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ON NON-DEVELOPABLE RULED SURFACES IN LORENTZ-MINKOWSKI 3-SPACES

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Dedicated to Professor Seong-Back Lee on the occasion of his retirement.

Abstract. In this paper, we classify ruled surfaces in Lorentz-Minkowski 3-spaces satisfying some algebraic equations in terms of the second Gaussian curvature, the mean curvature and the Gaussian curvature.

1. INTRODUCTION

The inner geometry of the second fundamental form has been a popular research topic for ages. It is readily seen that the second fundamental form of a surface is non-degenerate if and only if a surface is non-developable.

On a non-developable surface M, we can consider the Gaussian curvature K_{II} of the second fundamental form which is regarded as a new Riemannian metric. Therefore, K_{II} can be defined formally and it is the curvature of the Riemannian or pseudo-Riemannian manifold (M, II). Using classical notation, we denote the component functions of the second fundamental form by e, f and g. Thus we define the second Gaussian curvature by (cf. [2])

$$K_{II} = \frac{1}{(|eg| - f^2)^2}\left(\begin{vmatrix} -\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_t & \frac{1}{2}g_s \\ \frac{1}{2}e_t & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix} \right). \tag{1.1}$$

It is well known that a minimal surface has vanishing second Gaussian curvature but that a surface with vanishing second Gaussian curvature need not be minimal.

For the study of the second Gaussian curvature, D. Koutroufiotis ([10]) has shown that a closed ovaloid is a sphere if $K_{II} = cK$ for some constant c or if

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$K_{II} = \sqrt{K}$, where K is the Gaussian curvature. Th. Koufogiorgos and T. Hasanis ([9]) proved that the sphere is the only closed ovaloid satisfying $K_{II} = H$, where H is the mean curvature. Also, W. Kühnel ([11]) studied surfaces of revolution satisfying $K_{II} = H$. One of the natural generalizations of surfaces of revolution is the helicoidal surfaces. In [1] C. Baikoussis and Th. Koufogiorgos proved that the helicoidal surfaces satisfying $K_{II} = H$ are locally characterized by constancy of the ratio of the principal curvatures. On the other hand, D. E. Blair and Th. Koufogiorgos ([2]) investigated a non-developable ruled surface in a Euclidean 3-space $\mathbb{R}^3$ satisfying the condition

(1.2) $$aK_{II} + bH = \text{constant}, \quad 2a + b \neq 0,$$

along each ruling. Also, they proved that a ruled surface with vanishing second Gaussian curvature is a helicoid.

Recently, the second author ([16]) studied a non-developable ruled surface in a Euclidean 3-space $\mathbb{R}^3$ satisfying the conditions

(1.3) $$aH + bK = \text{constant}, \quad a \neq 0,$$

(1.4) $$aK_{II} + bK = \text{constant}, \quad a \neq 0,$$

along each ruling.
In particular, if it satisfies the condition (1.3), then a surface is called a linear Weingarten surface (see [12]).

On the other hand, in [7] the present authors investigated a non-developable ruled surface in a Lorentz-Minkowski 3-space satisfying the conditions (1.2), (1.3) and (1.4).

In this article, we will study a non-developable ruled surface in a Lorentz-Minkowski 3-space $\mathbb{L}^3$ satisfying the conditions

(1.5) $$aH^2 + 2bHK_{II} + cK_{II}^2 = \text{constant}, \quad a \neq 4(b - c), c \neq 0,$$

(1.6) $$aK^2 + 2bKK_{II} + cK_{II}^2 = \text{constant}, \quad c \neq 0,$$

(1.7) $$aH^2 + 2bHK + cK^2 = \text{constant}, \quad a \neq 0.$$

If a surface satisfies the equations (1.5), (1.6) and (1.7), then a surface is said to be a HK_{II}-quadric surface, KK_{II}-quadric surface and HK-quadric surface, respectively.

2. Preliminaries

Let $\mathbb{L}^3$ be a Lorentz-Minkowski 3-space with the scalar product of index 1 given by $\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is a standard rectangular coordinate system of $\mathbb{L}^3$. A vector x of $\mathbb{L}^3$ is said to be space-like if $\langle x, x \rangle > 0$ or $x = 0$, time-like if $\langle x, x \rangle < 0$ and light-like or null if $\langle x, x \rangle = 0$ and $x \neq 0$. A time-like or light-like vector in $\mathbb{L}^3$ is said to be *causal*. Now, we define a ruled surface M in a Lorentz-Minkowski 3-space $\mathbb{L}^3$. Let J_1 be an open interval in the real line $\mathbb{R}$. Let $\alpha = \alpha(s)$ be a curve in $\mathbb{L}^3$ defined on J_1 and $\beta = \beta(s)$ a transversal vector field along α. For an open interval J_2 of $\mathbb{R}$ we have the parametrization for M

$$x = x(s,t) = \alpha(s) + t\beta(s), \quad s \in J_1, \quad t \in J_2.$$

The curve $\alpha = \alpha(s)$ is called a base curve and $\beta = \beta(s)$ a director curve. In particular, the ruled surface M is said to be cylindrical if the director curve β is constant and non-cylindrical otherwise. First of all, we consider that the base curve α is space-like or time-like. In this case, the director curve β can be naturally chosen so that it is orthogonal to α. Furthermore, we have ruled surfaces of five different kinds according to the character of the base curve α and the director curve β as follows: If the base curve α is space-like or time-like, then the ruled surface M is said to be of type M_+ or type M_-, respectively. Also, the ruled surface of type M_+ can be divided into three types. In the case that β is space-like, it is said to be of type M_+^1 or M_+^2 if β' is non-null or light-like, respectively. When β is time-like, β' must be space-like by causal character. In this case, M is said to be of type M_+^3. On the other hand, for the ruled surface of type M_-, it is also said to be of type M_-^1 or M_-^2 if β' is non-null or light-like, respectively. Note that in the case of type M_- the director curve β is always space-like. The ruled surface of type M_+^1 or M_+^2 (resp. M_+^3, M_-^1 or M_-^2) is clearly space-like (resp. time-like). But, if the base curve α is a light-like curve and the vector field β along α is a light-like vector field, then the ruled surface M is called a *null scroll* (cf. [6]). Throughout the paper, we assume the ruled surface M under consideration is connected unless stated otherwise.

