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REGULARIZED RESOLVENT FAMILIES

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Abstract. In this paper we present the notion of regularized resolvent families, which generalizes the classes of regularized semigroups, regularized cosine families and resolvent families. We obtained some generation theorems and analyticity criterions for regularized resolvent families.

1. Introduction

Let X be a complex Banach space, $a \in L^1_{loc}(\mathbb{R}^+)$ (where $\mathbb{R}^+ = [0,\infty)$) be a scalar kernel $\not\equiv 0$, and A be a closed linear operator on X with dense domain $\mathcal{D}(A)$. We consider the Volterra equation

$$u(t) = f(t) + \int_0^t a(t-s)Au(s)\,ds, \quad t \geq 0 \tag{1.1}$$

where $f : \mathbb{R}^+ \to X$ is continuous.

The theory of the abstract Volterra equation has been developed rapidly due to its applications to many problems in mathematical physics, such as viscoelasticity and heat conduction in materials; see [14] and the references therein.

In 1980, Prato and Iannelli [4] first introduced the notion of resolvent families, which now plays a central role in the theory of Volterra equations. A family of strongly continuous bounded linear operators on X, $\{R(t)\}_{t\geq 0}$, is called a *resolvent family* for (1.1) if $R(t)$ commutes with A and satisfies the *resolvent equation*

$$R(t)x = x + \int_0^t a(t-s)AR(s)x\,ds, \quad t \geq 0,\ x \in \mathcal{D}(A).$$

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If the solution of (1.1) exists, it can be represented by the variation of parameters formula

$$u(t) = \frac{d}{dt}\int_0^t R(t-s)f(s)\,ds.$$

Recently there are some more general concepts such as integrated solution families [13] and $k(t)$-regularized resolvents [11]. The Laplace transform technique is a very useful tool for exponentially bounded resolvent families, integrated solution families and $k(t)$-regularized resolvents. However, in such cases, the resolvent set of the generator A is always nonempty.

In this paper we introduce the notion of C-regularized resolvent family, where C is an injective bounded operator on X, which allows the resolvent set of A being empty. The class of C-regularized resolvent families is a very natural extension of resolvent families (with $C = I$), C-regularized semigroups (with $a(t) \equiv 1$) and C-regularized cosine operators (with $a(t) \equiv t$) (see [4, 5, 15]). In section 2 we present some basic properties and generation theorems for regularized resolvent families. Theorem 3.3 and 3.2 give the analyticity criterions for C-regularized resolvent families, and Theorem 3.2 is even new for resolvent families. At last, we give several examples.

2. C-Regularized Resolvent Families and Their Basic Properties

Throughout this section, A is a densely defined closed operator on a complex Banach space X and $a \in L^1_{loc}(\mathbb{R}^+)$ is a scalar kernel $\not\equiv 0$ satisfying $\int_0^\infty e^{-\omega t}|a(t)|dt < \infty$. In the sequel, we will denote the range of a linear operator A by $\mathcal{R}(A)$, and by $\rho_C(A) = \{\lambda \in \mathbb{C} :\ \lambda - A$ is injective and $\mathcal{R}(C) \subset \mathcal{R}(\lambda - A)\}$ the *C-resolvent set* of A. If f is Laplace transformable, we denote by $\hat{f}(\lambda)$ the Laplace transform of f at λ.

Definition 2.1. A family $\{R(t)\}_{t\geq 0} \subset \mathbf{B}(X)$ is called a *C-regularized resolvent family for* (1.1) if the following conditions are satisfied:

(a) $R(\cdot)$ is strongly continuous on $\mathbb{R}^+$ and $R(0) = C$;

(b) $R(t)A \subset AR(t)$ for $t \geq 0$;

(c) For $x \in \mathcal{D}(A)$, $t \geq 0$, the C-resolvent equation

$$R(t)x = Cx + \int_0^t a(t-s)R(s)Axds \tag{2.1}$$

holds. If in addition, there are some constants $M, \omega \geq 0$ such that $\|R(t)\| \leq Me^{\omega t}$ for $t \geq 0$, then $\{R(t)\}_{t\geq 0}$ is called *exponentially bounded*.

We start with the following simple but important proposition.

Proposition 2.2. *Suppose $\{R(t)\}_{t\geq 0} \subset \mathbf{B}(X)$ is a C-regularized resolvent family for (1.1) satisfying $\|R(t)\| \leq Me^{\omega t}$. Then the following statements hold.*

(a) *For all $x \in X$, $t \geq 0$, $\int_0^t a(t-s)R(s)x\,ds \in \mathcal{D}(A)$ with*

$$R(t)x = Cx + A\int_0^t a(t-s)R(s)x\,ds \tag{2.2}$$

(b) *If $\mathrm{Re}\lambda > \omega$, then $\lambda - \lambda\hat{a}(\lambda)A$ is injective, $\mathcal{R}(\lambda - \lambda\hat{a}(\lambda)A) \subset \mathcal{R}(C)$, and*

$$(\lambda - \lambda\hat{a}(\lambda)A)^{-1}Cx = \int_0^\infty e^{-\lambda t}R(t)x\,dt \quad \textit{for } x \in X; \tag{2.3}$$

furthermore, if $\hat{a}(\lambda) \neq 0$, then $1/\hat{a}(\lambda) \in \rho_C(A)$. Moreover, $\{R(t)\}_{t\geq 0}$ is uniquely determined by A.

(c) $R(t)R(s) = R(s)R(t)$ *for $t, s \geq 0$.*

Proof. By Definition 2.1 $(b), (c)$ and the closedness of A, (2.2) holds for all $x \in \mathcal{D}(A)$, thus (a) follows from the denseness of $\mathcal{D}(A)$ and the closedness of A.

