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GENERALIZED R-KKM THEOREMS AND THEIR APPLICATIONS

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Abstract. We give some new generalized R-KKM theorems in the nonconvexity setting of topological spaces. As an application we answer a question posed by Isac et al. for the lower and upper bounds equilibrium problem in topological spaces.

1. Introduction and Preliminariesntroduction

In 1929, Knaster, Kuratowski and Mazurkiewicz [18], established closed version KKM theorem for finite topological vector space. Fan [10] proved that the assertion of the KKM theorem for infinite dimensional topological vector space. In 1987, Kim [16] and Shih et al. [23] obtained an open version KKM theorem. Shih et al. in two successive articles [24, 25] investigated the combinatorial foundation of KKM covering theorem and Shapley covering theorem. On the other hand, Kassay et al. [15] and Chang et al. [3] initiated the study of the so-called generalized KKM maps. In this setting, convexity assumption play a crucial role in solving this variety of problems. Horvath [11], replacing convex hulls by contractible subsets, gave a purely topological version of the KKM theorem. This has motivated other mathematicians to go into the question for generalized KKM theorems and minimax inequalities over topological spaces with no linear structure; see for example [2, 5-7, 17, 22, 26, 27] and references therein.

Our aim here is to derive some new versions of generalized R-KKM theorems due to Deng et al. [4] and some results in answering to problem Isac et al. [13] in topological spaces. Let Y be nonempty set, we denote by 2^Y the family of all nonempty subsets of Y, by $\mathcal{F}(Y)$ the family of all nonempty finite subsets of Y and $|A|$ the cardinality of $A \in \mathcal{F}(Y)$.

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For a topological space X, a subset A of X is said to be *compactly closed* (resp. *open*) in X if for any nonempty compact subset K of X, $A\bigcap K$ is closed (resp. open) in K. For any given subset A of X, the *compact closure of* A, denoted by $ccl(A)$, define as follows:

$$ccl(A)=\bigcap\{B\subseteq X: A\subseteq B,\ B \text{ is compactly closed in } X\}.$$

A multivalued map $F: Y\to 2^X$ is said to *transfer compactly closed-valued* on Y if for $y\in Y$ and for each nonempty compact subset K of X with $F(y)\bigcap K\neq\emptyset$, $x\notin F(y)\bigcap K$ implies that there exists a point $y'\in Y$ such that $x\notin cl_K(F(y')\bigcap K)$; see Ding [6]. If $B\subset X$, then the *upper inverse* of B under F is defined by

$$F^+(B)=\{y\in Y: F(y)\subset B\}.$$

A multiplied mapping $F: Y\to 2^X$ is said to be *generalized relatively KKM (R-KKM) mapping* if for each subset $N=\{y_0,....,y_n\}$ of Y, there exists a continuous mapping $\varphi_N:\Delta_n\to X$ such that, for each $e_{i_0},e_{i_1},...,e_{i_k}$, $\varphi_N(\Delta_k)\subseteq\bigcup_{j=0}^{k}F(y_{i_j})$; see Deng et al. [5].

Suppose that $F: Y\to 2^X$ is a generalized R-KKM mapping. If for each $y\in Y$ and each $N\in\mathcal{F}(Y)$, $\varphi_N(\Delta_n)\cap F(y)$ is closed (resp. open) in $\varphi_N(\Delta_n)$, where φ_N is the continuous mapping in touch with N in the definition of a generalized R-KKM mapping, then we call F *generalized finitely closed (resp. open)* valued.

Notice that, if F is compactly closed (resp. open) valued, then F is generalized finitely closed (resp. open) valued. Moreover, this concept generalize the notation of finitely closed (resp. open) topological vector space of Chang et al. [3], in L-convex space of Ding [7], in hyperconvex space of Kirk et al. [17] and in G-convex space of Tan [26].

Let X and Z be two topological spaces. An *admissible class* $\mathfrak{A}_c^\kappa(X,Z)$ of maps $T: X\to 2^Z$ (see Park et al. [22]) is one such that, for each compact subset K of X, there exists $\tilde{T}\in\mathfrak{A}_c(K,Z)$ satisfying $\tilde{T}(x)\subseteq T(x)$ for all $x\in K$, where $\mathfrak{A}_c$ consist of finite composites of maps $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathcal{C}$ of (single-valued) continuous function,

(ii) each $T\in\mathfrak{A}_c$ is u.s.c. and and compact valued, and

(iii) for any polytope P, each $T\in\mathfrak{A}_c(P,P)$ has a fixed point, where the intermediate spaces are suitably chosen.