On the other hand, many geometers have been interested in studying submanifolds of Euclidean and pseudo-Euclidean space in terms of the so-called finite type immersion ([3]). Also, such a notion can be extended to smooth maps on submanifolds, namely the Gauss map ([4]). In this regard, the authors defined pointwise finite type Gauss map ([6]). In particular, the Gauss map G on a submanifold M of a pseudo-Euclidean space $\mathbb{E}_s^m$ of index s is said to be of *pointwise 1-type* if $\Delta G = fG$ for some smooth function f on M where Δ denotes the Laplace operator defined on M. The authors showed that minimal non-cylindrical ruled surfaces in a Lorentz-Minkowski 3-space have pointwise 1-type Gauss map ([6]). Based on this fact, the authors proved the following theorem which will be useful to prove our theorems in this paper.

Theorem 2.1 ([6]). *Let M be a non-cylindrical ruled surface with space-like or time-like base curve in a Lorentz-Minkowski 3-space. Then, the Gauss map is of pointwise 1-type if and only if M is an open part of one of the following spaces: the space-like or time-like helicoid of the 1st, the 2nd and the 3rd kind, the space-like or time-like conjugate of Enneper's surface of the 2nd kind.*

3. Main Results

In this section we study ruled HK_{II}-quadric surface, KK_{II}-quadric surface and HK-quadric surface M in a Lorentz-Minkowski 3-space $\mathbb{L}^3$. Thus the ruled surface M under consideration must have the non-degenerate second fundamental form which automatically implies that M is non-developable.

Theorem 3.1. *Let M be a non-developable ruled surface with non-null base curve in a Lorentz-Minkowski 3-space. Then, M is a HK_{II}-quadric surface if and only if M is an open part of one of the following surfaces :*

(1) *the helicoid of the 1st kind as space-like or time-like surface,*
(2) *the helicoid of the 2nd kind as space-like or time-like surface,*
(3) *the helicoid of the 3rd kind as space-like or time-like surface,*
(4) *the conjugate of Enneper's surfaces of the 2nd kind as space-like or time-like surface.*

Proof. We consider two cases separately.

Case 1. Let M be a non-developable ruled surface of the three types M^1_+, M^3_+ or M^1_-. Then the parametrization for M is given by

$$x = x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle \beta, \beta \rangle = \varepsilon_1(= \pm 1), \langle \beta', \beta' \rangle = \varepsilon_2(= \pm 1)$ and $\langle \alpha', \beta' \rangle = 0$. In this case α is the striction curve of x, and the parameter is the arc-length on the (pseudo-)spherical curve β. And we have the natural frame $\{x_s, x_t\}$ given by $x_s = \alpha' + t\beta'$ and $x_t = \beta$. Then, the first fundamental form of the surface is given by $E = \langle \alpha', \alpha' \rangle + \varepsilon_2 t^2, F = \langle \alpha', \beta \rangle$ and $G = \varepsilon_1$. For later use, we define the smooth functions Q, J and D as follows:

$$Q = \langle \alpha', \beta \times \beta' \rangle \neq 0, \quad J = \langle \beta'', \beta' \times \beta \rangle, \quad D = \sqrt{|EG - F^2|}.$$

In terms of the orthonormal basis $\{\beta, \beta', \beta \times \beta'\}$ we obtain

$$\alpha' = \varepsilon_1 F\beta - \varepsilon_1\varepsilon_2 Q\beta \times \beta', \tag{3.1}$$

$$\beta'' = \varepsilon_1\varepsilon_2(-\beta + J\beta \times \beta'), \tag{3.2}$$

$$\alpha' \times \beta = \varepsilon_2 Q\beta', \tag{3.3}$$

which imply $EG - F^2 = -\varepsilon_2 Q^2 + \varepsilon_1\varepsilon_2 t^2$. And, the unit normal vector N is given by $N = \frac{1}{D}(\varepsilon_2 Q\beta' - t\beta \times \beta')$. Then, the components e, f and g of the second fundamental form are expressed as

$$e = \frac{1}{D}(\varepsilon_1 Q(F - QJ) - Q't + Jt^2), \quad f = \frac{Q}{D} \neq 0, \quad g = 0.$$

Therefore, using the data described above and (1.1), we obtain

$$\begin{aligned} K_{II} &= \frac{1}{f^4}\left(ff_t(f_s - \frac{1}{2}e_t) - f^2(-\frac{1}{2}e_{tt} + f_{st})\right) \\ &= \frac{1}{2Q^2D^3}\left(Jt^4 + \varepsilon_1 Q(F - 2QJ)t^2 + 2\varepsilon_1 Q^2Q't + Q^3(F + QJ)\right). \end{aligned} \tag{3.4}$$

Furthermore, the mean curvature H is given by

$$\begin{aligned} H =& \frac{1}{2}\frac{Eg - 2Ff + Ge}{|EG - F^2|} \\ =& \frac{1}{2D^3}\left(\varepsilon_1 Jt^2 - \varepsilon_1 Q't - Q(F + QJ)\right). \end{aligned} \tag{3.5}$$