(b) By Fubini's theorem and (2.2), we have

$$\begin{aligned}\int_0^\infty e^{-\lambda t}R(t)x\,dt &= \int_0^\infty e^{-\lambda t}Cx\,dt + A\int_0^\infty \Big(e^{-\lambda t}\int_0^t a(t-s)R(s)x\,ds\Big)dt \\ &= \lambda^{-1}Cx + A\int_0^\infty R(s)x\Big(\int_s^\infty e^{-\lambda t}a(t-s)\,dt\Big)\,ds \\ &= \lambda^{-1}Cx + A\hat{R}(\lambda)\hat{a}(\lambda)x,\end{aligned}$$

where $x \in X$ and $\mathrm{Re}\lambda > \omega$. Hence,

$$(\lambda - \lambda\hat{a}(\lambda)A)\hat{R}(\lambda)x = Cx \quad \text{for } x \in X \text{ and } \mathrm{Re}\lambda > \omega. \tag{2.4}$$

Since $R(t)$ commutes with A, we get

$$\hat{R}(\lambda)(\lambda - \lambda\hat{a}(\lambda)A)x = Cx \quad \text{for } x \in \mathcal{D}(A) \text{ and } \mathrm{Re}\lambda > \omega,$$

so that the injectivity of C implies that $\lambda - \lambda\hat{a}(\lambda)A$ is injective. Thus we see that (2.3) holds for $\mathrm{Re}\lambda > \omega$ and $1/\hat{a}(\lambda) \in \rho_C(A)$ if $\hat{a}(\lambda) \neq 0$. Moreover, by the uniqueness of the Laplace transform, we know that $\{R(t)\}_{t\geq 0}$ is uniquely determined by A.

(c) We first prove that $R(t)C = CR(t)$ for all $t \geq 0$. By Definition 2.1(b), we have $CA \subset AC$. Thus, we get from (2.4) that

$$(\lambda - \lambda\hat{a}(\lambda)A)\hat{R}(\lambda)C = C^2 = (\lambda C - \lambda\hat{a}(\lambda)CA)\hat{R}(\lambda) = (\lambda - \lambda\hat{a}(\lambda)A)C\hat{R}(\lambda),$$

where Re$\lambda > \omega$. Thus $\hat{R}(\lambda)C = C\hat{R}(\lambda)$, so, by the uniqueness of the Laplace transform, we have $R(t)C = CR(t)$. Using a similar procedure one can show that (c) holds. ■

Necessary and sufficient conditions for the existence of a resolvent family have been studied by Da Prato and Lannelli in [4]. In the following, we will give the generation theorem for exponentially bounded C-regularized resolvent families by using the Laplace transform.

Theorem 2.3. *Suppose that* $\{R(t)\}_{t\geq 0} \subset \mathbf{B}(X)$ *is strongly continuous and satisfies* $\|R(t)\| \leq Me^{\omega t}$. *Then* $\{R(t)\}_{t\geq 0}$ *is a* C*-regularized resolvent family for* (1.1) *if and only if the following conditions hold:*

(a) $CA \subset AC$*;*

(b) $1/\hat{a}(\lambda) \in \rho_C(A)$ *for* $\lambda > \omega$ *with* $\hat{a}(\lambda) \neq 0$*;*

(c) $(\lambda - \lambda\hat{a}(\lambda)A)^{-1}Cx = \int_0^\infty e^{-\lambda t}R(t)x\, dt$ *for all* $x \in X$ *and* $\lambda > \omega$.

Proof. The necessity follows immediately from Definition 2.1 and Proposition 2.2.

Conversely, suppose that (a)-(c) hold. Let $x \in \mathcal{D}(A)$ and $\omega < \lambda$, then

$$(\lambda - \lambda\hat{a}(\lambda)A)^{-1}CAx = \int_0^\infty e^{-\lambda t}R(t)Ax\, dt,$$

which means

$$A(\lambda - \lambda\hat{a}(\lambda)A)^{-1}Cx = A\Big(\int_0^\infty e^{-\lambda t}R(t)x\, dt\Big) = \int_0^\infty e^{-\lambda t}R(t)Ax\, dt,$$

it thus follows from the inversion formulas for Laplace transform and the closedness of A (see cf. Theorem 1.10 in Ch 1. of [16]) that $R(t)x \in \mathcal{D}(A)$ with $AR(t)x = R(t)Ax$.

To prove (2.1), let $\rho_C(A) \ni \lambda > \omega$ and $x \in \mathcal{D}(A)$. Then, by (a) and (c), we obtain

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t}(R(t)x - Cx)dt \\
&= \hat{R}(\lambda)x - \lambda^{-1}Cx \\
&= (\lambda - \lambda\hat{a}(\lambda)A)^{-1}Cx - \lambda^{-1}Cx \\
&= (\lambda - \lambda\hat{a}(\lambda)A)^{-1}\hat{a}(\lambda)ACx \\
&= (\lambda - \lambda\hat{a}(\lambda)A)^{-1}C\hat{a}(\lambda)Ax = \hat{R}(\lambda)\hat{a}(\lambda)Ax \\
&= \int_0^\infty e^{-\lambda t}\Big(\int_0^t a(t-s)R(s)Ax\, ds\Big)\, dt.
\end{aligned}$$

The uniqueness theorem of the Laplace transform and the strongly continuity of $R(t)$ yield the C-regularized resolvent equation (2.1). Finally, by (2.1) we also get $R(0) = C$ and thus complete the proof. ■

Next, we will give a Hille-Yosida-type characterization of C-regularized resolvent family.

Theorem 2.4. (1.1) *admits a C-regularized resolvent family* $\{R(t)\}_{t\geq 0} \subset \mathbf{B}(X)$ *satisfying* $\|R(t)\| \leq Me^{\omega t}$ *if and only if the following conditions hold:*

(a) $CA \subset AC$*;*

(b) $1/\hat{a}(\lambda) \in \rho_C(A)$ *for* $\lambda > \omega$ *with* $\hat{a}(\lambda) \neq 0$*;*

(c) $\|((\lambda - \lambda\hat{a}(\lambda)A)^{-1}C)^{(n)}\| \leq Mn!(\lambda - \omega)^{-(n+1)}$, $\lambda > \omega$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Proof. Suppose that (1.1) admits a C-regularized resolvent family $R(t)$. Differentiating n times with respect to λ on both sides of the identity (2.3) yields (c). By the necessity part of Theorem 2.3 or by Definition 2.1 and Proposition 2.2(b), we see that the necessity holds. It remains to prove the sufficiency. By (c) and the Arendt-Widder Theorem given in [1], there is a family $\{F(t)\}_{t\geq 0} \subset \mathbf{B}(X)$ satisfying $F(0) = 0$,

$$\|F(t+h) - F(t)\| \leq Mhe^{\omega(t+h)}, \quad \forall t, h \geq 0 \tag{2.5}$$

such that

$$(\lambda - \lambda\hat{a}(\lambda)A)^{-1}Cx = \lambda \int_0^\infty e^{-\lambda t}F(t)x\,dt, \quad \forall\ x \in X \text{ and } \lambda > \omega. \tag{2.6}$$