2. Generalized R-KKM Results

The KKM theorem is very important tool in the study of the equilibrium problem. We give some new versions of the generalized R-KKM theorem. The following result is well-known; see [11, 18, 23].

The KKM Principle. *Let D be the set of vertices of Δ_n and $F : D \to \Delta_n$ be a KKM map (that is, $coA \subseteq F(A)$ for each $A \subseteq D$) with closed (resp. open) values. Then, $\cap_{z\in D} F(z) \neq \emptyset$.*

Theorem 2.1. *Let X be a topological space and Y be a nonempty set. Assume that $F : Y \to 2^X$ is a generalized R-KKM mapping and generalized finitely closed (resp. open) valued. Then, the family $\{F(y) : y \in Y\}$ has the finite intersection property (More precisely, for each $N = \{y_0, y_1, ...y_n\} \in \mathcal{F}(Y)$, $\varphi_N(\Delta_n) \cap (\cap_{y\in N} F(y)) \neq \emptyset$, where φ_N is the continuous mapping in touch with N in definition of a generalized R-KKM mapping).*

Proof. Suppose that $N = \{y_0, y_1, ...y_n\} \in \mathcal{F}(Y)$. Then, from definition of generalized R-KKM mapping, there exists a continuous mapping $\varphi_N : \Delta_n \to X$ such that for each $e_{i_0}, e_{i_1}, ..., e_{i_k}$, $\varphi_N(\Delta_k) \subseteq \bigcup_{j=0}^{k} F(y_{i_j})$. Therefore, $co(e_{i_0}, ...e_{i_k}) \subseteq \bigcup_{j=0}^{k} \varphi_N^{-1}(F(y_{i_j}) \bigcap \varphi_N(\Delta_n))$ and $\varphi_N^{-1}(F(y_{i_j}) \bigcap \varphi_N(\Delta_n))$ is closed (resp. open) in Δ_n. Consequently, the map $e_i \to \varphi_N^{-1}(F(y_i) \bigcap \varphi_N(\Delta_n))$ from $\{e_0, e_1, ..., e_n\}$ to Δ_n is a KKM map. Thus, by the KKM principle $\bigcap_{i=0}^{n} \varphi_N^{-1}(F(y_i) \bigcap \varphi_N(\Delta_n)) \neq \emptyset$ and therefore $\varphi_N(\Delta_n) \bigcap(\bigcap_{i=0}^{n} F(y_i)) \neq \emptyset$.

Remark 2.2. Chang et al. [3, Theorem 3.1] showed a closed version of Theorem 2.1 for the case when Y is a convex subset of Hausdorff topological vector space with the finite topology. For an H-space X and $Y \subseteq X$ Theorem 2.1 was obtained by Chang et al. [2, Theorem 1]. If X is a hyperconvex metric space with finitely generated topology, Theorem 2.1 reduces to Kirk et al. [17, Theorem 2.1]. For an L-convex space Theorem 2.1 was obtained by Ding [7, Theorems 2.1 and 2.3]. Verma [29, Theorem 2.2] obtained the above theorem for generalized H-space. For a G-convex spaces Theorem 2.1 was obtained by Tan [26, Theorem 2.2] and Ding [6, Theorem 3.1]. Deng et al.[5, Theorem 3.1] obtained compactly closed version of Theorem 2.1. Moreover, the following example shows that the above theorem improves Theorem 3.1 of Deng et al.[5], Theorem 2.2 of Ding et al.[8] and Lemmas 4.1 and 4.2 of Huang [12] .