First of all, we suppose that $Q^2 - \varepsilon_1 t^2 > 0$. We now differentiate K_{II} and H with respect to t, the results are

$$\begin{aligned} (K_{II})_t =& \frac{1}{2Q^2D^5}\left(-\varepsilon_1 Jt^5 + Q(F + 2QJ)t^3 + 4Q^2Q't^2\right. \\ & \left. + \varepsilon_1 Q^3(5F - QJ)t + 2\varepsilon_1 Q^4Q'\right), \end{aligned} \tag{3.6}$$

$$H_t = \frac{1}{2D^5}\left(Jt^3 - 2Q't^2 - \varepsilon_1 Q(3F + QJ)t - \varepsilon_1 Q^2Q'\right). \tag{3.7}$$

Now, suppose that a non-developable ruled surface is HK_{II}-quadric surface. Then we have by (1.5)

$$aHH_t + b(H_tK_{II} + H(K_{II})_t) + cK_{II}(K_{II})_t = 0. \tag{3.8}$$

From (3.4)-(3.8) we have

$$aQ^4A_1 + bQ^2A_2 + cA_3 = 0, \tag{3.9}$$

where we put

$$\begin{aligned}
A_1 =& \varepsilon_1 J^2 t^5 - 3\varepsilon_1 JQ' t^4 + (4QJF - 2Q^2J^2 + 2\varepsilon_1 Q'^2)t^3 \\
&+ (2Q^2Q'J + 5QQ'F)t^2 + (Q^2Q'^2 + 4\varepsilon_1 Q^3JF + \varepsilon_1 Q^4J^2 \\
&+ 3\varepsilon_1 Q^2F^2)t + \varepsilon_1 Q^3Q'(F + QJ), \\
A_2 =& -Q'Jt^6 + (7\varepsilon_1 Q^2Q'J - \varepsilon_1 3QQ'F)t^4 \\
&+ (8Q^3JF - 4Q^2F^2 - 8\varepsilon_1 Q^2Q'^2)t^3 \\
&+ (-3Q^4Q'J - 18Q^3Q'F)t^2 \\
&+ (-8\varepsilon_1 Q^5JF - 8\varepsilon_1 Q^4F^2 - 4Q^4Q'^2)t - 3\varepsilon_1 Q^5Q'(QJ + F), \\
A_3 =& -\varepsilon_1 J^2t^9 + 4Q^2J^2t^7 + 2Q^2Q'Jt^6 \\
&+ (4\varepsilon_1 Q^3JF - 6\varepsilon_1 Q^4J^2 + \varepsilon_1 Q^2F^2)t^5 \\
&+ \varepsilon_1(6Q^3Q'F - 2Q^4Q'J)t^4 \\
&+ (4Q^6J^2 - 8Q^5JF + 6Q^4F^2 + 8\varepsilon_1 Q^4Q'^2)t^3 \\
&+ (16Q^5Q'F - 2Q^6Q'J)t^2 \\
&+ (4Q^6Q'^2 - \varepsilon_1 Q^8J^2 + 4\varepsilon_1 Q^7JF + 5\varepsilon_1 Q^6F^2)t \\
&+ 2\varepsilon_1 Q^7Q'(F + QJ).
\end{aligned} \tag{3.10}$$

From (3.10) we can obtain that the coefficient of the highest order t^9 of the equation (3.9) is

$$cJ^2 = 0.$$

Therefore, one finds $J = 0$ since $c \neq 0$, which implies (3.10) becomes

$$\begin{aligned}
A_1 =& 2\varepsilon_1 Q'^2t^3 + 5QQ'Ft^2 + (Q^2Q'^2 + 3\varepsilon_1 Q^2F^2)t + \varepsilon_1 Q^3Q'F, \\
A_2 =& -3\varepsilon_1 QQ'Ft^4 + (-8\varepsilon_1 Q^2Q'^2 - 4Q^2F^2)t^3 - 18Q^3Q'Ft^2 \\
&+ (-8\varepsilon_1 Q^4F^2 - 4Q^4Q'^2)t - 3\varepsilon_1 Q^5Q'F, \\
A_3 =& \varepsilon_1 Q^2F^2t^5 + 6\varepsilon_1 Q^3Q'Ft^4 + (6Q^4F^2 + 8\varepsilon_1 Q^4Q'^2)t^3 \\
&+ 16Q^5Q'Ft^2 + (4Q^6Q'^2 + 5\varepsilon_1 Q^6F^2)t + 2\varepsilon_1 Q^7Q'F.
\end{aligned} \tag{3.11}$$

By (3.11) the coefficient of the highest order t^5 of the equation (3.9) is

$$cQ^2F^2 = 0,$$

which implies $F = 0$. Therefore, (3.11) implies

$$\begin{aligned} A_1 =& 2\varepsilon_1 {Q'}^2 t^3 + Q^2 {Q'}^2 t, \\ A_2 =& -8\varepsilon_1 Q^2 {Q'}^2 t^3 - 4Q^4 {Q'}^2 t, \\ A_3 =& 8\varepsilon_1 Q^4 {Q'}^2 t^3 + 4Q^6 {Q'}^2 t. \end{aligned} \tag{3.12}$$

From (3.9) and (3.12) we have

$$ {Q'}^2 (a - 4b + 4c) = 0. $$

Thus, we show that $J = F = Q' = 0$ when $a \neq 4(b - c)$. In this case the surface is minimal by (3.5). Since $EG - F^2 = \varepsilon_1\varepsilon_2 t^2 - \varepsilon_2 Q^2$ and $Q^2 - \varepsilon_1 t^2 > 0$, the surface is space-like or time-like when $\varepsilon_2 = -1$ or $\varepsilon_2 = 1$, respectively.