On the other hand, for $x \in \mathcal{D}(A)$ and $\lambda > \omega$, by $ACx = CAx$, Fubini's theorem and integration by parts, we obtain

$$\begin{aligned}
&\lambda \int_0^\infty e^{-\lambda t}\Big(tCx + \int_0^t a(t-s)F(s)Ax\,ds\Big)dt \\
&= \lambda^{-1}Cx + \lambda \int_0^\infty \Big(\int_s^\infty e^{-\lambda t}a(t-s)\,dt\Big)F(s)Ax\,ds \\
&= \lambda^{-1}Cx + \lambda\Big(\int_0^\infty e^{-\lambda t}a(t)\,dt\Big)\Big(\int_0^\infty e^{-\lambda s}F(s)Ax\,ds\Big) \\
&= \lambda^{-1}Cx + \hat{a}(\lambda)(\lambda - \lambda\hat{a}(\lambda)A)^{-1}CAx \\
&= (\lambda - \lambda\hat{a}(\lambda)A)^{-1}Cx.
\end{aligned}$$

Thus, by the uniqueness of the Laplace transform, we have

$$F(t)x = tCx + \int_0^t a(t-s)F(s)Ax\,ds, \quad t \geq 0.$$

Since $F(t)Ax$ is locally Lipschitz for each $x \in \mathcal{D}(A)$ and $a(t)$ is locally integrable, we have by Proposition 1.3.7 in [2] that $F(t)x$ is continuously differentiable on $\mathbb{R}^+$ for each $x \in \mathcal{D}(A)$. Define $R(t)x = F'(t)x$, $t \geq 0$, for $x \in \mathcal{D}(A)$; since $\mathcal{D}(A)$ is dense, $R(t)$ can be extended to a bounded operator by (2.5) and the Banach-Steinhaus Theorem. Moreover it is clear that $R(t)x$ is strongly continuous, $\|R(t)\| \leq Me^{\omega t}$ and $\hat{R}(\lambda) = (\lambda - \lambda\hat{a}(\lambda)A)^{-1}C$, from which and Theorem 2.3 we obtain that $R(t)$ is a C-regularized resolvent family for (1.1). ■

Remark 2.5. If $a(t) \equiv 1$ or $a(t) \equiv t$, then Theorem 2.3 is the generation theorem for generators of C-regularized semigroup [5, 6] or C-regularized cosine function [15], respectively. In the case where $C = I$, it is the generation theorem due essentially to Da Prato and Iannelli [4].

At last, we have the following proposition which is a consequence of Proposition 2.2.

Proposition 2.6. *Assume that $\{R(t)\}_{t\geq 0} \subset \mathbf{B}(X)$ is a C-regularized resolvent family for (1.1) and satisfies $\|R(t)\| \leq Me^{\omega t}$, and assume that AC is unbounded. Then $\hat{a}(\lambda) \neq 0$ and $1/\hat{a}(\lambda) \in \rho_C(A)$ for all λ with Re$\lambda > \omega$.*

Proof. Suppose that $\hat{a}(\lambda_0) = 0$ for some λ_0 with Re$\lambda_0 > \omega$. Let $H(\lambda) = (\lambda - \lambda\hat{a}(\lambda)A)^{-1}C$, then $H(\lambda_0) = \frac{1}{\lambda_0}C$. Since $\hat{a}(\lambda)$ is analytic and $a(t) \not\equiv 0$, λ_0 is an isolate zero of finite multiplicity. Therefore, if Γ is a small circle around λ_0, such that $\Gamma \subset \{\lambda \in \mathbb{C} :\ \text{Re}\lambda > \omega\}$ and $\hat{a}(\lambda) \neq 0$ on Γ, then on Γ,

$$AH(\lambda) = \frac{\lambda H(\lambda) - C}{\lambda\hat{a}(\lambda)}$$

is analytic, hence by Cauchy's formula we have

$$AC = A\lambda_0 H(\lambda_0) = A\Big(\frac{1}{2\pi i}\int_\Gamma \frac{\lambda H(\lambda)}{\lambda - \lambda_0}\Big)\, d\lambda = \frac{1}{2\pi i}\int_\Gamma \frac{\lambda H(\lambda) - C}{(\lambda - \lambda_0)\hat{a}(\lambda)}\, d\lambda,$$

and so AC is a bounded operator, a contradiction to our assumption that AC is unbounded. Thus $\hat{a}(\lambda) \neq 0$ for all Re$\lambda > \omega$ and the conclusion follows from Proposition 2.2(b). ■

3. Analyticity Criterion for C-Regularized Resolvent Families

In the following we will denote by $\Sigma_\theta := \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg\lambda| < \theta\}$ and $\omega + \Sigma_\theta = \{\omega + \lambda : \lambda \in \Sigma_\theta\}$. We suppose that

(H_0) A is an unbounded densely defined closed operator on X, C is injective and $CA \subseteq AC$. Let $0 \not\equiv a \in L^1_{loc}(\mathbb{R}^+)$ satisfying $\int_0^\infty e^{-\omega_0 t}|a(t)|\, dt < \infty$.

Definition 3.1. The C-regularized resolvent family $R(t)$ for (1.1) is *analytic*, if the function $R(\cdot) : \mathbb{R}^+ \to \mathbf{B}(X)$ admits analytic extension to a sector Σ_{θ_0} for some $0 < \theta_0 \le \pi/2$. We say that $R(z) \in H(\omega_0, \theta_0)$, if in addition, for each $0 < \theta < \theta_0$ and $\omega > \omega_0$ there is $M = M(\omega, \theta)$ such that

$$\|R(z)\| \le Me^{\omega \mathrm{Re} z}, \quad z \in \Sigma_\theta \tag{3.1}$$

The following two theorems are the main results of this section. Theorem 3.3 generalizes the result of Theorem 3.1 in Ch.1 of [14], Theorem 3.2 is even new for resolvent family.