Example 2.3. Suppose that $Y = \mathbb{N}$ and $X = [0, +\infty)$. Let $F : Y \to 2^X$ defined as follows: $F(y) = [0, y + 1)$ for each $y \in Y$, then $F(y)$ is not compactly closed for any $y \in Y$. Moreover, if for any $N = \{y_0, y_1, ..., y_n\} \in \mathcal{F}(Y)$ we define $\varphi_N : \Delta_n \to X$ by $\varphi_N(z) = \min N$ for each $z \in \Delta_n$. Then, for each $e_{i_0}, e_{i_1}, ..., e_{i_k}$, $\varphi_N(\Delta_k) = \{\min N\} \subseteq \bigcup_{j=0}^{k} F(y_{i_j})$. Therefore, F is a generalized R-KKM mapping. Also, F has generalized finitely closed valued and the class $\{F(y) : y \in Y\}$ has the finite intersection property.

As a consequence of the above theorem, we obtain the following result. The proof of this result is similar to that of Theorem 3.4 of Deng et al. [5].

Corollary 2.4. *Let X be a topological space and Y be a nonempty set. Suppose that $F : Y \to 2^X$ is transfer compactly closed-valued and $cclF$ is a generalized R-KKM mapping, where $cclF(y) = ccl(F(y))$ for all $y \in Y$. If there exists a finite subset M of Y such that $\bigcap_{y\in M} cclF(y)$ is nonempty and compact, then $\bigcap_{y\in Y} F(y) \neq \emptyset$.*

Motivated by recent works on KKM mapping with respected to a multivalued mapping; see [14, 21, 22] and references therein, we introduce the following definition. Let X be a topological space, Y, Z be two nonempty sets and $F : Y \to 2^Z$, $T : X \to 2^Z$ be two multivalued maps. The multivalued map F is said to be generalized R-KKM mapping with respect to T if for any $N = \{y_0, ..., y_n\} \in \mathcal{F}(Y)$, there exists a continuous mapping $\varphi_N : \Delta_n \to X$ such that for each $e_{i_0}, e_{i_1}, ..., e_{i_k}$,

$$T(\varphi_N(\Delta_k)) \subseteq \bigcup_{j=0}^{k} F(y_{i_j}).$$

Note that if $Y = Z$ and T is identity mapping on Y, then the above definition reduces to definition generalized R-KKM mapping.

Theorem 2.5. *Let X, Z be two topological spaces and Y be a nonempty set. Suppose that $T \in \mathfrak{A}_c^\kappa(X, Z)$, $F : Y \to 2^Z$ is a generalized R-KKM mapping with respect to T with compactly closed values. Then the class $\{F(y) \cap T(X) : y \in Y\}$ has the finite intersection property.*

Proof. Assume that there exists a finite subset $N = \{y_0, y_1, ..., y_n\}$ of Y such that $\bigcap_{i=0}^n (F(y_i) \cap T(X)) = \emptyset$. Thus, $T(X) \subseteq \bigcup_{i=0}^n V_i$, where $V_i = Z \setminus F(y_i)$. According to definition of generalized R-KKM mapping with respect to T, there exists a continuous mapping $\varphi_N : \Delta_n \to X$ such that for each $e_{i_0}, ..., e_{i_k}$, $T(\varphi_N(\Delta_k)) \subseteq \bigcup_{j=0}^k F(y_{i_j})$. Since $K = \varphi_N(\Delta_n)$ is a compact subset of X, then there exists $\tilde{T} \in \mathfrak{A}_c(K, Z)$ such that $\tilde{T}(x) \subseteq T(x)$ for all $x \in K$. Let $\{\psi_i : i = 0, ..., n\}$ be a partition of unity subordinated to the open cover $\{V_i \bigcap \tilde{T}(K) : i = 0, ..., n\}$ of $\tilde{T}(K)$. Let $\psi(z) = \sum_{i=0}^n \psi_i(z) e_i$, then by our assumptions on $\mathfrak{A}_c^\kappa(X, Z)$, the map $\psi\tilde{T}\varphi_N : \Delta_n \to 2^{\Delta_n}$ has a fixed point $e \in \psi\tilde{T}\varphi_N(e)$. So $\psi^{-1}(e) \bigcap \tilde{T}\varphi_N(e) \neq \emptyset$. If $z \in \psi^{-1}(e) \bigcap \tilde{T}\varphi_N(e)$ and $N_z = \{i : \psi_i(z) \neq 0\}$, then $i \in N_z$ if only if $z \in V_i$ and so $z \in \bigcap_{i\in N_z} V_i$. But $z \in \tilde{T}\varphi_N(\Delta_{N_z}) \subseteq \bigcup_{i\in N_z} F(y_i)$, this is a contradiction and the proof is complete.

