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A GENERALIZATION OF BESSEL'S INTEGRAL FOR THE BESSEL COEFFICIENTS

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Abstract. We derive an integral over the m-dimensional unit hypercube that generalizes Bessel's integral for $J_n(x)$. The integrand is $G(x\psi(\mathbf{t}))\exp(-2\pi\,\mathrm{i}\,\mathbf{n}\cdot\mathbf{t})$, where G is analytic, and $\psi(\mathbf{t}) = \mathrm{e}^{2\pi\mathrm{i}t_1} + \ldots + \mathrm{e}^{2\pi\mathrm{i}t_m} + \mathrm{e}^{-2\pi\mathrm{i}(t_1+\ldots+t_m)}$, while $\mathbf{n}$ is a set of non-negative integers. In particular, we consider the case when G is a hypergeometric function ${}_pF_q$.

1. Introduction

The series definition of the Bessel function

$$J_\nu(z) = \frac{\left(\frac{1}{2}z\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left[\begin{array}{c} \\ \nu+1 \end{array}\middle| -\tfrac{1}{4}z^2\right], \quad -\nu \notin \mathbb{N}, \tag{1}$$

and Bessel's integral representation

$$2\pi J_n(z) = \int_0^{2\pi} \exp(\mathrm{i}(z\sin\varphi - n\varphi))\mathrm{d}\varphi, \quad n \in \mathbb{Z}, \tag{2}$$

are well known and may be found in many textbooks; see, for instance, Ch. 7 in [1] or Ch. 6 in [2].

We are interested in establishing a multidimensional generalization of (2). However, it is more convenient to work within the framework of hypergeometric functions. Accordingly, we set $z = x\exp(-\frac{1}{2}\pi\mathrm{i})$ and $\varphi = \frac{1}{2}\pi + 2\pi t$ to obtain the equivalent representation

$$\frac{\left(\frac{1}{2}x\right)^n}{n!}\,{}_0F_1\left[\begin{array}{c} \\ n+1 \end{array}\middle| \tfrac{1}{4}x^2\right] = \int_0^1 \exp(x\cos(2\pi t) - 2\pi\mathrm{i}nt)\,\mathrm{d}t, \quad n \in \mathbb{N}_0. \tag{3}$$

In the sequel we shall establish a generalization of (3) in terms of an integral over the m-dimensional unit hypercube.

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2. The Generalized Integral

Let boldface letters denote m-dimensional vectors with the customary dot product. The analogue of the factor $\exp(-2\pi \mathrm{i} n t)$ may reasonably be expected to be $\exp(-2\pi \mathrm{i}\mathbf{n}\cdot\mathbf{t})$, where $n_1,\ldots,n_m \in \mathbb{N}_0$. It is less evident what should take the place of $\exp(x\cos(2\pi t))$. Some preliminary considerations indicated that we should consider $G(x\psi(\mathbf{t}))$, where

$$\psi(\mathbf{t}) = \exp(2\pi \mathrm{i} t_1) + \ldots + \exp(2\pi \mathrm{i} t_m) + \exp(-2\pi \mathrm{i}\,(t_1 + \ldots + t_m)), \tag{4}$$

while G is an analytic function. Introduce its Maclaurin expansion

$$G(\xi) = \sum_{k=0}^{\infty} \frac{g(k)}{k!}\xi^k, \quad |\xi| < R, \tag{5}$$

where for brevity $g(k)$ is written instead of the derivative $G^{(k)}(0)$. The integral to be investigated thus reads,

$$I = \int_0^1 \ldots \int_0^1 G(x\psi(\mathbf{t}))\exp(-2\pi \mathrm{i}\mathbf{n}\cdot\mathbf{t})\ \mathrm{d}t_1\cdots\mathrm{d}t_m. \tag{6}$$

This may, on account of (5), be written

$$I = \sum_{k=0}^{\infty} g(k)x^k L(k), \tag{7}$$

where

$$L(k) = \int_0^1 \ldots \int_0^1 \frac{[\psi(\mathbf{t})]^k}{k!}\ \exp(-2\pi \mathrm{i}\mathbf{n}\cdot\mathbf{t})\ \mathrm{d}t_1\cdots\mathrm{d}t_m. \tag{8}$$

