau\gamma\|x-x^*\|)^2}, \quad 0 \le \tau \le 1, \tag{4.8}$$

where $x^\tau = x^* + \tau(x - x^*)$*,* γ *is positive number. Let condition (3.2) hold and*

$$r = \frac{4\alpha + 1 - \sqrt{8\alpha^2 + 8\alpha + 1}}{4\gamma\alpha}, \tag{4.9}$$

where $\alpha = \frac{1-\upsilon_2}{\upsilon_1}$*, then Newton-like methods are convergent for all* $x_0 \in B(x^*, r)$*,*

$$q = \frac{\upsilon_1\gamma\|x_0 - x^*\|}{1 - 4\gamma\|x_0 - x^*\| + 2\,(\gamma\|x_0 - x^*\|)^2} + \upsilon_2 < 1 \tag{4.10}$$

and the inequality (3.4) holds.

Corollary 4.4. *Suppose* x^* *satisfies (1.1),* f *has a continuous derivative in* $B(x^*, r)$*,* $f'(x^*)^{-1}$ *exists and* $f'(x^*)^{-1}f'$ *satisfies the center Lipschitz condition with* L *average:*

$$\|f'(x^*)^{-1}f'(x) - I\| \le \frac{1}{(1-\gamma\|x-x^*\|)^2} - 1, \tag{4.11}$$

γ *is positive number. Let condition (3.2) hold and*

(4.12) $$r = \frac{3 + 4\alpha - \sqrt{8\alpha^2 + 16\alpha + 9}}{4\gamma(1 + \alpha)},$$

where $\alpha = \frac{1-\upsilon_2}{\upsilon_1}$. *Then Newton-like methods are convergent for all* $x_0 \in B(x^*, r)$,

(4.13) $$q = \frac{\upsilon_2 \left(1 - (1 + 2\gamma\|x_0 - x^*\|)(1 - \gamma\|x_0 - x^*\|)^2\right)}{\gamma\|x_0 - x^*\| \left(2(1 - \gamma\|x_0 - x^*\|)^2 - 1\right)} + \upsilon_2 < 1$$

and the inequality (3.9) holds.

Taking $\upsilon_1 = 1$ and $\upsilon_2 = 0$ in (3.2), we get the Newton method. For this case, the results of Corollaries 4.1 and 4.3 merge into Corollaries 6.1 and 6.3 by Wang [7]. Therefore, Corollaries $4.1 \sim 4.4$ generalize known results for Newton methods to the Newton-like framework.

Using Theorems $3.1 \sim 3.2$ and considering some functions L, we can derive the radii of convergence of Newton-like methods. In the following example, let c be a positive number.

Example 1. Taking

(4.14) $$L(u) = 2c\gamma(1 - \gamma u)^{-3},$$

we obtain that if the right-hand side in (4.8) is replaced by

(4.15) $$\frac{c}{(1 - \gamma\|x - x^*\|)^2} - \frac{c}{(1 - \tau\gamma\|x - x^*\|)^2},$$

we have

(4.16) $$r = \frac{2\alpha(c + 1) + c - \sqrt{c(4\alpha^2(c + 1) + 4\alpha(c + 1) + c)}}{2\alpha\gamma(c + 1)},$$

and

(4.17) $$q = \frac{\upsilon_1 c\gamma\|x_0 - x^*\|}{1 - 2(c + 1)\gamma\|x_0 - x^*\| + (c + 1)\left(\gamma\|x_0 - x^*\|\right)^2} + \upsilon_2.$$

If the right-hand side in (4.11) is replaced by

(4.18) $$\frac{c}{(1 - \gamma\|x - x^*\|)^2} - 1,$$

we have

(4.19) $$r = \frac{2\alpha + 3c + 2\alpha c - \sqrt{c\left(4\alpha + 4\alpha^2 + 9c + 12\alpha c + 4\alpha^2 c\right)}}{2\gamma(\alpha + 2c + \alpha c)},$$

and

$$q=\frac{\upsilon_1 c\left(1-(1+2\gamma\|x_0-x^*\|)(1-\gamma\|x_0-x^*\|)^2\right)}{\gamma\|x_0-x^*\|\left((1+c)(1-\gamma\|x_0-x^*\|)^2-c\right)}+\upsilon_2, \tag{4.20}$$

where $\alpha=\frac{1-\upsilon_2}{\upsilon_1}$.

