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EXISTENCE OF INVARIANT SUBSPACE FOR CERTAIN COMMUTATIVE BANACH ALGEBRAS OF OPERATORS

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Abstract. The main result presented in this paper is the existence of a nontrivial subspace of an $\mathcal{A}$-module Banach space X hyperinvariant for the commutative algebra $\mathcal{A}$. From this result we can deduce the 1952 theorem of J. Wermer [8] and some other classical results on the existence of a nontrivial invariant subspace.

Résumé. Le résultat principal de ce papier est l'existence d'un sous-espace nontrivial d'un $\mathcal{A}$-module de Banach X, hyperinvariant pour l'algèbre commutative $\mathcal{A}$. Ce résultat inclut le théorème de J. Wermer [8] de 1952, ainsi que d'autres résultats classiques sur l'existence de sous-espace invariant nontrivial.

1. Introduction

The *invariant subspace problem* is stated below:

> *Given a normed vector space X and a bounded linear operator T on X ($T \in \mathcal{L}(X)$), does there exist a nontrivial subspace $M \subset X$ such that $TM \subset M$?*

Since Beurling's paper in 1949 [2] "On 2 problems concerning linear transformation on a Hilbert space", there have been several hundred papers on the subject of existence of a nontrivial invariant subspace for a given operator. Most papers attempt to solve the problem in the positive direction. It took more than 30 years before the question was settled negatively when X is a Banach space. Separately, C. Read (1984) [6] and P. Enflo (1987) [5] (which had been going around already

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before 1981) constructed a Banach space X and an operator $T \in \mathcal{L}(X)$ which does not admit a nontrivial invariant subspace. Since in both constructions the Banach space X is nonreflexive, there might still be some hope that the problem has a positive answer when X is a reflexive Banach space or in particular if X is a Hilbert space. But it is still interesting to find conditions on a general Banach space X and T for which T admits a nontrivial invariant subspace.

Among the important results in the invariant subspace problem are contained in the theorem of J.Wermer, 1952 [8]:

(a.) If $||T^n||$, does not "'grow too fast"' with $n \in \mathbf{Z}$, (e.g. $||T^n|| = O(|n|^k)$ for some $k > 0$,) then T admits a nontrivial invariant subspace.

(b.) If
$$\sum_{n \in \mathbf{Z}} \frac{ln^+||T^n||}{1+n^2} < \infty$$
and the spectrum $\sigma(T)$ contains at least two elements then T admits a nontrivial invariant subspace.

We will prove in this paper a theorem about existence of hyperinvariant subspaces for some regular semisimple Banach algebras of operators (Theorem 2), which also contains Wermer's theorem [8] as a special case.

2. Remarks

1. If X is a vector space over $\mathbf{C}$ and $1 < \dim X < \infty$, the classical Jordan decomposition theorem guarantees existence of a nontrivial invariant subspace for any bounded linear operator T on X.
2. If X is a vector space over R, a bounded linear operator T on $\mathbf{R}$ may not admit a nontrivial invariant subspace. For example, let
$$X = \mathbf{R^2} \quad \text{and} \quad \mathbf{T} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
Any nontrivial subspace M of $\mathbf{R^2}$ is of dimension one and must therefore be generated by a single nonzero element, say $(a,b)^t$. For any element $x = (\lambda a, \lambda b)^t$ in M, $\quad Tx = (-\lambda a, \lambda b)^t$ is not in M. Therefore T does not admit a nontrivial invariant subspace.
3. If X is nonseparable and $0 \neq x \in X$, then
$$M = \overline{span}\{T^n x : n \in \mathbf{N}\} \neq X.$$
Therefore M is a nontrivial invariant subspace for T.

4. It is easy to see that T admits a nontrivial subspace M if and only if M is invariant for every operator S in the algebra $\mathcal{A}_T$ generated by T; in particular if λ is not in the spectrum of T, $(\lambda I - T)$ is invertible and admits M as an invariant subspace.

