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SOME NEW RESULTS ABOUT A SYMMETRIC D-SEMICLASSICAL LINEAR FORM OF CLASS ONE

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Abstract. We establish some properties concerning the linear form $\mathcal{B}[\nu]$ which is symmetric D-semiclassical of class 1. An integral representation is obtained. A connection with the D-classical Bessel one is discussed.

1. Introduction and First Results

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its topological dual. We denote by $\langle u, f\rangle$ the effect of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. We denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$ the moments of u. In particular, a linear form u is called symmetric if $\langle u, x^{2n+1}\rangle = 0,\ n \geq 0$.
For any linear form u, any polynomial g , let gu , be the linear form defined by duality

$$\langle gu, f\rangle := \langle u, gf\rangle \ , f \in \mathcal{P}. \tag{1.1}$$

For $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, the product uf is the polynomial

$$(uf)(x) := \langle u, \frac{xf(x) - \zeta f(\zeta)}{x - \zeta}\rangle. \tag{1.2}$$

The derivative $u' = Du$ of the linear form u is defined by

$$\langle u', f\rangle := -\langle u, f'\rangle \ , f \in \mathcal{P}. \tag{1.3}$$

We have [5]

$$(fu)' = f'u + fu'. \tag{1.4}$$

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Similarly, with the definitions

$$\langle h_a u, f\rangle := \langle u, h_a f\rangle = \langle u, f(ax)\rangle \ , u \in \mathcal{P}' \ , f \in \mathcal{P} \ , a \in \mathbb{C} - 0, \tag{1.5}$$

$$\langle \tau_b u, f\rangle := \langle u, \tau_{-b} f\rangle = \langle u, f(x+b)\rangle \ , u \in \mathcal{P}' \ , f \in \mathcal{P} \ , b \in \mathbb{C}. \tag{1.6}$$

The linear form u is called *regular* if we can associate with it a polynomial sequence $\{P_n\}_{n\geq 0}$, $\deg P_n = n$, such that

$$\langle u, P_m P_n\rangle = r_n \delta_{n,m} \ , \ n, m \geq 0 \ ; \ r_n \neq 0 \ , \ n \geq 0. \tag{1.7}$$

The polynomial sequence $\{P_n\}_{n\geq 0}$ is then said orthogonal with respect to u. Necessarily, $\{P_n\}_{n\geq 0}$ is an (OPS) whose any polynomial can be supposed monic (MOPS). Also, the (MOPS)$\{P_n\}_{n\geq 0}$ fulfils the recurrence relation

$$\begin{cases} P_0(x) = 1 \ , \ P_1(x) = x - \beta_0 \ , \\ P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x) \ , \ \gamma_{n+1} \neq 0 \ , \ n \geq 0. \end{cases} \tag{1.8}$$

From the linear application $p \mapsto (\theta_c p)(x) = \frac{p(x) - p(c)}{x - c}$, $p \in \mathcal{P}$, $c \in \mathbb{C}$, we define $(x-c)^{-1} u$ by

$$\langle (x-c)^{-1} u, p\rangle := \langle u, \theta_c p\rangle. \tag{1.9}$$

Finally, we introduce the operator $\sigma : \mathcal{P} \longrightarrow \mathcal{P}$ defined by $(\sigma f)(x) := f(x^2)$ for all $f \in \mathcal{P}$. Consequently, we define σu by duality

$$\langle \sigma u, f\rangle = \langle u, \sigma f\rangle \ , f \in \mathcal{P} \ , u \in \mathcal{P}'. \tag{1.10}$$

we have the two well known formulas[7]

$$f(x) \sigma u = \sigma(f(x^2) u), \tag{1.11}$$

$$\sigma u' = 2(\sigma(xu))'. \tag{1.12}$$

Let Φ monic and Ψ be two polynomials , $\deg \Phi = t$, $\deg \Psi = p \geq 1$. We suppose that the pair (Φ, Ψ) is *admissible* , i.e. when $p = t - 1$, writing $\Psi(x) = a_p x^p + ...$, then $a_p \neq n + 1$, $n \in \mathbb{N}$.