Example 2. Taking

$$L(u)=2c\gamma(1-\gamma u)^{-3/2}, \tag{4.21}$$

we obtain that if the right-hand side in (4.8) is replaced by

$$\frac{c}{\sqrt{1-\gamma\|x-x^*\|}}-\frac{c}{\sqrt{1-\tau\gamma\|x-x^*\|}}, \tag{4.22}$$

we have

$$r=\frac{\sqrt{16\alpha^3c(1+c)^2+(2\alpha c(2+c)-\alpha^2(1+2c)+c^2)^2}-(2\alpha c(2+c)-\alpha^2(1+2c)+c^2)}{2\gamma\alpha^2(c+1)^2}, \tag{4.23}$$

and

$$q=\frac{\upsilon_1 c\gamma\|x_0-x^*\|}{2-(c+2)\gamma\|x_0-x^*\|+(2-(c+1)\gamma\|x_0-x^*\|)\sqrt{1-\gamma\|x_0-x^*\|}}+\upsilon_2. \tag{4.24}$$

If the right-hand side in (4.11) is replaced by

$$\frac{c}{\sqrt{1-\gamma\|x-x^*\|}}-c, \tag{4.25}$$

we have

$$r=\begin{cases}\frac{\alpha^2+32\alpha c+8\alpha^2c+48c^2+32\alpha c^2+(\alpha-4c)\sqrt{\alpha^2+8\alpha c+16\alpha^2c+144c^2+192\alpha c^2+64\alpha^2c^2}}{2\gamma(\alpha^2+16\alpha c+8\alpha^2c+64c^2+64\alpha c^2+16\alpha^2c^2)}, & \text{if } c\leq\frac{\alpha}{4};\\ \frac{\alpha^2+32\alpha c+8\alpha^2c+48c^2+32\alpha c^2-(\alpha-4c)\sqrt{\alpha^2+8\alpha c+16\alpha^2c+144c^2+192\alpha c^2+64\alpha^2c^2}}{2\gamma(\alpha^2+16\alpha c+8\alpha^2c+64c^2+64\alpha c^2+16\alpha^2c^2)}, & \text{if } c>\frac{\alpha}{4},\end{cases} \tag{4.26}$$

and

$$q=\frac{4\upsilon_1 c\left(2(1-\gamma\|x_0-x^*\|)^2+\sqrt{1-\gamma\|x_0-x^*\|}\,(3\gamma\|x_0-x^*\|-2)\right)}{\gamma\|x_0-x^*\|\left((1+4c)\,(1-\gamma\|x_0-x^*\|)-4c\sqrt{1-\gamma\|x_0-x^*\|}\right)}+\upsilon_2, \tag{4.27}$$

where $\alpha=\frac{1-\upsilon_2}{\upsilon_1}$.

For the rest of this paper, we will assume that **X** and **Y** are finite dimensional spaces R^n.

We remark that the conditions (3.1), (3.2) and (4.1), (4.5) are affine invariant, as they are insensitive with respect to transformations of the mapping $f(x)$ of the form: $f(x) \longrightarrow Af(x)$, A an invertible matrix, as long as the same affine transformation is also valid for $B(x)$.

Since Newton's iterates are affine invariant, in [4, 5] convergence conditions were determined in affine invariant terms. We point out that, if the affine transformation is valid for B, i.e. $B(x) \rightarrow AB(x)$, then Theorems 3.1 and 3.2 represent an affine convergence analysis of Newton-like methods.