Next, by the multinomial theorem,

$$\begin{aligned}
&\frac{[\psi(\mathbf{t})]^k}{k!}\ \exp(-2\pi \mathrm{i}\mathbf{n}\cdot\mathbf{t}) \\
&= \sum_{\mathcal{J}_k} \frac{\exp[2\pi \mathrm{i}\,(\mu_1 t_1 + \ldots + \mu_m t_m - \mu_0\,(t_1 + \ldots + t_m) - (n_1 t_1 + \ldots + n_m t_m))]}{\mu_0!\,\mu_1\cdots\mu_m!} \\
&= \sum_{\mathcal{J}_k} \frac{\exp[2\pi \mathrm{i}\,(\mu_1 - \mu_0 - n_1)\,t_1]\cdots\exp[2\pi \mathrm{i}\,(\mu_m - \mu_0 - n_m)\,t_m]}{\mu_0!\,\mu_1!\cdots\mu_m!},
\end{aligned}$$

where the index set $\mathcal{J}_k$ is given by the inequalities

$$\mu_0 \geq 0,\ \mu_1 \geq 0,\ \ldots, \mu_m \geq 0,\ \mu_0 + \mu_1 + \ldots + \mu_m = k. \tag{9}$$

Hence,

$$\begin{aligned} L(k) &= \sum_{\mathcal{J}_k} \frac{1}{\mu_0!\,\mu_1!\cdots\mu_m!} \int_0^1 \exp\left[2\pi\mathrm{i}\,(\mu_1-\mu_0-n_1)\,t_1\right]\mathrm{d}t_1 \times \\ &\quad \cdots \times \int_0^1 \exp\left[2\pi\mathrm{i}\,(\mu_m-\mu_0-n_m)\,t_m\right]\mathrm{d}t_m \\ &= \sum_{\mathcal{J}_k} \frac{1}{\mu_0!\,\mu_1!\cdots\mu_m!}\,\delta(\mu_1,\mu_0+n_1)\cdots\delta(\mu_m,\mu_0+n_m) \\ &= \sum_{k=0}^{\infty} \frac{1}{\mu_0!\;(\mu_0+n_1)!\;\cdots(\mu_0+n_m)!}, \end{aligned}$$

where $\delta(\kappa,\lambda)$ is Kronecker's delta. The condition $k=\mu_0+\mu_1+\ldots+\mu_m$ implies that the last sum is empty unless we have

$$k = (m+1)\,\mu_0 + n_1 + \ldots + n_m \tag{10}$$

for some integer μ_0. Introducing for brevity

$$N = n_1 + \ldots + n_m \tag{11}$$

we may now state the result,

$$L(k) = \begin{cases} \dfrac{1}{\mu!\;(\mu+n_1)!\cdots(\mu+n_m)!}, & k=(m+1)\,\mu+N, \quad \mu\in\mathbb{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Inserting this into (7) we obtain

$$\begin{aligned} I &= \sum_{k=0}^{\infty} g(k)x^k L(k) = \sum_{\mu=0}^{\infty} \frac{g((m+1)\,\mu+N)x^{(m+1)\mu+N}}{\mu!\,(\mu+n_1)!\cdots(\mu+n_m)!} \\ &= \frac{x^N}{n_1!\cdots n_m!} \sum_{\mu=0}^{\infty} \frac{g((m+1)\,\mu+N)\,\left(x^{m+1}\right)^{\mu}}{\mu!\,(n_1+1)_\mu\cdots(n_m+1)_\mu}. \end{aligned}$$

Thus, the final result is,

$$\begin{aligned} &\int_0^1 \cdots \int_0^1 G(x\psi(\mathbf{t}))\exp\left(-2\pi\mathrm{i}\mathbf{n}\cdot\mathbf{t}\right)\,\mathrm{d}t_1\cdots\mathrm{d}t_m \\ &\quad = \frac{x^N}{n_1!\cdots n_m!} \sum_{\mu=0}^{\infty} \frac{g((m+1)\,\mu+N)\,\left(x^{m+1}\right)^{\mu}}{\mu!\,(n_1+1)_\mu\cdots(n_m+1)_\mu}, \end{aligned} \tag{13}$$

for $|x|$ sufficiently small.