But, $(\varepsilon_1, \varepsilon_2) = (-1, -1)$ is impossible because of the causal character. Let $(\varepsilon_1, \varepsilon_2) = (-1, 1)$. Then M is of the type M^3_+. Thus the surface is a helicoid of the 3rd kind according to Theorem 2.1. If $(\varepsilon_1, \varepsilon_2) = (1, \pm 1)$, then M is of the type M^1_+ or M^1_-. Hence the surface is a helicoid of the 1st kind or 2nd kind according to Theorem 2.1.

Next, we suppose that $Q^2 - \varepsilon_1 t^2 < 0$. In this case, we have

$$\begin{aligned} (K_{II})_t = & \frac{1}{2Q^2 D^5} \left(\varepsilon_1 J t^5 - Q(F + 2QJ)t^3 - 4Q^2 Q' t^2 \right. \\ & \left. + \varepsilon_1 Q^3 (-5F + QJ)t - 2\varepsilon_1 Q^4 Q' \right), \end{aligned} \tag{3.13}$$

$$ H_t = \frac{1}{2D^5} \left(-J t^3 + 2Q' t^2 - \varepsilon_1 Q(3F + QJ)t + \varepsilon_1 Q^2 Q' \right). \tag{3.14}$$

Thus, by the similar discussion as above we can also obtain $J = F = 0$ and $Q' = 0$ when $a \neq 4(b - c)$. Therefore, the surface is minimal. Since $EG - F^2 = -\varepsilon_2 (Q^2 - \varepsilon_1 t^2)$ and $Q^2 - \varepsilon_1 t^2 < 0$. Consequently, M is space-like or time-like according to $\varepsilon_2 = 1$ or $\varepsilon_2 = -1$, respectively.

In this case, $\varepsilon_1 = 1$. Therefore, M is of type M^1_+ or M^1_- depending on $\varepsilon_2 = \pm 1$. Thus, the surface is a helicoid of the 1st kind and the 2nd kind according to Theorem 2.1.

Case 2. Let M be a non-developable ruled surface of type M^2_+ or M^2_-. Then, the surface M is parametrized by

$$ x(s, t) = \alpha(s) + t\beta(s) $$

such that $\langle \beta, \beta \rangle = 1, \langle \alpha', \beta \rangle = 0, \langle \beta', \beta' \rangle = 0$ and $\langle \alpha', \alpha' \rangle = \varepsilon_1 (= \pm 1)$. We have put the non-zero smooth functions q and S as follows :

$$ q = ||x_s||^2 = \varepsilon \langle x_s, x_s \rangle = \varepsilon(\varepsilon_1 + 2St), \quad S = \langle \alpha', \beta' \rangle, $$

where ε denotes the sign of x_s. We note that $\beta \times \beta' = \beta'$. Then, the components of the induced pseudo-Riemannian metric on M are obtained by $E = \varepsilon q, F = 0$ and $G = 1$. For the moving frame $\{\alpha', \beta, \alpha' \times \beta\}$ we can calculate

$$\beta' = \varepsilon_1 S(\alpha' - \alpha' \times \beta), \quad \alpha'' = -S\beta - \varepsilon_1 R\alpha' \times \beta, \tag{3.15}$$

where $R = \langle \alpha'', \alpha' \times \beta \rangle$. Furthermore, using (3.15) we have

$$\langle \beta'', \alpha' \times \beta \rangle = S' + \varepsilon_1 SR, \quad \langle \alpha', \beta'' \rangle = S' + \varepsilon_1 SR.$$

The unit normal vector N is given by

$$N = \frac{1}{\sqrt{q}}(\alpha' \times \beta - t\beta'),$$

from which the coefficients of the second fundamental form are given by

$$e = \frac{1}{\sqrt{q}}(R + (S' + 2\varepsilon_1 SR)t), \quad f = \frac{S}{\sqrt{q}}, \quad g = 0.$$

On the other hand, the mean curvature H and the second Gaussian curvature K_{II} are obtained respectively by

$$H = \frac{1}{2q^{\frac{3}{2}}}(R + (S' + 2\varepsilon_1 SR)t), \tag{3.16}$$

$$K_{II} = \frac{\varepsilon_1 S'}{2Sq^{\frac{3}{2}}}. \tag{3.17}$$

Differentiating K_{II} and H with respect to t, we have

$$(K_{II})_t = \frac{-3}{2q^{\frac{5}{2}}}\varepsilon\varepsilon_1 S', \tag{3.18}$$

$$H_t = \frac{1}{2q^{\frac{5}{2}}}(\varepsilon\varepsilon_1 S' - \varepsilon SR - \varepsilon S(S' + 2\varepsilon_1 SR)t). \tag{3.19}$$

We suppose that a non-developable ruled surface is HK_{II}-quadric surface. Then, by (3.8), (3.16), (3.17), (3.18) and (3.19) we have

$$aSB_1 + bB_2 + cB_3 = 0, \tag{3.20}$$

where we put

$$\begin{aligned} B_1 =& -\varepsilon S(S' + 2\varepsilon_1 SR)^2 t^2 + (S' + 2\varepsilon_1 SR)(\varepsilon\varepsilon_1 S' - 2\varepsilon SR)t \\ & + \varepsilon\varepsilon_1 S'R - \varepsilon SR^2, \\ B_2 =& -4\varepsilon\varepsilon_1 SS'(S' + 2\varepsilon_1 SR)t - 4\varepsilon\varepsilon_1 SS'R + \varepsilon S'^2, \\ B_3 =& -3\varepsilon S'^2. \end{aligned} \tag{3.21}$$