Theorem 3.2. *Under the assumption* (H_0), (1.1) *admits an analytic* C-*regularized resolvent family* $R(z) \in H(\omega_0, \theta_0)$ *if and only if the following conditions hold:*

$(A1)$ $a(t)$ *admits analytic extension to* Σ_{θ_0} *(still denoted by* $a(z)$*) and for each* $\theta \in (-\theta_0, \theta_0)$, $a(te^{i\theta}) \in L^1_{loc}(\mathbb{R}^+)$ *and there are* $\omega_\theta, M_\theta \in \mathbb{R}^+$ *such that* $\|a(z)\| \le M_\theta e^{\omega_\theta \mathrm{Re} z}$ *for* $z \in \Sigma_\theta$ *with* $\mathrm{Re} z \ge 1$ *and* $za(z) \to 0$ *as* $z \to 0$ *in* Σ_θ*;*

$(A2)$ *For each* $\theta \in (-\theta_0, \theta_0)$, *let* $a_\theta(t) = a(te^{i\theta})$, $A_\theta = e^{i\theta}A$. *Then the equation*

$$u(t) = f(t) + \int_0^t a_\theta(t-s)A_\theta u(s)\, ds, \quad t \ge 0, \tag{3.2}$$

admits a C-*regularized resolvent family* $R_\theta(t)$*;*

$(A3)$ *For each* $\theta \in (-\theta_0, \theta_0)$, $\omega > \omega_0$, *there exists constant* $M = M(\omega, \theta)$ *such that* $\|R_\theta(t)\| \le Me^{\omega t \cos\theta}$.

In this case, $R_\theta(t) = R(te^{i\theta})$.

Theorem 3.3. *Let* (H_0) *hold. Then* (1.1) *admits an analytic* C-*regularized resolvent family* $R(z) \in H(\omega_0, \theta_0)$ *if and only if the following conditions hold:*

$(H1)$ $\hat{a}(\lambda)$ *admits meromorphic extension to* $\omega_0 + \Sigma_{\theta_0 + \pi/2}$*;*

$(H2)$ $1/\hat{a}(\lambda) \in \rho_C(A)$ *for all* $\lambda \in \omega_0 + \Sigma_{\theta_0+\pi/2}$ *with* $\hat{a}(\lambda) \ne 0$*;*

$(H3)$ $(\lambda - \lambda\hat{a}(\lambda)A)^{-1}C$ *has an analytic extension* $H(\lambda)$ *to* $\omega_0 + \Sigma_{\theta_0+\pi/2}$, *and for each* $\omega > \omega_0$ *and* $0 < \theta < \theta_0$ *there is a constant* $c = c(\omega, \theta)$ *such that*

$$\|H(\lambda)\| \le c/|\lambda - \omega| \quad \text{for all } \lambda \in \omega + \Sigma_{\theta+\pi/2}.$$

Before proving these two theorems, we need some preparations.

Lemma 3.4. *Let* $\omega \in \mathbb{R}$, $\alpha \in (0, \pi/2]$ *and* $F : (\omega, \infty) \to X$. *Then the following statements are equivalent:*

(a) F *has an analytic extension (still denoted by* F*) to* $\omega + \Sigma_{\alpha+\pi/2}$ *such that* $\|(\lambda - \omega)F(\lambda)\| \le M_\beta$ $(\lambda \in \omega + \Sigma_{\beta+\pi/2})$ *for every* $\beta \in (0, \alpha)$*;*

(b) *There exists an analytic function* $h : \Sigma_\alpha \to X$ *with* $\|h(z)\| \le M_\beta e^{\omega \mathrm{Re} z}$ $(z \in \Sigma_\beta)$ *for every* $\beta \in (0, \alpha)$ *such that*

$$F(\lambda) = \int_0^\infty e^{-\lambda z} h(z)\, dz \quad \textit{for } \lambda > \omega.$$

As a direct consequence of Lemma 3.4, we have

Corollary 3.5. *Suppose that* (H_0) *holds and* $R(z) \in H(\omega_0, \theta_0)$ *is an analytic* C*-regularized resolvent family for* (1.1). *Then* $H(\lambda) := \hat{R}(\lambda)$ *admits analytic extension to sector* $\omega_0 + \Sigma_{\theta_0 + \pi/2}$ *and for each* $\omega > \omega_0$, $0 < \theta < \theta_0$ *there is* $M = M(\omega, \theta)$ *such that*

$$\|H(\lambda)\| \le \frac{M(\omega,\theta)}{|\lambda - \omega|}, \quad \lambda \in \omega + \Sigma_{\theta+\pi/2}. \tag{3.3}$$

Although the proofs of the next two results are not new (see e.g. [14]), we write them out for completeness and better understanding.

Lemma 3.6. *Suppose that* (H_0) *holds and* (1.1) *has an analytic* C*-regularized resolvent family* $R(z) \in H(\omega_0, \theta_0)$. *Then* $\hat{a}(\lambda)$ *admits meromorphic extension to the sector* $\omega_0 + \Sigma_{\theta_0+\pi/2}$.

Proof. Let $H(\lambda) = \hat{R}(\lambda)$. For $\omega > \omega_0$, by Proposition 2.2(b),

$$H(\lambda) = (I - \hat{a}(\lambda)A)^{-1} C/\lambda, \quad \mathrm{Re}\lambda > \omega. \tag{3.4}$$

Choose $x \in \mathcal{D}(A)$, $x^* \in X^*$ such that $g(\lambda) := \langle \lambda H(\lambda) C x, x^* \rangle \not\equiv \langle C^2 x, x^* \rangle$. Otherwise we have $\lambda H(\lambda) \equiv C$, which leads to $\hat{a}(\lambda) \equiv 0$ or $A = 0$, in contradiction to the assumption (H_0) that $a \not\equiv 0$ and A is unbounded. By Corollary 3.5, $H(\lambda)$ is analytic on the sector $\omega_0 + \Sigma_{\theta_0+\pi/2}$, and so are $g(\lambda)$ and $g'(\lambda)$. For $\mathrm{Re}\lambda > \omega$,

$$\begin{aligned} g'(\lambda) &= \hat{a}'(\lambda) \langle (I - \hat{a}(\lambda)A)^{-2} C^2 A x, x^* \rangle \\ &= \hat{a}'(\lambda) \lambda^2 \langle H(\lambda)^2 A x, x^* \rangle \\ &= \hat{a}'(\lambda) h(\lambda), \end{aligned}$$