Definition 1.1. [5] *A linear form u is called D-semiclassical when it is regular and satisfies the equation*

$$(\Phi u)' + \Psi u = 0 \tag{1.13}$$

where the pair (Φ, Ψ) *is admissible. The corresponding orthogonal sequence* $\{P_n\}_{n\geq 0}$ *is called* D*-semiclassical.*

Remarks.

1. The D-semiclassical character is kept by shifting(see [6]). In fact, let $\{a^{-n}(h_a \circ \tau_{-b} P_n)\}_{n\geq 0}$, $a \neq 0$, $b \in \mathbb{C}$; when u satisfies (1.13) , then $h_{a^{-1}} \circ \tau_{-b} u$ fulfils the equation

$$\Big(a^{-t}\Phi(ax+b)\big(h_{a^{-1}} \circ \tau_{-b} u\big)\Big)' + a^{1-t}\Psi(ax+b)\big(h_{a^{-1}} \circ \tau_{-b} u\big) = 0. \tag{1.14}$$

2. The D-semiclassical linear form u is said to be of class $s = \max(p-1, t-2) \geq 0$ if and only if

$$\prod_{c \in \mathcal{Z}_\Phi} \Big\{ \Big|\Psi(c) + \Phi'(c)\Big| + \Big|\big\langle u, \theta_c \Psi + \theta_c^2 \Phi \big\rangle \Big| \Big\} > 0, \tag{1.15}$$

where $\mathcal{Z}_\Phi$ is the set of zeros of Φ. The corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ will be known as of class s[4].

3. When $s = 0$, the linear form u is usually called *D-classical* $\big($Hermite, Laguerre, Bessel, and Jacobi $\big)$[6].

Let us recall some characterizations of D-semiclassical orthogonal sequences which are needed in the sequel. $\{P_n\}_{n\geq 0}$ is D-semiclassical of class s, if and only if one of the following statements holds [4]

(1). $\{P_n\}_{n\geq 0}$ satisfies the following structure relation

$$\Phi(x)P'_{n+1}(x) = \frac{1}{2}\left(C_{n+1}(x) - C_0(x)\right) P_{n+1}(x) - \gamma_{n+1} D_{n+1}(x) P_n(x), n \geq 0, \tag{1.16}$$

where

$$C_{n+1}(x) = -C_n(x) + 2(x-\beta_n) D_n(x) \ , n \geq 0, \tag{1.17}$$

$$\begin{aligned} \gamma_{n+1} D_{n+1}(x) = {} & -\Phi(x) + \gamma_n D_{n-1}(x) + (x-\beta_n)^2 D_n(x) \\ & -(x-\beta_n) C_n(x), \ n \geq 0, \end{aligned} \tag{1.18}$$

$$\begin{cases} C_0(z) = -\Psi(z) - \Phi'(z), \\ D_0(z) = -(u\theta_0 \Phi)'(z) - (u\theta_0 \Psi)(z). \end{cases} \tag{1.19}$$

Φ, Ψ are the same parameters introduced in (1.13); β_n, γ_n are the coefficients of the three term recurrence relation (1.8). Notice that $D_{-1}(x) := 0$, $\deg C_n \leq s+1$ and

$\deg D_n \leq s,\ n \geq 0$.
(2). Each polynomial $P_{n+1},\ n \geq 0$ satisfies a second order linear differential equation

$$J(x,n)P''_{n+1}(x) + K(x,n)P'_{n+1}(x) + L(x,n)P_{n+1}(x) = 0\ ,\ n \geq 0, \tag{1.20}$$

with

$$\begin{cases} J(x,n) = \Phi(x)D_{n+1}(x)\ , \\ K(x,n) = D_{n+1}(x)\Big(\Phi'(x) + C_0(x)\Big) - D'_{n+1}(x)\Phi(x)\ , \\ L(x,n) = \frac{1}{2}\Big(C_{n+1}(x) - C_0(x)\Big)D'_{n+1}(x) - \\ \qquad -\frac{1}{2}\Big(C'_{n+1} - C'_0\Big)(x)D_{n+1}(x) - D_{n+1}(x)\Sigma_n(x)\ ,\ n \geq 0, \end{cases} \tag{1.21}$$

and

$$\Sigma_n(x) := \sum_{k=0}^{n} D_k(x)\ ,\ n \geq 0. \tag{1.22}$$

Φ, C_n, D_n are the same in the previous characterization. Notice that
$\deg J(.,n) \leq 2s+2$, $\deg K(.,n) \leq 2s+1$ and $\deg L(.,n) \leq 2s$.