By (3.20) and (3.21) we have

$$S' = -2\varepsilon_1 SR, \quad R^2(a - 4b + 4c) = 0,$$

since $a \neq 0$. Thus, we have $S' = 0$ and $R = 0$ when $a \neq 4(b - c)$. Consequently, the surface M is minimal by (3.16), that is, it is a conjugate of Enneper's surface of the 2nd kind as space-like or time-like surface according to Theorem 2.1. This completes the proof. ∎

Remark. In Theorem 3.1, if $a = 4(b - c)$, then, $J = F = 0$ with arbitrary Q' in Case 1 and $S' = -2\varepsilon_1 SR$ with arbitrary R in Case 2 imply the equation $K_{II} = -2H$.
In Case 1, we have

$$\alpha' = -\varepsilon_1\varepsilon_2 Q\beta \times \beta',$$

$$\beta'' = -\varepsilon_1\varepsilon_2\beta,$$

because of $J = F = 0$.

(1). $(\varepsilon_1, \varepsilon_2) = (1, 1)$. Without loss of generality, we may assume $\beta(0) = (0, 0, 1)$. Then we have

$$\beta(s) = (d_1 \sin s, d_2 \sin s, \cos s + d_3 \sin s)$$

for some constants d_1, d_2, d_3 satisfying $-d_1^2 + d_2^2 + d_3^2 = 1$. Since $\langle\beta, \beta\rangle = 1$, we have $-d_1^2 + d_2^2 = 1$ and $d_3 = 0$. From this we can obtain

$$\beta(s) = (d_1 \sin s, \pm\sqrt{1 + d_1^2}\sin s, \cos s),$$

for some constant d_1. Therefore, we have

$$\alpha(s) = (\mp\sqrt{1 + d_1^2}, -d_1, 0)f(s) + \mathbb{E},$$

where $f(s) = \int Q(s)ds$ and $\mathbb{E} = (e_1, e_2, e_3)$ is constant vector. Thus, the surface M has the parametrization of the form

$$\begin{aligned} x(s,t) =& (\mp\sqrt{1 + d_1^2}f(s) + td_1 \sin s + e_1, \\ & - d_1 f(s) \pm t\sqrt{1 + d_1^2}\sin s + e_2, t\cos s + e_3), \end{aligned} \tag{3.22}$$

where d_1 is constant, $f(s) = \int Q(s)ds$ and (e_1, e_2, e_3) is constant vector.

If $d_1 = 0$, then the surface M is a conoid of the 3rd kind (See [7]).

(2). $(\varepsilon_1, \varepsilon_2) = (1, -1)$. Without loss of generality, we may assume $\beta(0) = (0, 0, 1)$. Then we have

$$\beta(s) = (d_1 \sinh s, \pm\sqrt{d_1^2 - 1}\sinh s, \cosh s),$$

where $d_1 \leq -1$ or $d_1 \geq 1$. Therefore, we have

$$\alpha(s) = (\mp\sqrt{d_1^2 - 1}, d_1, 0) f(s) + \mathbb{E},$$

where $f(s) = \int Q(s)ds$ and $\mathbb{E} = (e_1, e_2, e_3)$ is constant vector. Thus, the parametrization for the surface M is given by

$$\begin{aligned} x(s,t) =& (\mp\sqrt{d_1^2 - 1} f(s) + t d_1 \sinh s + e_1, \\ & d_1 f(s) \pm t\sqrt{d_1^2 - 1}\sinh s + e_2, t\cosh s + e_3), \end{aligned} \tag{3.23}$$

where $d_1 \leq -1$ or $d_1 \geq 1$, $f(s) = \int Q(s)ds$ and (e_1, e_2, e_3) is constant vector.

If $d_1 = \pm 1$, then the surface M is a conoid of the 1st kind (See [7]).

(3). $(\varepsilon_1, \varepsilon_2) = (-1, 1)$. We may assume $\beta(0) = (1, 0, 0)$. Then we have

$$\beta(s) = (\cosh s, d_2 \sinh s, \pm\sqrt{1 - d_2^2}\sinh s),$$

where $-1 \leq d_2 \leq 1$. Therefore, we have

$$\alpha(s) = (0, \pm\sqrt{1 - d_2^2}, -d_2) f(s) + \mathbb{E},$$

where $f(s) = \int Q(s)ds$ and $\mathbb{E} = (e_1, e_2, e_3)$ is constant vector. Thus, the surface M is parametrized by

$$\begin{aligned} x(s,t) =& (t\cosh s + e_1, \pm\sqrt{1 - d_2^2} f(s) + t d_2 \sinh s + e_2, \\ & - d_2 f(s) \pm t\sqrt{1 - d_2^2}\sinh s + e_3), \end{aligned} \tag{3.24}$$

where $-1 \leq d_2 \leq 1$, $f(s) = \int Q(s)ds$ and (e_1, e_2, e_3) is constant vector.

If $d_2 = 0$ or $d_2 = \pm 1$, then the surface M is a conoid of the 2nd kind (See [7]).

(4). $(\varepsilon_1, \varepsilon_2) = (-1, -1)$ is impossible because of the causal character.

For specific functions $f(s)$ and appropriate intervals of s and t in (3.22), (3.23) and (3.24), we have the graphs shown in Figures 1, 2 and 3, respectively.