where $h(\lambda) := \lambda^2 \langle H(\lambda)^2 A x, x^* \rangle$ is analytic on $\omega_0 + \Sigma_{\theta_0+\pi/2}$. We claim that $h(\lambda) \not\equiv 0$. Otherwise, $g(\lambda)$ is a constant. This and the fact that $g(\lambda) \to \langle C^2 x, x^* \rangle$ as $\lambda \to \infty$ (because $R(t)x \to Cx$ as $t \to 0^+$) imply that $g(\lambda) \equiv \langle C^2 x, x^* \rangle$, which is a contradiction to our choice of x and x^*. So we know that $\hat{a}'(\lambda) = g(\lambda)/h(\lambda)$ admits meromorphic extension to the same sector. Moreover,

$$\begin{aligned} g'(\lambda) &= \hat{a}'(\lambda) \langle (I - \hat{a}(\lambda)A)^{-2} C^2 A x, x^* \rangle \\ &= [\hat{a}'(\lambda)/\hat{a}(\lambda)] \cdot \langle (I - \hat{a}(\lambda)A)^{-2} C^2 x - (I - \hat{a}(\lambda)A)^{-1} C^2 x, x^* \rangle \\ &= [\hat{a}'(\lambda)/\hat{a}(\lambda)] \cdot \langle \lambda^2 H(\lambda)^2 x - \lambda H(\lambda) C x, x^* \rangle \\ &= [\hat{a}'(\lambda)/\hat{a}(\lambda)] \cdot k(\lambda), \end{aligned}$$

where $k(\lambda) := \langle \lambda^2 H(\lambda)^2 x - \lambda H(\lambda)Cx, x^*\rangle$ is analytic on $\omega_0 + \Sigma_{\theta_0+\pi/2}$; this implies that

$$\hat{a}(\lambda) = \hat{a}'(\lambda)/g'(\lambda) \cdot k(\lambda) = k(\lambda)/h(\lambda), \quad \text{Re}\lambda > \omega.$$

From which we know that $\hat{a}(\lambda)$ can be extended meromorphically to $\omega_0 + \Sigma_{\theta_0+\pi/2}$. ■

Proposition 3.7. *Suppose that* (H_0) *holds and* $R(z) \in H(\omega_0, \theta_0)$ *is an analytic* C*-regularized resolvent family for* (1.1)*. Then* $R(z)x \to Cx$ *for* $z \in \Sigma_\theta$ *as* $z \to 0$*, for each* $x \in X$ *and* $0 < \theta < \theta_0$.

Proof. Choose $x \in D(A)$ and $x^* \in X^*$ such that $\langle C^2Ax, x^*\rangle \neq 0$. Consider $f(z) = \langle R(z)x, x^*\rangle$. From the facts that $f(t) \to f(0) = \langle Cx, x^*\rangle$ as $t \to 0^+$, and f is analytic and uniformly exponentially bounded on Σ_θ for every $0 < \theta < \theta_0$, one knows that

$$\lambda\hat{f}(\lambda) = \langle \lambda H(\lambda)x, x^*\rangle \to \langle Cx, x^*\rangle$$

as $|\lambda| \to \infty$ when $\lambda \in \omega + \Sigma_{\theta+\pi/2}$ for any $\omega > \omega_0$; similarly, considering $(R*R)(z)$, we have

$$\langle \lambda^2 H(\lambda)^2 x, x^*\rangle \to \langle C^2x, x^*\rangle,$$

as $|\lambda| \to \infty$ when $\lambda \in \omega + \Sigma_{\theta+\pi/2}$. Now from

$$k(\lambda) = \langle \lambda H(\lambda)(\lambda H(\lambda)x - Cx), x^*\rangle \to 0$$

and

$$h(\lambda) = \lambda^2\langle H(\lambda)^2 Ax, x^*\rangle \to \langle C^2Ax, x^*\rangle,$$

we obtain that

$$\hat{a}(\lambda) = k(\lambda)/h(\lambda) \to 0,$$

as $|\lambda| \to \infty$ uniformly in $\lambda \in \omega + \Sigma_{\theta+\pi/2}$. Since $\hat{R}(\lambda) = H(\lambda)$, by the inversion formula for Laplace transform we have for $x \in X$, $z \in \Sigma_\theta$, $\theta_1 \in (\theta, \theta_0)$, $\omega > \omega_0$, and $R > 0$ that

$$R(z)x = (2\pi i)^{-1} \int_\Gamma e^{\lambda z} H(\lambda)x \, d\lambda,$$

where

$$\Gamma := \{\omega + re^{-i(\pi/2+\theta_1)} : R \le r < \infty\} \cup \{\omega + Re^{i\phi} : |\phi| \le \pi/2 + \theta_1\}$$
$$\cup \{\omega + re^{i(\pi/2+\theta_1)} : R \le r < \infty\}$$

is oriented counterclockwise. Moreover, by (3.4) we have for $x \in \mathcal{D}(A)$ and $\text{Re}\lambda > \omega$,

$$\lambda H(\lambda)(x - \hat{a}(\lambda)Ax) = Cx;$$

the above identity holds also for λ in $\omega_0 + \Sigma_{\theta_0+\pi/2}$ when $\hat{a}(\lambda)$ has an analytic extension at λ. And since $\hat{a}(\lambda) \to 0$ as $|\lambda| \to \infty$ uniformly in $\lambda \in \omega + \Sigma_{\theta+\pi/2}$, by choosing $z = te^{i\varphi}$, $R = 1/t$, and $\delta = \sin(\theta_1 - \theta)$, we obtain when t small enough that

$$R(z)x - Cx = (2\pi i)^{-1} \int_\Gamma e^{\lambda z} H(\lambda)\hat{a}(\lambda)Ax\, d\lambda,$$

which yields that

$$\begin{aligned} &\|R(z)x - Cx\| \\ \leq\ & \frac{M}{2\pi} \int_\Gamma e^{\mathrm{Re}(\lambda z)} |\lambda - \omega|^{-1} |\hat{a}(\lambda)| \|Ax\|\, d\lambda \\ \leq\ & \frac{M}{\pi} e^{\omega t} \Big[\int_R^\infty r^{-1} |\hat{a}(\lambda)| e^{-\delta r}\, dr + \int_0^\pi e^{\cos\varphi} |\hat{a}(\lambda)|\, d\varphi \Big] \|Ax\| \\ \leq\ & Me^{\omega t} \|Ax\| \cdot \sup\{|\hat{a}(\lambda)| : |\lambda| \geq t^{-1}, \lambda \in \omega + \Sigma_{\theta_1+\pi/2}\} \to 0 \end{aligned}$$

as $|z| = t \to 0$. Thus $R(z)x \to Cx$ as $z \to 0$, $z \in \Sigma_\theta$ for each $x \in \mathcal{D}(A)$, our assertion follows from the Banach-Steinhauss theorem. ∎

The proof of the following result is the same as Corollary 2.3 in Ch.1 of [14].