3. The Hypergeometric Case

Assume now that G is a hypergeometric function,

$$G(\xi) = {}_pF_q\left[\begin{array}{c} a_1, \dots, a_p \\ c_1, \dots, c_q \end{array}\middle|\, \xi\right]; \tag{14}$$

we then have

$$g(k) = \frac{(a_1)_k \cdots (a_p)_k}{(c_1)_k \cdots (c_q)_k}. \tag{15}$$

Furthermore, by the multiplication formula for the Pochhammer symbol we obtain

$$\begin{aligned}(\alpha)_{N+(m+1)\mu} &= (\alpha)_N\,(\alpha+N)_{(m+1)\mu} \\ &= (\alpha)_N\,(m+1)^{(m+1)\mu}\left(\frac{\alpha+N}{m+1}\right)_\mu\left(\frac{\alpha+N+1}{m+1}\right)_\mu\cdots\left(\frac{\alpha+N+m}{m+1}\right)_\mu,\end{aligned}$$

and, by insertion, we arrive at the desired integral formula:

$$\begin{aligned}&\int_0^1\cdots\int_0^1 {}_pF_q\left[\begin{array}{c} a_1, \dots, a_p \\ c_1, \dots, c_q \end{array}\middle|\, x\psi(\mathbf{t})\right]\,\exp\left(-2\pi\mathrm{i}\mathbf{n}\cdot\mathbf{t}\right)\,\mathrm{d}t_1\cdots\mathrm{d}t_m \\ &= \frac{(a_1)_N\cdots(a_p)_N\,x^N}{(c_1)_N\cdots(c_q)_N\,n_1!\cdots n_m!} \\ &\quad\times{}_{(m+1)p}F_{(m+1)q+m}\times\left[\begin{array}{c}\mathcal{P}_\mathrm{N}\\ \mathcal{P}_\mathrm{D}\end{array}\middle|\,\left[x\,(m+1)^{p-q}\right]^{m+1}\right],\end{aligned} \tag{16}$$

where the parameter sets are given as follows

$$\begin{aligned}\mathcal{P}_\mathrm{N} &= \{\Delta(m+1, a_1+N), \dots, \Delta(m+1, a_p+N)\}, \\ \mathcal{P}_\mathrm{D} &= \{\Delta(m+1, c_1+N), \dots, \Delta(m+1, c_q+N), n_1+1, \dots, n_m+1\}\end{aligned} \tag{17}$$

with, as usual,

$$\Delta(\nu, \alpha) = \left\{\frac{\alpha}{\nu}, \frac{\alpha+1}{\nu}, \dots, \frac{\alpha+\nu-1}{\nu}\right\}. \tag{18}$$

As to the hypergeometric functions in (16) we must, in general, require $p \le q+1$. Moreover, in the case $p = q+1$ they are hypergeometric *series* for $|x|\,(m+1) < 1$; otherwise, analytic continuations have to be considered.

4. Particular Cases

We note some results obtained by further specialization.

4.1. Assume that one of the numerator parameters $a_1, \ldots, a_p$ equals a negative integer $-M$. If $M < N$, the right-hand memeber of (16) vanishes. If $N \le M \le N + m$, the hypergeometric function on the right-hand side of (16) reduces to unity and we are left with the prefactor.