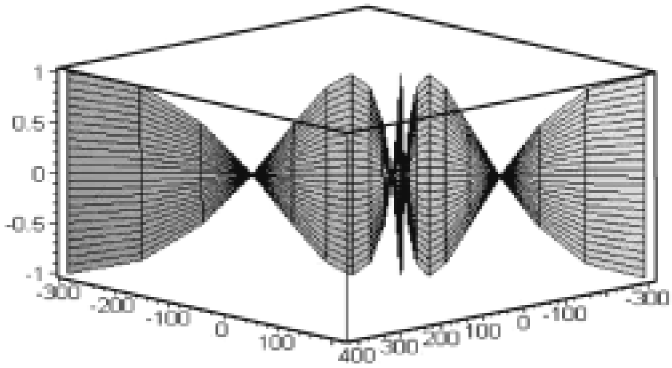

Fig. 1.

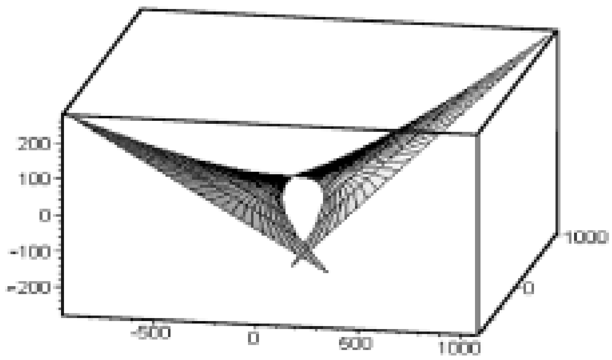

Fig. 2.

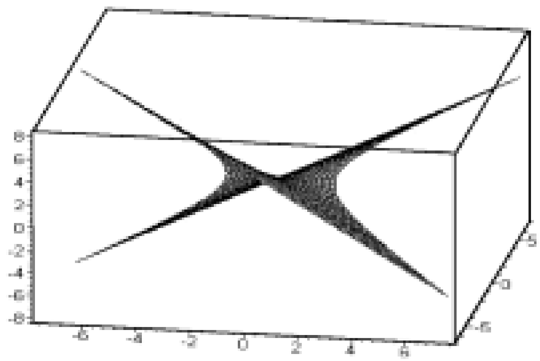

Fig. 3.

Theorem 3.2. *Let M be a non-developable ruled surface with non-null base curve in a Lorentz-Minkowski 3-space. Then, M is a HK-quadric surface if and only if M is an open part of one of the following surfaces:*

(1) *the helicoid of the 1st kind as space-like or time-like surface,*

(2) *the helicoid of the 2nd kind as space-like or time-like surface,*

(3) *the helicoid of the 3rd kind as space-like or time-like surface,*

(4) *the conjugate of Enneper's surfaces of the 2nd kind as space-like or time-like surface.*

Proof. In order to prove the theorem, we split it into two cases.

Case 1. As is described in Theorem 3.1 we assume that the non-developable ruled surface M of the three types M^1_+, M^3_+ or M^1_- is parametrized by

$$x = x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle \beta, \beta \rangle = \varepsilon_1 (= \pm 1), \langle \beta', \beta' \rangle = \varepsilon_2 (= \pm 1)$ and $\langle \alpha', \beta' \rangle = 0$. Using the same notations given in Theorem 3.1 the Gaussian curvature K is given by

$$K = \langle N, N \rangle \frac{eg - f^2}{EG - F^2} = \frac{Q^2}{D^4}. \tag{3.25}$$

Differentiating K with respect to t we obtain

$$K_t = \frac{4\varepsilon_1 Q^2 t}{D^6}. \tag{3.26}$$

Suppose that the surface M is HK-quadric. Then the equation (1.6) implies

$$aHH_t + b(H_t K + HK_t) + cKK_t = 0. \tag{3.27}$$

First of all, we assume that $Q^2 - \varepsilon_1 t^2 > 0$. Then, by substituting (3.5), (3.7), (3.25) and (3.26) into (3.27) it follows that

$$a^2 A_5^2 D^4 + (8acA_5A_6 - 4b^2A_4^2)D^2 + 16c^2A_6^2 = 0, \tag{3.28}$$

where we put

$$\begin{aligned}
A_4 &= 5Q^2Jt^3 - 6Q^2Q't^2 - (7\varepsilon_1 Q^3F + 5\varepsilon_1 Q^4 J)t - \varepsilon_1 Q^4 Q',\\
A_5 &= \varepsilon_1 J^2 t^5 - 3\varepsilon_1 Q'Jt^4 + (2\varepsilon_1 Q'^2 - 4QJF - 2Q^2J^2)t^3 + 3\varepsilon_1 Q^2F^2\\
&\quad + (2Q^2Q'J + 5QQ'F)t^2 + (Q^2Q'^2 + 4\varepsilon_1 Q^3JF + \varepsilon_1 Q^4 J^2)t\\
&\quad + \varepsilon_1 Q^3 Q'(QJ + F),\\
A_6 &= 4\varepsilon_1 Q^4 t.
\end{aligned} \tag{3.29}$$

From (3.29) we obtain that the coefficient of the highest order of the equation (3.28) is

$$a^2 J^4 = 0.$$

This equation implies $J = 0$ since $a \neq 0$ and (3.29) becomes

$$\begin{aligned} A_4 &= -6Q^2Q't^2 - 7\varepsilon_1 Q^3 Ft - \varepsilon_1 Q^4 Q', \\ A_5 &= 2\varepsilon_1 Q'^2 t^3 + 5QQ'Ft^2 + (Q^2Q'^2 + 3\varepsilon_1 Q^2F^2)t + \varepsilon_1 Q^3 Q'F, \\ A_6 &= 4\varepsilon_1 Q^4 t. \end{aligned} \tag{3.30}$$

By (3.28) and (3.30) we have $Q' = 0$, which implies $F = 0$. Thus, the mean curvature H is identically zero.