Corollary 3.8. *Suppose that* $R(z) \in H(\omega_0, \theta_0)$ *is an analytic* C*-regularized resolvent family for* (1.1). *Then* $a(t)$ *admits analytic extension to* Σ_{θ_0}. *Furthermore, on each sector* Σ_θ, $0 < \theta < \theta_0$, *there is a decomposition of the form*

$$a(z) = \sum_j p_j(z) e^{\lambda_j z} + a_1(z), \quad z \in \Sigma_\theta, \tag{3.5}$$

where the λ_j *denote the finitely many poles of* $\hat{a}(\lambda)$ *contained in* $\overline{\omega + \Sigma_{\theta+\pi/2}}$, *the* $p_j(z)$ *are polynomials, and* $a_1(z)$ *is analytic in* Σ_θ *and satisfies*

$$|a_1(z)| \leq CMe^{\omega \mathrm{Re} z}/|z|, \quad z \in \Sigma_\theta, \tag{3.6}$$

$$za_1(z) \to 0, \quad as\ z \to 0, \quad z \in \Sigma_{\theta_0}. \tag{3.7}$$

Lemma 3.9. *Suppose that* $R(z)$ *is an analytic* C*-regularized resolvent family for* (1.1) *of type* (ω_0, θ_0). *For* $\theta \in (-\theta_0, \theta_0)$ *let* $a_\theta(t) := a(te^{i\theta})$ *be the holomorphic extension of* $a(t)$ *on* Σ_{θ_0} *given by Corollary 3.8, then* $a_\theta(t)$ *is locally integrable on* $\mathbb{R}^+$ *and the identity*

$$R(te^{i\theta})x = Cx + \int_0^t a((t-s)e^{i\theta}) R(se^{i\theta}) e^{i\theta} Ax\, ds$$

holds for $x \in \mathcal{D}(A)$. *That is,* $R_\theta(t) := R(te^{i\theta})$ *is the* C*-regularized resolvent family for* (3.2).

Proof. First from (3.5), (3.7) and the integrability of $a(t)$, we know that $a_\theta(t)$ is locally integrable; next from (3.5) and (3.6), we have that $a_\theta(t)$ is Laplace transformable for λ large enough and

$$\hat{a}_\theta(\lambda) = \int_0^\infty e^{-\lambda t} a_\theta(t)\, dt = e^{-i\theta} \int_0^\infty e^{-\lambda e^{-i\theta} t} a(t)\, dt = e^{-i\theta} \hat{a}(\lambda e^{-i\theta}). \tag{3.8}$$

For $\mathrm{Re}\lambda > \omega$, we have

$$(\lambda - \lambda \hat{a}(\lambda) A)^{-1} C x = \int_0^\infty e^{-\lambda t} R(t) x dt$$

for all $x \in X$. So for λ large enough, we have

$$(\lambda e^{-i\theta} - \lambda e^{-i\theta} \hat{a}(\lambda e^{-i\theta}) A)^{-1} C x = \int_0^\infty e^{-\lambda e^{-i\theta} t} R(t) x\, dt,$$

hence

$$\begin{aligned}(\lambda - \lambda \hat{a}_\theta(\lambda) A_\theta)^{-1} C x &= \int_0^\infty e^{-\lambda e^{-i\theta} t} R(t) x e^{-i\theta}\, dt \\ &= \int_{\Gamma_\theta} e^{-\lambda z} R(e^{i\theta} z) x\, dz \\ &= \int_0^\infty e^{-\lambda t} R_\theta(t) x\, dt,\end{aligned}$$

since $R(z)$ is analytic on Σ_θ and (3.1) holds, where $\Gamma_\theta := \{re^{-i\theta} : r \geq 0\}$. Our result is obtained by Theorem 2.3. ■

Now we are in the position to give the proofs of the two main results.

Proof of Theorem 3.2. Necessity follows from Corollary 3.8 and Lemma 3.9.

Sufficiency, Let $R(z) = R_\theta(t)$ for $z = te^{i\theta} \in \Sigma_{\theta_0}$, where $t \geq 0$, $\theta \in (-\theta_0, \theta)$. We only need to show that $R(z)$ is analytic and satisfies (3.1). For each $\theta \in (-\theta_0, \theta_0)$, by $(A3)$ we can define analytic family

$$H_\theta(\lambda) = \int_0^\infty e^{-\lambda t} R_\theta(t) dt$$

for Re$\lambda > \omega \cos\theta$. Moreover, by $(A2)$ and Proposition 2.2, for Re$\lambda > \max\{\omega \cos\theta, \omega_\theta\}$,

$$\begin{aligned}H_\theta(\lambda) &= (\lambda - \lambda \hat{a}_\theta(\lambda) A_\theta)^{-1} C = (\lambda - \lambda \hat{a}(\lambda e^{-i\theta}) A)^{-1} C \\ &= e^{-i\theta} (\lambda e^{-i\theta} - \lambda e^{-i\theta} \hat{a}(\lambda e^{-i\theta}) A)^{-1} C,\end{aligned}$$

since (3.8) also holds under the assumption $(A1)$. Note that $\Phi : \lambda \to \lambda e^{i\theta}$ maps the region $\left\{\lambda \in \mathbb{C} : -\frac{\pi}{2} - \theta < \arg(\lambda - \omega) < \frac{\pi}{2} - \theta\right\}$ to $\{\lambda \in \mathbb{C} : -\frac{\pi}{2} < \arg(\lambda - \omega\cos\theta) < \frac{\pi}{2}\}$ (which equals to $\left\{\lambda \in \mathbb{C} : -\frac{\pi}{2} < \arg(\lambda - \omega e^{i\theta}) < \frac{\pi}{2}\right\}$), we have when $0 < \theta_1 < \theta < \theta_0$ that

$$e^{i\theta} H_\theta(\lambda e^{i\theta}) = e^{i\theta_1} H_{\theta_1}(\lambda e^{i\theta_1}) = (\lambda - \lambda\hat{a}(\lambda)A)^{-1}C$$