In [1], the authors give the description of symmetric D-semiclassical linear forms of class 1. There are three canonical cases for Φ

$$\Phi(x) = x, \qquad \Phi(x) = x(x^2-1), \qquad \Phi(x) = x^3.$$

The first and the second canonical cases are well known. They are respectively the generalized Hermite $\mathcal{H}(\mu)$ and The symmetric generalized Gegenbauer $\mathcal{G}(\alpha,\beta)$[1,3]. So, the aim of this paper is to give some new results concerning the third case. It's the linear form $\mathcal{B}[\nu]$, symmetric D-semiclassical of class 1 for $\nu \neq -n-1$, $n \geq 0$. We have[1]

$$\begin{cases} \beta_n = 0\ ,\ \gamma_{n+1} = \frac{1}{16}\dfrac{1-2\nu-(-1)^n(2n+2\nu+1)}{(n+\nu)(n+\nu+1)}\ ,\ n \geq 0, \\ \Big(x^3\mathcal{B}[\nu]\Big)' - \Big\{2(\nu+1)x^2 + \frac{1}{2}\Big\}\mathcal{B}[\nu] = 0. \end{cases} \tag{1.23}$$

Taking into account the functional equation in (1.23), it is easy to see that the moments of $\mathcal{B}[\nu]$ are

$$(\mathcal{B}[\nu])_{2n} = \frac{(-1)^n\Gamma(\nu+1)}{2^{2n}\Gamma(n+\nu+1)}\ ,\ (\mathcal{B}[\nu])_{2n+1} = 0\ ,\ n \geq 0, \tag{1.24}$$

where Γ is the gamma function. In accordance of (1.17)-(1.19) and (1.22) and after some calculation we get

$$\begin{cases} C_n(x) = (2n+2\nu-1)x^2 + \frac{1}{2}(-1)^n, \\ D_n(x) = 2(n+\nu)x, \\ \Sigma_n(x) = (n+1)(n+2\nu)x, \end{cases} \quad , \; n \geq 0. \tag{1.25}$$

Therefore, with (1.20)-(1.21), the second order linear differential equation satisfied by $P_{n+1},\ n \geq 0$ is

$$\begin{aligned} &x^4 P''_{n+1}(x) + x\big\{(2\nu+1)x^2 + \frac{1}{2}\big\}P'_{n+1}(x) \\ &\quad -\big\{(n+1)(n+2\nu+1)x^2 + \frac{(1+(-1)^n)}{4}\big\}P_{n+1}(x) = 0. \end{aligned} \tag{1.26}$$

Proposition 1.2. *Let $\{P_n\}_{n\geq 0}$ be the (MOPS) with respect to the linear form $\mathcal{B}[\nu]$. Then, every polynomial P_{n+1}, $n \geq 1$ have simple zeros.*

Proof. First, the (MOPS)$\{P_n\}_{n\geq 0}$ is of class 1. Taking into account the structure relation (1.16), we can deduce the following: if c is a zero of order η of $P_{n+1}, n \geq 1$ with $\eta \geq 2$, then $\eta \leq 2$ and c is a zero of order $\eta - 1 = 1$ of D_{n+1}.[4] Second,the (MOPS)$\{P_n\}_{n\geq 0}$ is symmetric then $P_n(-x) = (-1)^n P_n(x), n \geq 0$[3] and according to (1.8) with $\beta_n = 0$, $n \geq 0$ we get

$$P_{2n+1}(0) = 0\ ,\ P_{2n}(0) = (-1)^n \prod_{k=0}^{n} \gamma_{2k-1} \neq 0\ ,\ n \geq 0\ ,\ \gamma_{-1} := 1. \tag{1.27}$$

To establish the desired result, it is sufficient to prove that $P'_{2n+1}(0) \neq 0, n \geq 0$ since the above, the expression of the polynomial D_{n+1} in (1.25), and (1.27). Differentiating (1.16), then taking $x = 0$ and $n \to 2n$, and after an easy computation we obtain $P'_{2n+1}(0) = \frac{n+\nu}{2n+\nu} P_{2n}(0) \neq 0$, $n \geq 0$ ■

In [1], an integral representation of the last case is not given. See also [2]. In the next section, we are going to give an integral representation for $\mathcal{B}[\nu]$. Moreover, the relationship with the D-classical Bessel linear form is obtained.