4.2. For $m = 1$, we obtain $\psi(t) = 2\cos(2\pi t)$, and the formula (16) yields,

$$\begin{aligned} &\int_0^1 {}_pF_q\left[\begin{array}{c} a_1, \ldots, a_p \\ c_1, \ldots, c_q \end{array}\middle| 2x\cos(2\pi t)\right] \exp\left(-2\pi \mathrm{i} n t\right) \mathrm{d}t \\ &= \frac{(a_1)_n \cdots (a_p)_n \; x^n}{(c_1)_n \cdots (c_q)_n \; n!} \times \\ &\times {}_{2p}F_{2q+1}\left[\begin{array}{c} \frac{1}{2}(a_1+n), \frac{1}{2}(a_1+n+1), \ldots, \frac{1}{2}(a_p+n), \frac{1}{2}(a_p+n+1) \\ \frac{1}{2}(c_1+n), \frac{1}{2}(c_1+n+1), \ldots, \frac{1}{2}(c_q+n), \frac{1}{2}(c_q+n+1), n+1 \end{array}\middle| 4^{p-q}x^2\right]. \end{aligned} \tag{19}$$

We may, furthermore, take $p = 0 = q$, and replace x with $\frac{1}{2}x$. This leads to (3).

4.3. Let $m = 2$, $p = 1$, $q = 0$, $a_1 = \frac{1}{2}$, $\mathbf{n} = (n, 2n)$. Moreover, let $x \to \frac{1}{3}$; then on the right-hand side of (16) a ${}_3F_2[1]$ appears to which Watson's theorem applies. After a few steps involving elementary properties of the Pochhammer symbol, and the duplication formula for the Gamma function, we arrive at the formula

$$\begin{aligned} &\int_0^1\int_0^1 \frac{\exp(-2\pi \mathrm{i} n\,(t_1+2t_2))}{\sqrt{1-\frac{1}{3}\left[\exp(2\pi \mathrm{i}\; t_1)+\exp(2\pi \mathrm{i} t_2)+\exp(-2\pi \mathrm{i}\,(t_1+t_2))\right]}}\,\mathrm{d}t_1\,\mathrm{d}t_2 \\ &= \frac{\pi\left(\frac{1}{6}\right)_n\left(\frac{5}{6}\right)_n}{4^n\left[\Gamma(\frac{1}{2}n+\frac{7}{12})\,\Gamma(\frac{1}{2}n+\frac{11}{12})\right]^2} = \frac{1}{4\pi}\Gamma\left[\begin{array}{c} \frac{1}{2}n+\frac{1}{12}, \frac{1}{2}n+\frac{5}{12} \\ \frac{1}{2}n+\frac{7}{12}, \frac{1}{2}n+\frac{11}{12} \end{array}\right]. \end{aligned} \tag{20}$$

4.4. The case $m = 3$, $p = 1$, $q = 0$, $a_1 = 1$, $\mathbf{n} = (n, n, 2n)$, and $x \to \frac{1}{4}$, is reminiscent of the preceding one. A parameter cancellation takes place, and we obtain a ${}_3F_2[1]$ to which we can, again, apply Watson's theorem. The formula obtained reads,

$$\begin{aligned} &\int_0^1\int_0^1\int_0^1 \frac{\exp(-2\pi \mathrm{i} n\,(t_1+t_2+2t_3))\;\;\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3}{1-\frac{1}{4}\left[\exp(2\pi \mathrm{i} t_1)+\exp(2\pi \mathrm{i} t_2)+\exp(2\pi \mathrm{i} t_3)+\exp(-2\pi \mathrm{i}\,(t_1+t_2+t_3))\right]} \\ &= \frac{\pi\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{4^n\left[\Gamma(\frac{1}{2}n+\frac{5}{8})\,\Gamma(\frac{1}{2}n+\frac{7}{8})\right]^2} = \frac{1}{2\sqrt{2}\pi}\Gamma\left[\begin{array}{c} \frac{1}{2}n+\frac{1}{8}, \frac{1}{2}n+\frac{3}{8} \\ \frac{1}{2}n+\frac{5}{8}, \frac{1}{2}n+\frac{7}{8} \end{array}\right]. \end{aligned} \tag{21}$$

5. Further Generalization

One might consider a function G of several variables (see, e.g., [3]) in such a way that the integrand would involve (for example) $G(x_1\psi(\mathbf{t}), \ldots, x_r\psi(\mathbf{t}))$. Although the corresponding investigation would proceed along similar lines, and the function L would again be useful, the resulting expressions would be rather bulky; we shall, therefore, leave this approach aside.

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