for $\lambda \in \left\{\lambda \in \mathbb{C} : -\frac{\pi}{2} - \theta_1 < \arg(\lambda - \omega') < \frac{\pi}{2} - \theta\right\}$ where $\omega' > \max\{\omega, \omega_\theta / \cos\theta, \omega_{\theta_1} / \cos\theta_1\}$. And similar identity holds for $-\theta_0 < \theta < \theta_1 < 0$. Since each $H_\theta(\lambda)$ is analytic, the function

$$\begin{aligned} H(\lambda) &= e^{i\theta} H_\theta(\lambda e^{i\theta}), \\ &\text{when } \lambda \in \left\{\lambda \in \mathbb{C} : -\frac{\pi}{2} - \theta < \arg(\lambda - \omega) < \frac{\pi}{2} - \theta\right\}, \theta \in (-\theta_0, \theta_0) \end{aligned}$$

is well-defined on $\omega + \Sigma_{\theta_0 + \pi/2}$ and analytic. Moreover, for each $0 < \theta < \theta_0$, if $\lambda \in \omega + \Sigma_{\theta+\pi/2}$ with $\arg(\lambda - \omega) < 0$, then $\lambda = \omega + re^{i(-\theta+\alpha)}$ for some $r > 0$ and $-\pi/2 < \alpha < \theta$, which falls into region $\left\{\lambda \in \mathbb{C} : -\frac{\pi}{2} - \theta_1 < \arg(\lambda - \omega) < \frac{\pi}{2} - \theta_1\right\}$ where $\theta_1 = (\theta + \theta_0)/2$, so that

$$\begin{aligned} \|H(\lambda)\| &= \|H_{\theta_1}(\lambda e^{i\theta_1})\| = \|\int_0^\infty e^{-\lambda e^{i\theta_1} t} R_{\theta_1}(t)dt\| \\ &\leq \frac{M(\theta_1, \omega)}{|\mathrm{Re}(\lambda e^{i\theta_1}) - \omega\cos\theta_1|} = \frac{M(\theta_1, \omega)}{r\cos(\theta_1 - \theta + \alpha)} = \frac{M(\theta_1, \omega)}{|\lambda - \omega|\cos(\theta_1 - \theta + \alpha)} \\ &\leq \frac{M(\theta_1, \omega)}{|\lambda - \omega| \min\{\sin(\theta_1 - \theta), \cos\theta_1\}} =: \frac{M_{\theta,\omega}}{|\lambda - \omega|}, \end{aligned}$$

and similar inequality holds for $\lambda \in \omega + \Sigma_{\theta+\pi/2}$ with $\arg(\lambda - \omega) > 0$, thus by Lemma 3.4, there exists analytic function $F(z) : \Sigma_{\theta_0} \to \mathbf{B}(X)$ such that $\|F(z)\| \leq M_\theta e^{\omega \mathrm{Re} z}(z \in \Sigma_\theta)$ for $0 < \theta < \theta_0$ and

$$H(\lambda) = \int_0^\infty e^{-\lambda z} F(z)dz \text{ for } \lambda > \omega.$$

Since $F(z)$ is analytic, we have

$$\begin{aligned} H_\theta(\lambda) &= e^{-i\theta} H(\lambda e^{-i\theta}) = e^{-i\theta} \int_0^\infty e^{-\lambda t e^{-i\theta}} F(t)dt \\ &= \int_0^\infty e^{-\lambda t} F(e^{i\theta} t)dt, \end{aligned}$$

thus by the uniqueness of Laplace transform we have $R_\theta(t) = F(te^{i\theta})$, which means that $R(z) = F(z)$ is analytic and satisfies (3.1). ∎

Proof of Theorem 3.3. Necessity. Lemma 3.6 gives $(H1)$. For $\omega > \omega_0$, let $H(\lambda) = \hat{R}(\lambda)$ for Re$\lambda > \omega$. Then by Proposition 2.2, $\lambda H(\lambda) = (I - \hat{a}(\lambda)A)^{-1}C$; and by Corollary 3.5, $H(\lambda)$ admits analytic extension to $\omega_0 + \Sigma_{\theta_0+\pi/2}$ and satisfies (3.3). This gives (H3).

Now let $\Omega = \{\lambda \in \omega_0 + \Sigma_{\theta_0+\pi/2} : \lambda \neq 0, \hat{a}(\lambda) \neq 0, \infty\}$. Then for $x \in \mathcal{D}(A)$, we have

$$\lambda H(\lambda)(I - \hat{a}(\lambda)A)x = Cx \qquad \text{for all } \lambda \in \Omega \tag{3.9}$$

since this relation holds for Re$\lambda > \omega$. On the other hand, since $R_\theta(t) = R(te^{i\theta})$ is the C-regularized resolvent family for (3.2) by Theorem 3.2, we have $R(z)$ commutes with A for every $z \in \Sigma_{\theta_0}$. Moreover, from the proof of Theorem 3.2 we know that, for $\lambda \in \{\lambda \in \mathbb{C} : -\pi/2 - \theta < \arg(\lambda - \omega) < \pi/2 - \theta\}$, $\theta \in (-\theta_0, \theta_0)$, $\lambda e^{i\theta} \in \omega \cos\theta + \Sigma_{\pi/2} = \omega e^{i\theta} + \Sigma_{\pi/2}$ and

$$H(\lambda) = e^{i\theta} H_\theta(\lambda e^{i\theta}) = e^{i\theta} \int_0^\infty e^{-\lambda e^{i\theta} t} R_\theta(t)dt,$$

thus $H(\lambda)$ commutes with A for every $\lambda \in \omega_0 + \Sigma_{\theta_0+\pi/2}$ since $R(z)$ commutes with A and θ is arbitrary. Therefore, $AH(\lambda)x = H(\lambda)Ax$ is analytic for $x \in \mathcal{D}(A)$, $\lambda \in \Omega$ and

$$\lambda(I - \hat{a}(\lambda)A)H(\lambda)x = Cx \qquad \text{for all } \lambda \in \Omega. \tag{3.10}$$

From (3.9) and (3.10) we know that $H(\lambda) = (\lambda - \lambda\hat{a}(\lambda)A)^{-1}C$ for all $\lambda \in \Omega$.