Because of the above remarks, the invariant subspace problem is meaningful and interesting only under the following assumptions.

1. X is a vector space over **C**	2. $\dim X = \infty$.
3. X is separable	4. T^{-1} exists.

3. $\mathcal{A}$-Module Banach Space

Another way of looking at the invariant subspace problem is by considering $\mathcal{L}(X)$ as a Banach algebra. Our result is obtained through an algebraic approach.

(a) Definitions and Notations

The following definitions and notations are based on the paper of Datry-Muraz [DM]. Let X be a Banach space and $\mathcal{A}$ a commutative Banach algebra, where the product in $\mathcal{A}$ is denoted by $*$. X is an **$\mathcal{A}$-module Banach space** if there exists a homomorphism

$$\pi : \mathcal{A} \to \mathcal{L}(X)$$

such that $\pi(a * b) = \pi(a)\pi(b)$ and $||\pi(a)||_{\mathcal{L}(X)} \leq ||a||_{\mathcal{A}}$. And we write $a \star x = \pi(a)x$. X is said to be **order free** if for each $x \in X$, $x \neq 0$, there exists $a \in \mathcal{A}$ such that $a \star x \neq 0$. We denote by Δ the spectrum of $\mathcal{A}$ which consists of the nonzero complex homomorphisms on $\mathcal{A}$, called the characters of $\mathcal{A}$. If $C(\Delta)$ denotes the continuous functions on Δ, the **Gelfand morphism** is the map

$$\wedge : \mathcal{A} \to C(\Delta).$$

We write $\wedge(a) = \hat{a}$ and define $\hat{a}(\phi) = \phi(a)$ for all $\phi \in \Delta$. The kernel of the Gelfand morphism is the radical, Rad($\mathcal{A}$), of $\mathcal{A}$. It is defined by

$$Rad(\mathcal{A}) = \{a \in \mathcal{A} : \lim_{n\to\infty} ||a^n||^{1/n} = 0\}.$$

If $Rad(\mathcal{A}) = 0$ then the Gelfand morphism is injective and the algebra $\mathcal{A}$ is said to be **semisimple**. $\mathcal{A}$ is said to be **regular**(see [3]) if for each $\gamma_1 \in \Delta$ and each closed set $E \subset \Delta \setminus \{\gamma_1\}$ there is $a \in \mathcal{A}$ for which $\hat{a}(\gamma_1) = 1$ and $\hat{a}(\gamma) = 0$ for all $\gamma \in E$.

(b) Some Properties of Ideals of $\mathcal{A}$

Given an ideal I of $\mathcal{A}$,

$$Z(I) = \{\gamma \in \Delta : \hat{a}(\gamma) = 0 \quad \text{for all } a \in I\} = \cap_{a \in I} \hat{a}^{-1}(\{0\}).$$

Also given $F \subset \Delta$,

$$I(F) = \{a \in \mathcal{A} : \hat{a}(\gamma) = 0 \quad \text{for all } \gamma \in F\}.$$

In general, $I(Z(I)) \neq I$. And it is clear that

$$I \subset J \ \text{ implies } Z(I) \supset Z(J) \text{ and } F \subset G \text{ implies } I(F) \supset I(G).$$

(c) The Beurling Spectrum

Let $\mathcal{A}$ be a semisimple commutative algebra and X an $\mathcal{A}$-module Banach space. Given an $x \in X$, we write

(1) $I_x = \{a \in \mathcal{A} : a \star x = 0\}$.

(2) $sp(x) = \{\phi \in \Delta : \hat{a}(\phi) = 0, \forall a \in I_x\} = \cap_{a \in I_x} \hat{a}^{-1}(\{0\}) = Z(I_x)$

This is called the **Beurling spectrum** of x [4]. Similarly, for any subset M of X :

(3) $I_M = \{a \in \mathcal{A} : a \star x = 0, \forall x \in M\}$.

(4) $sp(M) = \{\gamma \in \Delta : \hat{a}(\gamma) = 0, \forall a \in I_M\}$.