2. An Integral Representation for $\mathcal{B}[\nu]$

Let u be a D-semiclassical linear form satisfying (1.13). We are looking for an integral representation of u and consider

$$\langle u, f\rangle = \int_{-\infty}^{+\infty} U(x) f(x) dx\ ,\ f \in \mathcal{P}, \tag{2.1}$$

where we suppose the function U to be absolutely continuous on $\mathbb{R}$, and is decaying as fast as its derivative U'. From (1.13) we get

$$\int_{-\infty}^{+\infty} \big((\Phi U)' + \Psi U\big) f(x)dx - \Phi(x)U(x)f(x)\big]_{-\infty}^{+\infty} = 0, \ \ f \in \mathcal{P}.$$

Hence, from the assumptions on U, the following conditions hold

$$\Phi(x)U(x)f(x)\big]_{-\infty}^{+\infty} = 0 \ , \ f \in \mathcal{P}, \tag{2.2}$$

$$\int_{-\infty}^{+\infty} \big((\Phi U)' + \Psi U\big) f(x)dx = 0 \ , \ f \in \mathcal{P}. \tag{2.3}$$

Condition (2.3) implies

$$(\Phi U)' + \Psi U = \omega g, \tag{2.4}$$

where $\omega \neq 0$ arbitrary and g is a locally integrable function with rapid decay representing the null-form (see[8])

$$\int_{-\infty}^{+\infty} x^n g(x)dx = 0 \ , \ n \geq 0. \tag{2.5}$$

Conversely, if U is a solution of (2.4) verifying the hypothesis above and the condition

$$\int_{-\infty}^{+\infty} U(x)dx \neq 0, \tag{2.6}$$

then (2.2)-(2.3) are fulfilled and (2.1) defines a linear form u which is a solution of (1.13).

Now, the linear form u is $\mathcal{B}[\nu]$, $\nu \neq -n-1 \ , \ n \geq 0$ with

$$\Phi(x) = x^3 \quad , \quad \Psi(x) = -2(\nu+1)x^2 - \frac{1}{2}.$$

Equation (2.4) becomes

$$\big(x^3 U\big)' - \{2(\nu+1)x^2 + \frac{1}{2}\}U = \omega g(x). \tag{2.4$'$}$$

For instance, let $g(x) = -|x|s(x^2), \ x \in \mathbb{R}$ [8] where s is the Stieltjes function [8,9]

$$s(x) = \begin{cases} 0, & x \leq 0, \\ \mathrm{e}^{-x^{\frac{1}{4}}} \sin x^{\frac{1}{4}}, & x > 0. \end{cases} \tag{2.7}$$

A possible solution of (2.4)’ is the even function

$$(2.8)\qquad U(x)=\begin{cases}0, & x=0,\\ \omega|x|^{2\nu-1}\mathrm{e}^{-\frac{1}{4x^2}}\displaystyle\int_{|x|}^{+\infty}t^{-2\nu-1}\,\mathrm{e}^{\frac{1}{4t^2}}s(t^2)dt, & x\in\mathbb{R}-\{0\}.\end{cases}$$

First, condition (2.2) is fulfilled, for we have

$$|x^3U(x)|\le|\omega||x|^{2\Re\nu+2}\mathrm{e}^{-\frac{1}{4x^2}}\int_{|x|}^{+\infty}t^{-2\Re\nu-1}\,\mathrm{e}^{\frac{1}{4t^2}}\,\mathrm{e}^{-t^{\frac{1}{2}}}\,dt=o\big(\mathrm{e}^{-\frac{1}{2}|x|^{\frac{1}{2}}}\big),\ |x|\to+\infty.$$