Next we consider the poles of $\hat{a}(\lambda)$. If $\lambda_0 \in \omega_0 + \Sigma_{\theta_0+\pi/2}$ is a pole of $\hat{a}(\lambda)$ of order n, then there is an analytic function $b(\lambda)$ near λ_0 and $a_n \neq 0$ such that

$$\hat{a}(\lambda) = a_n(\lambda - \lambda_0)^{-n} + (\lambda - \lambda_0)^{-n+1} b(\lambda),$$

then from (3.9) and (3.10) we know that $\lambda H(\lambda)$ has a zero of order n at λ_0; letting $\lambda \to \lambda_0$ in (3.9) and (3.10) we obtain

$$-a_n H_n Ax = -a_n AH_n x = Cx \qquad \text{for } x \in \mathcal{D}(A),$$

where $H_n := \lim_{\lambda\to\lambda_0} \lambda H(\lambda)(\lambda - \lambda_0)^{-n}$, which implies that $0 \in \rho_C(A)$. And if $\omega_0 < 0$ then it is easy to see that $\lambda = 0$ must be a pole of $\hat{a}(\lambda)$ from (3.9) and (3.10). Thus (H2) is proved.

Sufficiency. It follows from $(H3)$ and Lemma 3.4 that for every $\omega > \omega_0$ and $\theta < \theta_0$ there exists an analytic family $R(z) : \Sigma_\theta \to \mathbf{B}(X)$ with $\|R(z)\| \leq c(\omega, \theta)e^{\omega Rez}$ such that

$$H(\lambda) = \int_0^\infty e^{-\lambda z} R(z)dz$$

for $\lambda > \omega$. Since $H(\lambda) = (\lambda - \lambda\hat{a}(\lambda)A)^{-1}C$ for $\lambda > \omega$, it follows from Theorem 2.3

that $R(t)$ is a C-regularized resolvent family for (1.1). And from Definition 3.1, we know that $R(z) \in H(\omega_0, \theta_0)$. ■

From the proof of Theorem 3.3, we have

Corollary 3.10. *Suppose that* $R(z) \in H(\omega_0, \theta_0)$ *is an analytic* C*-regularized resolvent family for* (1.1)*. Then* (3.4) *can be extended to* $\omega_0 + \Sigma_{\theta_0 + \pi/2}$.

4. Examples

In this section, we will give several examples.

Example 4.1. Suppose that $\mathcal{R}(C)$ is dense. For the kernel

$$a(t) = t^{\beta-1}/\Gamma(\beta), \quad t > 0$$

where $\beta \in (0, 2)$, the Volterra equation (1.1) has a bounded analytic C-regularized resolvent family for (1.1) of angle θ_0 if and only if $\rho_C(A) \supset \Sigma_{\beta(\theta_0+\pi/2)}$ and

$$\|\mu(\mu - A)^{-1}C\| \leq M, \quad \mu \in \Sigma_{\beta(\theta_0+\pi/2)}. \tag{4.1}$$

In fact, since $\hat{a}(\lambda) = \lambda^{-\beta}$ for $\mathrm{Re}\lambda > 0$, $\hat{a}(\lambda)$ admits analytic extension to the complex plane sliced along the negative real axis. Moreover, $1/\hat{a}(\lambda)$ maps the sector $\Sigma_{\theta_0+\pi/2}$ onto the sector $\Sigma_{\beta(\theta_0+\pi/2)}$. Thus by Theorem 3.3, there is a bounded analytic C-regularized resolvent family for (4.1) if and only if $\Sigma_{\beta(\theta_0+\pi/2)} \subset \rho_C(A)$ and

$$\|(\lambda - \lambda\hat{a}(\lambda)A)^{-1}C\| \leq M/|\lambda|, \quad \lambda \in \Sigma_{\theta_0+\pi/2},$$

which is exactly (4.1).

So the C-regularized resolvent family extends the conditions on resolvents of A for a resolvent family to its C-resolvents. Also, since we only need a sector (of angle probably less than $\pi/2$) contained in the C-resolvent set of A, the class of C-regularized resolvent families is not a trivial generalization of C-regularized semigroups.

Recall that a C^∞-function $f : (0, \infty) \to \mathbb{R}$ is called *completely monotonic* if $(-1)^n f^{(n)}(\lambda) \geq 0$ for all $\lambda > 0$, $n \in \mathbb{N}_0$, and *Bernstein function* if $f(\lambda) \geq 0$ and $f'(\lambda)$ is completely monotonic. Suppose that $\hat{a}(\lambda) \neq 0$ for all $\lambda > 0$. Then $a(t)$ is called *completely positive* if $1/\lambda\hat{a}(\lambda)$ is completely monotonic and $1/\hat{a}(\lambda)$ is a Bernstein function. Similarly as the proof of Theorem 3.7 in [7], we can show

Lemma 4.2 *Let* A *be the generator of an exponentially bounded* C*-regularized semigroup and* $a(t)$ *be completely positive. Then* (1.1) *admits a* C*-regularized resolvent family.*

And the following improves slightly Example 3.10 in [7].

Example 4.3. Consider the equation

$$\frac{\partial u}{\partial t}(t,x)=Au(t,x)-\alpha\int_0^t e^{-\alpha(t-s)}Au(s,x)\,ds,\quad \alpha\ge 0, 0\le t\le T, x\in\mathbb{R} \tag{4.2}$$
$$u(0,x)=u_0(x).$$

This equation equals to the Volterra equation

$$u(t,x)=f(x)+\int_0^t a(t-s)Au(s,x)ds \tag{4.3}$$

with $f(x)=u_0(x), a(t)=e^{-\alpha t}$. Thus $a(t)$ is completely positive, since $1/\hat{a}(\lambda)=\lambda+\alpha$ is a Bernstein function and $1/\lambda\hat{a}(\lambda)=\frac{\alpha}{\lambda}+1$ is completely monotonic. Now choose $A=a\frac{\partial^3}{\partial x^3}+b\frac{\partial}{\partial x}(a,b\in\mathbb{R}\backslash\{0\})$, then (4.2) is just the K-dV equation

$$\frac{\partial u}{\partial t}(t,x)=\alpha\frac{\partial^3 u}{\partial x^3}(t,x)+b\frac{\partial u}{\partial x}(t,x).$$

From [8], we know that A generates a r-times integrated semigroup for $r>|\frac{1}{2}-\frac{1}{p}|$, it thus follows from [9] that A generates a $(\omega-A)^{-r}$-regularized semigroup for some $\omega>0$. Therefore, by Lemma 4.2, there is a C-regularized resolvent family for (4.3). So, for $u_0(x)\in W^{r+3,p}(\mathbb{R})$, the equation (4.2) has a solution.

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