5. Applications to Determinate Convergence Ball of Newton Iterative Method

Clearly many developed Newton iterative methods [3, 8] can be classified as Newton-like methods. Theorems 3.1 and 3.2 establish convergence analysis for many methods and make us explicitly see how big the convergence field, comparing with Newton's method.

In [8] Newton-arithmetic mean (Newton-AM) method for solving the system of nonlinear equations

$$f(x) = 0, \quad f : D \subseteq R^n \longrightarrow R^n \tag{5.1}$$

were studied by E.Galligani. Now we will show how to determinate the convergence ball of Newton-AM method using Theorem 3.1.

E.Galligani considered the following two splittings of the matrix $f'(x)$

$$f'(x) = M_1(x) - N_1(x) = M_2(x) - N_2(x), \tag{5.2}$$

where the spectral radius $\rho(M_1(x)^{-1}N_1(x)) < 1$, $\rho(M_2(x)^{-1}N_2(x)) < 1$.

Combining the splittings (5.2), Newton-AM method can be described as follows:

Choose the initial guess x_0
For $n = 0$ *step 1 until convergence do*

$$w_n^{(0)} = 0$$

For $j = 1, \cdots, j_n$ *do*

$$\begin{aligned} M_1(x_n)z_1 &= N_1(x_n)w_n^{(j-1)} - f(x_n) \\ M_2(x_n)z_2 &= N_2(x_n)w_n^{(j-1)} - f(x_n) \\ w_n^{(j)} &= \frac{1}{2}(z_1 + z_2) \end{aligned}$$

Set $x_{n+1} = x_n + w_n^{(j_n)}$.

Here $\{j_n\}$ denotes a sequence of positive integers.

In fact, at each outer iteration n, the Newton-AM method generates the vectors

$$\begin{aligned} w_n^{(1)} &= -M(x_n)^{-1}f(x_n), \\ w_n^{(2)} &= -(H(x_n)+I)M(x_n)^{-1}f(x_n), \\ &\vdots \\ w_n^{(j_n)} &= -\left(\sum_{j=0}^{j_n-1}(H(x_n))^j\right)M(x_n)^{-1}f(x_n), \end{aligned}$$

where

$$M(x)^{-1} = \frac{1}{2}\left(M_1(x)^{-1} + M_2(x)^{-1}\right), \tag{5.3}$$

$$H(x) = \frac{1}{2}\left(M_1(x)^{-1}N_1(x) + M_2(x)^{-1}N_2(x)\right) = I - M(x)^{-1}f'(x). \tag{5.4}$$

If we set

$$B(x_n)^{-1} = \left(\sum_{j=0}^{j_n-1}(H(x_n))^j\right)M(x_n)^{-1}, \tag{5.5}$$

then we have

$$x_{n+1} = x_n - B(x_n)^{-1}f(x_n). \tag{5.6}$$

Thus, the Newton-AM method can be regarded as a class Newton-like method in which $f'(x_n)$ has been replaced by the matrix $B(x_n)$ given by (5.5).

We now assume that the matrices $M_1(x_n)$, $M_2(x_n)$, $M(x_n)$ are all nonsingular and $H(x)$ is convergent at $x_n \in D$ (i.e., the spectral radius $\rho(H(x_n)) < 1$). Thus from (5.5),

$$\begin{aligned} B(x_n)^{-1} &= \left(I - (H(x_n))^{j_n}\right)(I - H(x_n))^{-1}M(x_n)^{-1} \\ &= \left(I - (H(x_n))^{j_n}\right)\left(M(x_n)^{-1}f'(x_n)\right)^{-1}M(x_n)^{-1} \\ &= \left(I - (H(x_n))^{j_n}\right)f'(x_n)^{-1}. \end{aligned} \tag{5.7}$$

Now combining the above discussion and Theorem 3.1 we will give the convergence ball of Newton-AM method.