The following theorem is similar to Theorem 4 of Lin [21] which including Theorems 2.1 and 2.2 of Kalmoun et al. [14] in our context.

Theorem 2.6. *Let X be a topological space, Y be a nonempty set and Z be a Hausdorff topological space. Suppose that $T \in \mathfrak{A}_c^\kappa(X, Z)$ and $F : Y \to 2^Z$*

is transfer compactly closed valued. Assume that $cclF$ is a generalized R-KKM mapping with respect to T and there exist a finite subset M of Y and a compact subset K of Z such that $\bigcap_{y\in M} cclF(y) \subseteq K$, then $(\bigcap_{y\in Y} F(y)) \cap K \cap cl(T(X)) \neq \emptyset$.

Proof. Let $(\bigcap_{y\in Y} F(y)) \cap K \cap cl(T(X)) = \emptyset$. Thus there exists a nonempty finite subset $A \subseteq Y$ such that $K \cap cl(T(X)) \subseteq \cup_{y\in A}(Z \setminus cclF(y))$. Suppose that $N = A \cup M = \{y_0, ..., y_n\}$, then by our assumptions $T(X) \subseteq \bigcup_{i=0}^{n}(Z \setminus cclF(y_i)) := \bigcup_{i=0}^{n} V_i$. Now we are in a position that we can use the argument in the proof of Theorem 2.5 to obtain a contradiction.

In the sequel we give an open-valued version of a generalized R-KKM Theorem for topological spaces, which is an improvement of Theorem 11 of Park et al.[22] and Lemma 4.3 of Huang [12].

Theorem 2.7. *Let X be a topological space, Y be a nonempty set and Z be a Hausdorff topological space. Suppose that $T \in \mathfrak{A}_c^{\kappa}(X, Z)$, $F : Y \to 2^Z$ is generalized finitely open valued and $T^+F : Y \to 2^X$ is a generalized R-KKM mapping, then the class $\{F(y) : y \in Y\}$ has the finite intersection property.*

Proof. Suppose that the conclusion does not hold, then there exists a finite subset $N = \{y_0, y_1, ..., y_n\}$ of Y such that $\bigcap_{i=0}^{n} F(y_i) = \emptyset$. Now, from definition of generalized R-KKM mapping, there exists a continuous mapping $\varphi_N : \Delta_n \to X$ such that for any $e_{i_0}, e_{i_1}, ..., e_{i_k}$,

$$\varphi_N(\Delta_k) \subseteq \bigcup_{j=0}^{k} T^+F(y_{i_j}). \tag{2.1}$$

Since $\varphi_N(\Delta_n)$ is a compact subset of X, there exists $\tilde{T} \in \mathfrak{A}_c(\varphi_N(\Delta_n), Z)$ such that $\tilde{T}(x) \subseteq T(x)$ for any $x \in \varphi_N(\Delta_n)$,

$$\bigcap_{i=0}^{n} \tilde{T}^+F(y_i) = \emptyset, \ \textit{and} \ \varphi_N(\Delta_n) = \bigcup_{i=0}^{n}(\varphi_N(\Delta_n) \setminus \tilde{T}^+F(y_i)) := \bigcup_{i=0}^{n} C_i.$$

Now, for each $t \in \Delta_n \subseteq \bigcup_{i=0}^{n} \varphi_N^{-1}(C_i)$, let $N_t = \{i \in \{0, 1, ..., n\} : t \in \varphi_N^{-1}(C_i)\}$ and $S(t) = co\{e_i : i \in N_t\} = \Delta_{N_t}$. Thus, $O = \Delta_n \setminus \varphi_N^{-1}(\bigcup_{i\notin N_t} C_i)$ is an open neighborhood of t in Δ_n. If $t' \in O$, then $N_{t'} \subseteq N_t$ and so $S(t') \subseteq S(t)$. Consequently, $S : \Delta_n \to 2^{\Delta_n}$ is an upper semicontinuous map with non empty compact convex values. Hence, by Kakutani's fixed point Theorem, there exists a point $t_0 \in \Delta_n$ such that $t_0 \in S(t_0) = \Delta_{N_{t_0}}$. But $t_0 \in \bigcap_{i\in N_{t_0}} \varphi_N^{-1}(C_i)$. Hence,

$$\varphi_N(t_0) \in \bigcap_{i\in N_{t_0}} (\varphi_N(\Delta_n) \setminus \tilde{T}^+F(y_i)) \subseteq \bigcap_{i\in N_{t_0}} (\varphi_N(\Delta_n) \setminus T^+F(y_i)),$$

which contradicts (2.1).