And for any subset $F \subset \Delta$, we write

(5) $X(F) = \{x \in X : sp(x) \subset F\}$.

(d) Some Properties of Spectrum

(1) For all $x \in X$ and $a_1 \in \mathcal{A}$, $\ a_1 \notin I_x$

$$sp(a_1 \star x) \subset sp(x) \cap cl\{\gamma \in \Delta : \hat{a}_1(\gamma) \neq 0\}.$$

(2) For any $a_1 \in \mathcal{A}$,

$$sp(a_1 \star X) \subset sp(X) \cap cl\{\gamma \in \Delta : \hat{a}_1(\gamma) \neq 0\}.$$

Proof of 1 and 2. By definition, it follows that $I_x \subset I_{a_1 \star x}$ and $sp(a_1 \star x) \subset sp(x)$. Let $F = cl\{\gamma \in \Delta : \hat{a_1}(\gamma) \neq 0\}$. For every $b \in I(F)$,

$$b * (a_1 \star x) = (b * a_1) \star x = 0$$

because, $\hat{b}(\gamma)\hat{a}(\gamma) = 0$, for all $\gamma \in \Delta$. Thus,

$$I(F) \subset I_{a_1 \star x} \ \text{ and so } \ sp(a_1 \star x) = Z(I_{a_1 \star x}) \subset Z(I(F)) \subset F.$$

The proof of 2 follows because

$$\begin{aligned} sp(a_1 \star X) &= \cap_{x \in X} sp(a_1 \star x) \\ &\subset \cap_{x \in X} (sp(x) \cap F) \\ &= (\cap_{x \in X} sp(x)) \cap F = sp(X) \cap F. \end{aligned}$$

The following results give some properties when the spectrum is empty.

(3) $x \in X(\emptyset)$ if and only if $I_x = \mathcal{A}$.

Proof. Let $x \in X(\emptyset)$. By definition of $X(\emptyset)$, $sp(x) = \emptyset$. If $I_x \neq \mathcal{A}$, then I_x is a proper ideal of $\mathcal{A}$, and hence there is a maximal ideal I_M which contains I_x. We know that this maximal ideal is the kernel of some $\phi \in \Delta$, i.e., $I_x \subset I_M = \ker \phi$. So, for any $a \in I_x$, $\phi(a) = 0$ and so $\phi \in sp(x)$. This is a contradiction because $sp(x) = \emptyset$.

Conversely, let $I_x = \mathcal{A}$. Suppose there exists a $\phi \in sp(x)$. Then $\hat{a}(\phi) = 0$, for all $a \in \mathcal{A}$ and this implies that $\phi = 0$. This contradiction completes the proof.

(4) $X(\emptyset) = \{0\}$ if and only if X is of order free.

Proof. Let $X(\emptyset) = \{0\}$. Then for any nonzero $x \in X$, $x \notin X(\emptyset)$ and therefore by 3, I_x is strictly contained in $\mathcal{A}$. Hence there exists an $a \in \mathcal{A} \setminus I_x$ which means that $a \star x \neq 0$. X is therefore of order free.

Conversely, let $x \in X(\emptyset)$. Again by 3, $I_x = \mathcal{A}$, i.e., $a \star x = 0$ for all $a \in \mathcal{A}$. Since X is of order free, $x = 0$.

(e) Remarks

By 1, for any $F \in \Delta$, the space $X(F)$ is an $\mathcal{A}$-invariant subspace.

By 4, if X is not order free, the space $X(\emptyset)$ is an $\mathcal{A}$-invariant subspace, different of (0).

2. Existence of a Nontrivial Invariant Subspace

(a) When sp(X) has a finite number of elements

If $sp(X) = \emptyset$ 3 implies that $I_x = \mathcal{A}$, for every $x \in X$, i.e. $a * x = 0$ for all $a \in \mathcal{A}$, $x \in X$. In this case therefore every subspace of X is invariant for $\mathcal{A}$, because $\mathcal{A}$ annihilates every $x \in X$.
If $sp(X) = \{\gamma_0\}$.