Next, we suppose that $Q^2 - \varepsilon_1 t^2 < 0$. In this case, by using (3.14) and (3.26) we can also show that the surface M is minimal. Consequently, by the proof of Theorem 3.1 the surface M is an open part of one of the helicoid of the 1st kind, 2nd kind and 3rd kind as space-like or time-like surface.

Case 2. Let M be a non-developable ruled surface of type M^2_+ or M^2_-. In this case, the curve α is space-like or time-like and β space-like but β' is light-like. We also use the notations given in Theorem 3.1. On the other hand, the Gaussian curvature K is obtained by

$$K = \frac{S^2}{q^2}, \tag{3.31}$$

and the differentiation of K with respect to t is given by

$$K_t = -\frac{4\varepsilon S^3}{q^3}. \tag{3.32}$$

Suppose that the surface M is HK-quadric. Then by (3.16), (3.19), (3.27), (3.31) and (3.32) we get

$$a^2q^4B_5^2 + 8acB_5B_6 - 4b^2qB_4^2 + 16c^2B_6^2 = 0, \tag{3.33}$$

where

$$\begin{aligned} B_4 &= (S' + 2\varepsilon_1 SR)(4S^4 - \varepsilon S^3)t + \varepsilon\varepsilon_1 S^2S' - \varepsilon S^3R - 4\varepsilon_1 S^3S', \\ B_5 &= -\varepsilon S(S' + 2\varepsilon_1 SR)^2t^2 + (S' + 2\varepsilon_1 SR)(\varepsilon\varepsilon_1 S' - 2\varepsilon SR)t \\ &\quad + \varepsilon\varepsilon_1 S'R - \varepsilon SR^2, \\ B_6 &= -4\varepsilon S^5. \end{aligned} \tag{3.34}$$

By (3.33) and (3.34) we show that $S' = 0$, $R = 0$ and $c = 0$. (3.16) implies that the mean curvature H is identically zero. Consequently, by the proof of Theorem 3.1 the surface M is a conjugate of Enneper's surface of the 2nd kind as space-like or time-like surface. This completes the proof. ∎

Combining the results of Theorems 3.1, 3.2 and Theorems in [6, 7], we have

Theorem 3.3. *Let M be a non-developable ruled surface with non-null base curve in a Lorentz-Minkowski 3-space. Then, the following are equivalent :*

(1) *M has pointwise 1-type Gauss map.*

(2) *M satisfies the equation $aK_{II} + bH = costant$, $a, b \in \mathbb{R} - \{0\}, 2a - b \neq 0$, along each ruling.*

(3) *M satisfies the equation $aH + bK = costant$, $a \neq 0, b \in \mathbb{R}$, along each ruling.*

(4) *M satisfies the equation $aH^2 + 2bHK_{II} + cK_{II}^2 =$ constant, $a \neq 4(b - c)$, along each ruling.*

(5) *M satisfies the equation $aH^2 + 2bHK + cK^2 =$ constant, $a \neq 0$, along each ruling.*

Theorem 3.4. *Let $\alpha(s) + t\beta(s)$ be a non-developable ruled surface with non-null base curve in a Lorentz-Minkowski 3-space. Then, M is a KK_{II}-quadric surface if and only if M is an open part of one of the following surfaces Then, we have the following:*

1. *Non-cylindrical ruled surfaces such that $\beta'(s)$ is non-null are parts of one of the following surfaces:*

   (1) *the helicoid of the 1st kind as space-like or time-like surface,*

   (2) *the helicoid of the 2nd kind as space-like or time-like surface,*

   (3) *the helicoid of the 3rd kind as space-like or time-like surface.*

2. *Non-cylindrical ruled surfaces such that $\beta'(s)$ is null have vanishing second Gaussian curvature.*

Proof. In order to prove the theorem, we also split it into two cases.

Case 1. As is described in Theorem 3.1 we assume that the ruled surface M of the three types M_+^1, M_+^3 or M_-^1 is assumed to be parametrized by

$$x = x(s, t) = \alpha(s) + t\beta(s)$$

such that $\langle\beta, \beta\rangle = \varepsilon_1(= \pm 1), \langle\beta', \beta'\rangle = \varepsilon_2(= \pm 1)$ and $\langle\alpha', \beta'\rangle = 0$. Likewise by Theorem 3.1 and 3.2 the second Gaussian curvature K_{II} and the Gaussian curvature K are given by (3.4) and (3.25), respectively. Suppose that the surface M is KK_{II}-quadric. First, we suppose that $Q^2 - \varepsilon_1 t^2 > 0$. Then, from (1.7) we have

$$aKK_t + b(K_tK_{II} + K(K_{II})_t) + cK_{II}(K_{II})_t = 0, \tag{3.35}$$

from which we get by (3.4), (3.6), (3.25) and (3.26)