3. Some Applications

In 1999, Isac et al. [13] raised the following open problem which is closed related to the equilibrium problem. Given a closed nonempty subset K in a locally convex semireflexive topological space, a mapping $f : K \times K \to \mathbb{R}$ and two real numbers α, β where $\alpha \le \beta$, it is interesting to know that under what conditions there exists an $\bar{x} \in K$ such that

$$\alpha \le f(\bar{x}, y) \le \beta \ \forall y \in K. \tag{3.1}$$

First, Li [20] gave some answers to the open problem (3.1) by introducing and using the concept of extremal subsets. Then Chadli et al. [1] gave some answers to this open problem by a method different from that Li used. More recently Fakhar et al. [9] obtained an answer to the question of lower and upper bounds equilibrium problem in G-convex spaces.

In this section, we give some solutions of equilibrium problem with lower and upper bounds (3.1) on topological spaces. First, from Theorem 2.1, we obtain the following result.

Theorem 3.1. *Let X be a topological space, and $\alpha, \beta \in \mathbb{R}$ with $\alpha \le \beta$. Assume that f, g_1 and g_2 are real valued bifunctions on $X \times X$ satisfying the following conditions:*

1. *$g_1(x,x) \ge \alpha$ and $g_2(x,x) \le \beta$, for all $x \in X$,*
2. *for each $x \in X$, the set $\{y \in X : \alpha \le f(x,y) \le \beta\} \cap \psi(\Delta_n)$ is closed (resp. open) in $\psi(\Delta_n)$ for any Δ_n and any continuous mapping $\psi : \Delta_n \to X$,*
3. *there exists a finite subset M of X such that for each $y \in X$ there is a point $x \in M$ such that $\alpha \le f(x,y) \le \beta$,*
4. *for every finite subset $N \subseteq X$ with $|N| = n+1$ there exists a continuous mapping $\varphi_N : \Delta_n \to X$ such that any $y \in X$ and subset $J \subseteq N \bigcap \{x \in X : \alpha \le f(x,y) \le \beta\}$ with $|J| = k+1$, $\varphi_N(\Delta_k) \subseteq \{x \in X : g_1(x,y) < \alpha \text{ or } g_2(x,y) > \beta\}$.*

Then, there exists $\bar{x} \in X$ such that $\alpha \le f(\bar{x}, y) \le \beta$ for each $y \in X$.

Proof. Let F, $\hat{F} : X \to 2^X$ be defined by

$$F(x) = \{y \in X : \alpha \le f(x,y) \le \beta\}$$

and

$$\hat{F}(x) = X \setminus F(x).$$

If conclusion of theorem is not true, then $\hat{F}(x)$ is nonempty for all $x \in X$. But by condition (3), $\bigcap_{x\in M} \hat{F}(x) = \emptyset$. From condition (2) and Theorem 2.1, $\hat{F}$ is not a generalized R-KKM mapping. Therefore, there exist a finite subset $N = \{x_0, ..., x_n\} \subseteq X$ and $e_{i_0}, e_{i_1}, ..., e_{i_k}$ such that $\varphi_N(\Delta_k) \not\subseteq \bigcup_{j=0}^{k} \hat{F}(x_{i_j})$, where φ_N is the continuous mapping in touch with N in condition (4). Hence, there exists $\bar{x} \in \varphi_N(\Delta_k) \bigcap (\bigcap_{j=0}^{k} F(x_{i_j}))$. Thus, by condition (4), $g_1(\bar{x}, \bar{x}) < \alpha$ or $g_2(\bar{x}, \bar{x}) > \beta$ which contradicts condition (1).