Theorem 1. *Let $x \in X$ be such that $sp(x) = \{\gamma_0\}$. If $\{\gamma_0\}$ is a set of spectral synthesis(i.e., $Z(I) = \{\gamma_0\}$ if and only if $I = I(\{\gamma_0\})$) then for every $a \in \mathcal{A}$, $a \star x = \hat{a}(\gamma_0)x$.*

By hypothesis, the ideal $I_x = \{a \in \mathcal{A} : a \star x = 0\}$ satisfies

$$sp(x) = Z(I_x) = \{\gamma \in \Delta : \hat{a}(\gamma) = 0 \text{ for all } a \in I_x\}$$
$$= \cap_{a \in I_x} \hat{a}^{-1}(0) = \{\gamma_0\}.$$

Since $R(a)\{\gamma_0\}$ is a set of spectral synthesis, $I_x = \{a \in \mathcal{A} : \hat{a}(\gamma_0) = 0\}$ and hence for every $b \in \mathcal{A},\ b - \hat{b}(\gamma_0)e \in I_x$ (e is the identity in $\mathcal{A}$) and therefore, $(b - \hat{b}(\gamma_0)e) \star x = 0 = b \star x - \hat{b}(\gamma_0)x$. So for every $a \in \mathcal{A} \setminus I_x,\ x$ is an eigenvector corresponding to the eigenvalue $\hat{a}(\gamma_0)$.

If $sp(X)$ is a finite set with more than one element.

Let $\gamma_1 \in sp(X)$. For every $a \in \mathcal{A}$ and $x \in X$, the element $a * x - \hat{a}(\gamma_1)x$ is in X with

$$sp(\{a \star x - \hat{a}(\gamma_1)x : a \in \mathcal{A},\ x \in X\}) \subset sp(X) \setminus \{\gamma_1\}.$$

Thus the space generated by $\{a \star x - \hat{a}(\gamma_1)x : a \in \mathcal{A},\ x \in X\}$ is a nontrivial invariant subspace of X for $\mathcal{A}$, contained in $X(sp(X)\ \{\gamma_1\})$.

(b) When $\mathcal{A}$ is a Regular Algebra

In the preceding paragraph we saw the existence of a nontrivial subspace invariant when $sp(X)$ is empty or has a finite number of elements, even if $\mathcal{A}$ is not regular. The following theorem, covers the case when $sp(X)$ has an infinite number of elements but here, regularity of $\mathcal{A}$ is required.

Theorem 2. *Let $\mathcal{A}$ be a regular semisimple commutative Banach algebra and X an $\mathcal{A}-$module Banach space with $card\,(sp(X)) \geq 2$. Then X has a nontrivial subspace which is invariant for $\mathcal{A}$.*

Proof. If $\{\gamma_1, \gamma_2\} \subset sp(X)$, by regularity of $\mathcal{A}$, for each closed set K_2 where $\gamma_2 \in K_2$ and $\gamma_1 \notin K_2$, there exists $a_1 \in \mathcal{A}$ such that

$$\hat{a}(\gamma) = 0 \text{ if } \gamma \in K_2 \quad \text{and} \quad \hat{a}_1(\gamma_1) = 1.$$

K_2 is taken such that K_2 contains an open neighborhood of γ_2.

Now, let $X_1 = a_1 \star X$. Suppose that $a_1 \star x = 0$ for all $x \in X$ (i.e., $a_1 \in I_X$), by the choice of $a_1,\ \hat{a}_1(\gamma_1) = 1$. Therefore, $\gamma_1 \notin sp(X)$. This is a contradiction. Hence $a_1 \star X = X_1 \neq \{0\}$ and by 2,

$$sp(X_1) \subset sp(X) \cap \overline{\{\gamma : \hat{a}_1(\gamma) \neq 0.\}}$$

Since $\hat{a}_1(\gamma) = 0$ on the neighborhood of γ_2, then $\gamma_2 \notin sp(X_1)$. Thus $X_1 \neq X$. Moreover, $\mathcal{A} \star X_1 = \mathcal{A} \star (a_1 \star X) = a_1 \star (\mathcal{A} \star X) \subset a_1 \star X = X_1$. Hence X_1 is a nontrivial subspace of X which is invariant for $\mathcal{A}$. This completes the proof.