$$c^2A_9^2D^4 + 8acA_7A_9D^2 + 16a^2Q^8A_7^2 - 4b^2Q^8A_8^2D^2 = 0, \tag{3.36}$$

where

$$\begin{aligned} A_7 &= 4\varepsilon_1 Q^4 t, \\ A_8 &= 3\varepsilon_1 J t^5 + (5QF - 6Q^2J)t^3 + 12Q^2Q't^2 \\ &\quad + (9\varepsilon_1 Q^3 F + 3\varepsilon_1 Q^4 J)t + 2\varepsilon_1 Q^4 Q', \\ A_9 &= -\varepsilon_1 J^2 t^9 + 4Q^2J^2t^7 + 2Q^2Q'Jt^6 \\ &\quad + (4\varepsilon_1 Q^3 JF - 6\varepsilon_1 Q^4 J^2 + \varepsilon_1 Q^2 F^2)t^5 \\ &\quad + (6\varepsilon_1 Q^3 Q'F - 2\varepsilon_1 Q^4 Q'J)t^4 \\ &\quad + (6Q^4F^2 - 8Q^5JF + 4Q^6J^2 + 8\varepsilon_1 Q^4 Q'^2)t^3 \\ &\quad + (16Q^5Q'F - 2Q^6Q'J)t^2 \\ &\quad + (4Q^6Q'^2 + 5\varepsilon_1 Q^6 F^2 + 4\varepsilon_1 Q^7 JF - \varepsilon_1 Q^8 J^2)t \\ &\quad + 2\varepsilon_1 Q^7 Q'(F + QJ). \end{aligned} \tag{3.37}$$

Similarly to Case 1 of Theorem 3.1 we can obtain $J = 0$, $F = 0$, $Q' = 0$ and $a = 0$. Therefore the mean curvature H is identically zero by the help of (3.5). Thus, the surface M is minimal.

Next, we suppose that $Q^2 - \varepsilon_1 t^2 < 0$. In this case, we can also show that M is minimal. Consequently, the surface M is an open part of one of the helicoids of the 1st kind, 2nd kind and 3rd kind as space-like or time-like surfaces depending on Case 1 of Theorem 3.1.

Case 2. Let M be a non-developable ruled surface of type M_+^2 or M_-^2. In this case, the curve α is space-like or time-like and β space-like but β' is light-like. Suppose that the surface M is KK_{II}-quadric. Then we have by (3.35)

$$c^2q^2B_9^2 + (8acSB_7B_9 - 4b^2S^2B_8^2)q + 16a^2S^2B_7^2 = 0, \tag{3.38}$$

where

$$\begin{aligned} B_7 &= -4\varepsilon S^5, \\ B_8 &= -7\varepsilon\varepsilon_1 S^2 S', \\ B_9 &= -3\varepsilon S'^2, \end{aligned} \tag{3.39}$$

which imply $S' = 0$ and $a = 0$. Thus, from (3.17) the second Gaussian curvature K_{II} is identically zero. This completes the proof. ■

Combining the results of Theorems 3.4 and Theorems in [7], we have

Theorem 3.5. *Let M be a ruled surface with non-null base curve in a Lorentz-Minkowski 3-space with non-degenerate second fundamental form. Then, the following are equivalent:*

(1) *M satisfies the equation $aK_{II} + bK =$ constant, $a \neq 0$, along each ruling.*

(2) *M satisfies the equation $aK^2 + 2bKK_{II} + cK_{II}^2 =$ constant, $c \neq 0$, along each ruling.*

Finally, we investigate the relations between the second Gaussian curvature, the Gaussian curvature and the mean curvature of null scrolls in $\mathbb{L}^3$.

Theorem 3.5. *Let M be a null scroll in a Lorentz-Minkowski 3-space. Then, M satisfies the equations $K = H^2, K_{II} = H^{-1}$.*

Proof. Let $\alpha = \alpha(s)$ be a light-like curve in $\mathbb{L}^3$ and $\beta = \beta(s)$ be a light-like vector field along α. Then, the null scroll M is parametrized by

$$x = x(s,t) = \alpha(s) + t\beta(s)$$

such that $\langle \alpha', \alpha' \rangle = 0$, $\langle \beta, \beta \rangle = 0$ and $\langle \alpha', \beta \rangle = 1$. Furthermore, without loss of generality, we may choose α as a null geodesic of M. We then have $\langle \alpha'(s), \beta'(s) \rangle = 0$ for all s. The induced Lorentz metric on M is given by $E = \langle \beta', \beta' \rangle t^2$, $F = 1$, $G = 0$ and the unit normal vector N is obtained by

$$N = \alpha' \times \beta + t\beta' \times \beta.$$

Thus, the component functions of the second fundamental form are given by

$$e = \langle \alpha'' + t\beta'', N \rangle, \quad f = \langle \beta', \alpha' \times \beta \rangle = Q, \quad g = 0,$$

which imply $H = Q$ and $K = Q^2$.

If $\langle \beta', \beta' \rangle = 0$, then β' is either the zero vector or a null vector. If β' is the zero vector, the surface is flat because of $f = Q = 0$. Therefore, β' is a null vector and there is a non-zero smooth function ρ such that $\beta = \rho\beta'$. It is a contradiction by the properties of α and β.

Since it is described in Section 2, β' cannot be a time-like vector and thus we can choose the parameter s in such a way that $\langle \beta', \beta' \rangle = 1$. Let $\{\alpha', \beta, \beta'\}$ be a null frame in $\mathbb{L}^3$. Then, the vector β'' can be expressed by

$$\beta'' = -\alpha' + \langle \alpha', \beta'' \rangle \beta,$$

from which

$$e_{tt} = 2\langle \beta'', N_t \rangle = 2\langle \beta'', \beta' \times \beta \rangle = 2Q.$$

Therefore, using (1.1) and the above equations the second Gaussian curvature K_{II} is given by

$$K_{II} = \frac{1}{2Q^2} e_{tt} = \frac{1}{Q}.$$

Thus, it easily follows that $K_{II} = \frac{1}{H}$ holds everywhere on a null scroll. This completes the proof. ■

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