Further, when $x\to+\infty$

$$|U(x)|\le|\omega|x^{2\Re\nu-1}\int_{x}^{+\infty}t^{-2\Re\nu-1}\,\mathrm{e}^{-t^{\frac{1}{2}}}\,dt=o\big(\mathrm{e}^{-\frac{1}{2}x^{\frac{1}{2}}}\big),$$

and when $x\to+0$

$$|U(x)|\le|\omega|x^{2\Re\nu-1}\mathrm{e}^{-\frac{1}{4x^2}}\int_{x}^{1}t^{-2\Re\nu-1}\,\mathrm{e}^{\frac{1}{4t^2}}\,dt+o(1),$$

we apply l’Hospital’s rule to the ratio

$$\lim_{x\to+0}\frac{\displaystyle\int_{x}^{1}t^{-2\Re\nu-1}\,\mathrm{e}^{\frac{1}{4t^2}}\,dt}{x^{-2\Re\nu+1}\,\mathrm{e}^{\frac{1}{4x^2}}}=\lim_{x\to+0}\frac{x}{(2\Re\nu-1)x^2+\frac{1}{2}}=0,$$

so $\lim\limits_{x\to+0}U(x)=0=U(0)$.

Consequently, $U\in\mathrm{L}_1$.

Condition (2.6) now becomes

$$(2.9)\quad\int_{-\infty}^{+\infty}U(x)dx=2\omega\int_{0}^{+\infty}\xi^{-2\nu-1}\,\mathrm{e}^{\frac{1}{4\xi^2}}s(\xi^2)\Big(\int_{0}^{\xi}x^{2\nu-1}\mathrm{e}^{-\frac{1}{4x^2}}\,dx\Big)d\xi=\omega S_\nu\neq0$$

with

$$(2.10)\qquad S_\nu=2\int_{0}^{+\infty}t^{-4\nu-1}\,\mathrm{e}^{\frac{1}{4t^4}}\,\varphi_{\nu-\frac{3}{2}}(t^2)\mathrm{e}^{-t}\sin t\,dt,$$

$$(2.11)\qquad \varphi_\nu(t)=\int_{0}^{t}x^{2\nu+2}\mathrm{e}^{-\frac{1}{4x^2}}\,dx.$$

Let us establish some results about S_ν

Lemma 2.1. *We have for* $\nu \geq -1$

$$\frac{1}{4}t^2\varphi_\nu(t) \leq \varphi_{\nu+1}(t) \leq t^2\varphi_\nu(t),\ t \geq 0, \tag{2.12}$$

$$2\frac{t^{2\nu+5}}{1+2(2\nu+5)t^2}\mathrm{e}^{-\frac{1}{4t^2}} \leq \varphi_\nu(t) \leq 4\frac{t^{2\nu+5}}{2+(2\nu+5)t^2}\mathrm{e}^{-\frac{1}{4t^2}},\ t \geq 0. \tag{2.13}$$

Proof. It is easy to prove (2.12) from (2.11) and monotonicity.
From (2.11), we have upon integration by parts

$$\varphi_\nu(t) = 2t^{2\nu+5}\mathrm{e}^{-\frac{1}{4t^2}} - 2(2\nu+5)\varphi_{\nu+1}(t),\ \nu \in \mathbb{C},\ t \geq 0. \tag{2.14}$$

Now, in accordance of (2.12) and (2.14) we obtain the desired result (2.13) ■

Proposition 2.2. *We have the following expression for* $m \geq 1,\ \nu \in \mathbb{C}$

$$S_\nu = (-1)^m 2^{2m+1}\prod_{k=1}^{m}(\nu+k)\int_0^{+\infty} t^{-4\nu-1}\mathrm{e}^{\frac{1}{4t^4}}\varphi_{\nu-\frac{3}{2}+m}(t^2)\mathrm{e}^{-t}\sin t dt. \tag{2.15}$$

Proof. From (2.14), and using the Stieltjes representation (2.5) of the null-form, we get

$$S_\nu = -2^3(\nu+1)\int_0^{+\infty} t^{-4\nu-1}\mathrm{e}^{\frac{1}{4t^4}}\varphi_{\nu-\frac{1}{2}}(t^2)\mathrm{e}^{-t}\sin t dt.$$

Suppose (2.15) for $m \geq 1$ fixed. From (2.14) where $\nu \to \nu + m$ and $t \to t^2$

$$\varphi_{\nu+m-\frac{3}{2}}(t^2) = 2t^{4(\nu+m+1)}\mathrm{e}^{-\frac{1}{4t^4}} - 4(\nu+m+1)\varphi_{\nu+m-\frac{1}{2}}(t^2),$$

hence easily (2.15) for $m \to m+1$ ■

Corollary 2.3. *We have* $S_{-n-1} = 0,\ n \geq 0.$
This result is consistent with the fact that the linear form $\mathcal{B}[\nu]$ is not regular for these values of ν.