If $\mathcal{A}$ is a semisimple regular commutative subalgebra of $\mathcal{L}(X)$, then X can be considered naturally as an $\mathcal{A}$-module Banach space and hence X is necessarily order free.

The corollary below follows immediately from the theorem.

Corollary 1. *Let $\mathcal{A}$ be a commutative semisimple regular Banach subalgebra of the algebra $\mathcal{L}(X)$ of all bounded linear operators on X with $card(\Delta) \geq 2$. Then each $T \in \{\mathcal{A}\}'$ (the commutants of $\mathcal{A}$) admits a nontrivial invariant subspace.*

Remark. Let T be a bounded linear operator on X and $\mathcal{A}_T$ the sub- Banach algebra of $\mathcal{L}(X)$ generated by T; if $\mathcal{A}_T$ is semisimple and regular then T admits a nontrivial hyperinvariant subspace.

5. Applications

1. Proof of Wermer's theorem

It will be seen that the 1952 existence theorem of Wermer[8] can be deduced from the result of the previous section. Given a bounded linear operator T on a Banach space X, a Banach algebra $\mathcal{A}$ which is a certain weighted l^1-space, will be constructed and let this act on X, making X an $\mathcal{A}$-module Banach space. Let w be a weight function defined on $\mathbf{Z}$ and set

$$l^1(w) = \{a = (a_k) \in l^1(\mathbf{Z}) : \sum_{k\in\mathbf{Z}} |a_k| w(k) < \infty.\}$$

If $a = (a_k)$ and $b = (b_k)$, a multiplication in $l^1(w)$ is defined by

$$a * b = c \text{ where the } nth \text{ term of } c \text{ is } \ c_n = \sum_{k\in\mathbf{Z}} a_{n-k} b_k.$$

This makes $l^1(w)$ a commutative Banach algebra. The action of $l^1(w)$ on X is defined by

$$a \star x = \sum_{k\in\mathbf{Z}} a_k T^k x$$

making X an $l^1(w)$-module Banach space if $w(n)$ is defined by

$$w(n) = \max\{1, \|T^n\|, \|T^{-1}\|\}.$$

Then for some constant C, by the hypothesis of Wermer,

$$\sum_{n\neq 0} \frac{ln^+ w(n)}{n^2} < C \sum_{n\in\mathbf{Z}} \frac{ln^+ \|T^n\|}{1+n^2} \leq \infty.$$

By the Wiener-Domar Theorem [3], [?]

The subalgebra $l^1(w)$ of $l^1(\mathbf{Z})$ is regular and semisimple if and only if

$$\sum_{n\neq 0} \frac{ln^{+}w(n)}{n^2} < \infty.$$

it follows that for $w(n)$ defined above, $l^1(w)$ is a regular semisimple subalgebra of $l^1(\mathbf{Z}.)$ Hence by the Theorem 2, X has a nontrivial subspace M invariant under the action of $l^1(w)$. In particular for $a = (a_k)$ where $a_k = 0$ if $k \neq 1$ and $a_1 = 1,\ a \star x = Tx$ for all $x \in X$. Thus M is invariant for T. This proves Wermer's theorem.

2. If T is a normal operator, classical result [?] shows that the algebra $\mathcal{A}_{T^*T}$ generated by T^* and T is a C^*-algebra which is identified with the space $C(\sigma(T))$ of continuous functions on the spectrum of T. Since $C(\sigma(T))$ is regular and hence is semisimple, the well known result that a normal operator (in particular a self-adjoint operator) admits a nontrivial hyperinvariant subspace follows from Corollary 1.

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