Theorem 5.1. *Suppose x^* satisfies (5.1), f has a continuous derivative in $B(x^*, r)$, $f'(x^*)^{-1}$ exists and $f'(x^*)^{-1}f'$ satisfies the radius Lipschitz condition*

(3.1) *with* L *average.* $M_1(x)$, $M_2(x)$, $M(x)$, $H(x)$ *are invertible for all* $x \in B(x^*, r)$ *and*

$$\|I - H(x)\| \leq \upsilon_1, \quad \|H(x)\| \leq \upsilon_2 \leq 1. \tag{5.8}$$

Let $r > 0$ *satisfy*

$$\frac{\upsilon_1 \int_0^r L(u)udu}{r(1-\upsilon_2)\left(1 - \int_0^r L(u)du\right)} + \upsilon_2 \leq 1. \tag{5.9}$$

Then Newton-AM method is convergent for all $x_0 \in B(x^*, r)$ *and*

$$\|x_{n+1} - x^*\| \leq \frac{\upsilon_1 \displaystyle\int_0^{\rho(x_0)} L(u)udu}{\rho(x_0)^2(1-\upsilon_2)\left(1 - \displaystyle\int_0^{\rho(x_0)} L(u)du\right)} \|x_n - x^*\|^2 + \upsilon_2\|x_n - x^*\|, \quad n = 1, 2, \cdots, \tag{5.10}$$

where

$$q = \frac{\upsilon_1 \displaystyle\int_0^{\rho(x_0)} L(u)udu}{\rho(x_0)(1-\upsilon_2)\left(1 - \displaystyle\int_0^{\rho(x_0)} L(u)du\right)} + \upsilon_2 \tag{5.11}$$

is less than 1.

Proof. In fact, From (5.7) and (5.8), we have

$$\|B(x_k)^{-1}f'(x_k) - I\| = \|\left(H(x_k)\right)^{j_k}\| \leq \|H(x_k)\|^{j_k} \leq \upsilon_2,$$

$$\begin{aligned}\|B(x_k)^{-1}f'(x_k)\| &= \|I - \left(H(x_k)\right)^{j_k}\| \\ &\leq \|I - H(x_k)\| \cdot \sum_{j=0}^{j_k-1} \|H(x^{(k)})\|^j \leq \frac{\|I - H(x_k)\|}{1 - \|H(x_k)\|} \leq \frac{\upsilon_1}{1-\upsilon_2}.\end{aligned}$$

By Theorem 3.1, we obtain the result immediately. ■

Taking $L(u)$ as a positive constant, we can have the following corollary directly.

Corollary 5.1. *Suppose* x^* *satisfies (5.1),* f *has a continuous derivative in* $B(x^*, r)$, $f'(x^*)^{-1}$ *exists and* $f'(x^*)^{-1}f'$ *satisfies the radius Lipschitz condition*

(5.1) with $L(u)$ a positive constant. $M_1(x)$, $M_2(x)$, $M(x)$, $H(x)$ are invertible for all $x \in B(x^*, r)$ and (5.8) holds, Let

$$r = \frac{2(1-\upsilon_2)^2}{L\upsilon_1 + 2L(1-\upsilon_2)^2} > 0. \tag{5.12}$$

Then Newton-AM method is convergent for all $x_0 \in B(x^*, r)$ and for

$$q = \frac{\upsilon_1 L\|x_0 - x^*\|}{2(1-\upsilon_2)\,(1 - L\|x_0 - x^*\|)} + \upsilon_2 < 1 \tag{5.13}$$

the inequality (5.10) holds.

Under affine invariant Lipschitz condition:

$$\|f'(u)^{-1}(f'(v) - f'(u))\| \leq L\|u - v\|, \qquad \forall u, v \in B(x^*, r), \tag{5.14}$$

E.Galligani [8,Theorem 1] gave the convergence analysis of Newton-AM method. It is clear that the conditions (3.1) and (4.1) are different from the condition (5.14). That is to say, under others affine invariant Lipschitz condition, Theorem 5.1 and Corollary 5.1 show the convergence analysis of Newton-AM method, so Theorem 5.1 and Corollary 5.1 extend the results of E.Galligani and expand the application field of Newton-AM method.

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