Proposition 2.4. *For* $\nu \geq \frac{1}{2}$ *, we have* $S_\nu > 0.$

Proof. First, we need the following lemma [8]. ■

Lemma 2.5. *Consider the following integral*

$$S = \int_0^{+\infty} F(t)\sin t dt \tag{2.16}$$

where we suppose $F(t) \geq 0$, continuous, increasing in $0 < t \leq \bar{t}$ and decreasing to zero for $t > \bar{t}$. Then,

$$(2.17) \qquad 0 < \bar{t} \leq \pi, \quad \int_0^\pi \big(F(t) - F(t+\pi)\big) \sin t dt \geq 0 \Longrightarrow S > 0.$$

Now, denoting $F(t) = F_\nu(t) = f_\nu(t)\mathrm{e}^{-t}$ with $f_\nu(t) = t^{-4\nu-1}\mathrm{e}^{\frac{1}{4t^4}}\varphi_{\nu-\frac{3}{2}}(t^2)$. We have from (2.13)

$$(2.13)' \qquad \frac{2t^3}{1+4(\nu+1)t^4} \leq f_\nu(t) \leq \frac{2t^3}{1+(\nu+1)t^4}, \ t \geq 0, \ \nu \geq \frac{1}{2}.$$

Then,

$$(2.18) \qquad \frac{2t^3}{1+4(\nu+1)t^4}\mathrm{e}^{-t} \leq F_\nu(t) \leq \frac{2t^3}{1+(\nu+1)t^4}\mathrm{e}^{-t}, \ t \geq 0, \ \nu \geq \frac{1}{2}.$$

Consequently, $F_\nu(t) > 0$ for $t > 0$, $F_\nu(0) = 0$ and $\lim_{t\to+\infty} F_\nu(t) = 0$ which implies that F_ν has a maximum for $t = \bar{t}$ defined by $f'_\nu(\bar{t}) = f_\nu(\bar{t})$.
Hence,

$$(2.19) \qquad f_\nu(\bar{t}) = \frac{2\bar{t}^3}{1+(4\nu+1)\bar{t}^4+\bar{t}^5}$$

since

$$f'_\nu(t) = \frac{2}{t^2} - \Big\{\frac{4\nu+1}{t} + \frac{1}{t^5}\Big\} f_\nu(t), \, t > 0.$$

From the first inequality of (2.13)' and by virtue of (2.19) necessarily $\bar{t} \leq 3$.
Therefore the implication (2.17) is true if the following is verified

$$(2.20) \qquad \int_0^\pi \sin t \frac{(\pi+t)^3}{1+(\nu+1)(\pi+t)^4}\mathrm{e}^{-t-\pi} dt \leq \int_0^\pi \sin t \frac{t^3}{1+4(\nu+1)t^4}\mathrm{e}^{-t} dt.$$

The function $t \mapsto \frac{t^3}{1+(\nu+1)t^4}$ is decreasing for $t \geq t_1 = (\frac{3}{\nu+1})^{\frac{1}{4}}$ and from $\nu \geq \frac{1}{2}$ we have easily $t_1 < \frac{\pi}{2}$. We have successively

$$\int_0^\pi \sin t \frac{(\pi+t)^3}{1+(\nu+1)(\pi+t)^4}\mathrm{e}^{-t-\pi} dt \leq \mathrm{e}^{-\pi}\frac{\pi^3}{1+(\nu+1)\pi^4}\frac{\mathrm{e}^{-\pi}+1}{2}.$$

On the other hand

$$\int_{t_1}^\pi \sin t \frac{t^3}{1+4(\nu+1)t^4}\mathrm{e}^{-t} dt = \int_{t_1}^\pi \sin t \frac{t^3}{1+(\nu+1)t^4}\frac{1+(\nu+1)t^4}{1+4(\nu+1)t^4}\mathrm{e}^{-t} dt$$

$$\geq \tfrac{1}{4}\frac{\pi^3}{1+(\nu+1)\pi^4}\int_{\frac{\pi}{2}}^{\pi}\sin t\,\mathrm{e}^{-t}dt$$
$$\geq \tfrac{1}{8}\mathrm{e}^{-\pi}\frac{\pi^3}{1+(\nu+1)\pi^4}(1+\mathrm{e}^{\frac{\pi}{2}}).$$

Thus, (2.20) is fulfilled if

$$\mathrm{e}^{-\pi}\frac{\pi^3}{1+(\nu+1)\pi^4}\frac{(1+\mathrm{e}^{-\pi})}{2} \leq \int_0^{t_1}\sin t\frac{t^3}{1+4(\nu+1)t^4}\mathrm{e}^{-t}dt + \tfrac{1}{8}\mathrm{e}^{-\pi}\frac{\pi^3}{1+(\nu+1)\pi^4}\big(1+\mathrm{e}^{\frac{\pi}{2}}\big). \tag{2.21}$$

But, $1+\mathrm{e}^{-\pi} < \frac{1}{4}(1+\mathrm{e}^{\frac{\pi}{2}})$, therefore the inequality (2.21) is satisfied and the proposition is proved ■

Finally, for $f \in \mathcal{P}$, $\nu \geq \frac{1}{2}$

$$\langle \mathcal{B}[\nu], f\rangle = S_\nu^{-1}\int_{-\infty}^{+\infty}\frac{1}{x^2}\int_{|x|}^{+\infty}\Big(\frac{|x|}{t}\Big)^{2\nu+1}\exp\Big(\frac{1}{4t^2}-\frac{1}{4x^2}\Big)s(t^2)dt f(x)dx. \tag{2.22}$$

Let now $\mathcal{B}(\alpha)$, $\alpha \neq -\frac{n}{2}$, $n \geq 0$ be the Bessel D-classical linear form. We have[4]

$$\Big(x^2\mathcal{B}(\alpha)\Big)' - 2(\alpha x+1)\mathcal{B}(\alpha) = 0. \tag{2.23}$$

In the following proposition we are going to establish the connection between $\mathcal{B}[\nu]$ and $\mathcal{B}(\alpha)$.

Proposition 2.6. *We have*

$$\sigma\mathcal{B}[\nu] = h_{\frac{1}{8}}\mathcal{B}\big(\frac{\nu+1}{2}\big), \ \ \nu \neq -n-1 \ , \ \ n \geq 0. \tag{2.24}$$

Proof. From (1.23) we have

$$\Big(x^3\mathcal{B}[\nu]\Big)' - \Big\{2(\nu+1)x^2 + \frac{1}{2}\Big\}\mathcal{B}[\nu] = 0. \tag{2.25}$$

Applying the operator σ to the both sides of (2.25) and in accordance of (1.11)-(1.12) we get

$$\Big(x^2\sigma\mathcal{B}[\nu]\Big)' - \Big\{(\nu+1)x + \frac{1}{4}\Big\}\sigma\mathcal{B}[\nu] = 0. \tag{2.25$'$}$$

Moreover, the linear form $\mathcal{B}[\nu]$ is symmetric and regular then $\sigma\mathcal{B}[\nu]$ is regular[3,7]. So, on the one hand, taking into account (2.25)' the linear form $\sigma\mathcal{B}[\nu]$ is D-classical.

On the other hand, from (2.23) with $\alpha = \frac{\nu+1}{2}$, $\mathcal{B}\Big(\frac{\nu+1}{2}\Big)$ satisfies the functional equation

$$(2.23)' \qquad \Big(x^2\mathcal{B}\big(\frac{\nu+1}{2}\big)\Big)' - 2\Big(\frac{\nu+1}{2}x+1\Big)\mathcal{B}\big(\frac{\nu+1}{2}\big) = 0.$$

Formula (1.14) with the choice $a = 8, b = 0$ yields to

$$(2.23)'' \qquad \Big(x^2 h_{\frac{1}{8}}\mathcal{B}\big(\frac{\nu+1}{2}\big)\Big)' - \Big((\nu+1)x+\frac{1}{4}\Big)h_{\frac{1}{8}}\mathcal{B}\big(\frac{\nu+1}{2}\big) = 0.$$

Consequently, we obtain (2.24) ■

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