Remark 3.2. In the case where the set $\{y \in X : \alpha \leq f(x, y) \leq \beta\}$ is compactly closed (resp. open). Then, condition (2) of the above theorem trivially holds.

The following theorem is an analogous result to that of Theorem 2.1 of Chadli et al. [1] and Theorem 3.1 of Li [20] in topological spaces which improves Theorem 3.1 of Congjun [4] for the case, when $f = g_1 = g_2$.

Theorem 3.3. *Let X be a topological space and K a nonempty compact subset of X. Assume that f, g_1 and g_2 are real valued bifunctions on $X \times X$ satisfying the following conditions:*

(1) *$g_1(x, x) \geq \alpha$ and $g_2(x, x) \leq \beta$, for all $x \in X$,*
(2) *for every $x \in X$ and $N = \{y_0, ..., y_n\} \in \mathcal{F}(X)$ there exists a continuous mapping $\varphi_N : \Delta_n \to X$ such that if $\{y_{i_0}, ..., y_{i_k}\} \subseteq \{y \in X : f(x, y) < \alpha$ or $f(x, y) > \beta\} \cap N$, then $\varphi_N(\Delta_k) \subseteq \{y \in X : g_1(x, y) < \alpha$ or $g_2(x, y) > \beta\}$,*
(3) *for each $y \in X$, the set $\{x \in X : \alpha \leq f(x, y) \leq \beta\}$ is compactly closed,*
(4) *for each $A \in \mathcal{F}(X)$, there exists a nonempty compact subset L_A of X containing A such that $\varphi_N(\Delta_k) \subseteq L_A$ for any $N \in \mathcal{F}(L_A)$ and any $\Delta_k \subseteq \Delta_n$, where φ_N is the continuous mapping in touch with N in condition (2) and the set $\{y \in L_A : f(x, y) < \alpha$ or $f(x, y) > \beta\}$ is nonempty for each $x \in X \setminus K$.*

Then there exists $\bar{x} \in K$ such that $\alpha \leq f(\bar{x}, y) \leq \beta$ for each $y \in X$.

Proof. Let $F : X \to 2^X$ be defined as follows:

$$F(y) = \{x \in X : \alpha \leq f(x, y) \leq \beta\}.$$

Then, it is enough to show that the family $\{F(y) \bigcap K : y \in X\}$ has the finite intersection property. If $A \in \mathcal{F}(X)$, then there exists a nonempty compact subset L_A of X which satisfies in condition (4). Suppose that $F_A : L_A \to 2^{L_A}$ defined as $F_A(y) = F(y) \bigcap L_A$, then $F_A(y)$ is closed in L_A. Conditions (1), (2) and (4) imply F_A is a generalized R-KKM mapping. Thus, by Theorem 2.1 there is a point $\hat{x} \in \bigcap_{y\in L_A} F_A(y)$. But condition (4) implies that $\hat{x} \in \bigcap_{y\in A}(F(y) \bigcap K)$.

As a consequence of the above theorem we obtain the following corollary. This result improves some minimax theorems, including Theorems 2.3, 2.4 of Verma [28] and also Theorem 2.6 of Verma [29].

Corollary 3.4. *Let X be a compact topological space and c a real number. Suppose that f is a real valued bifunction defined on $X \times X$ such that:*

(1) *for each $x \in X$, $f(x,x) \leq c$,*

(2) *for each $N \in \mathcal{F}(X)$, with $|N| = n+1$ there exists a continuous mapping $\varphi_N : \Delta_n \to X$ such that for each $J \subseteq N$ with $|J| = k+1$ and any $x \in \varphi_N(\Delta_k)$, $\min_{y \in J} f(x,y) \leq c$,*

(3) *f is l.s.c. in the first variable.*

Then there exists $\bar{x} \in X$ such that $f(\bar{x}, y) \leq c$ for all $y \in X$.

Proof. Define $f_1, g_1 : X \times X \to \mathbb{R}$ by $f_1(x,y) = g_1(x,y) = e^{f(x,y)}$ for all $(x,y) \in X \times X$. If $\alpha = 0$ and $\beta = e^c$, then it is easy to see that all conditions of Theorem 3.3 are satisfied for f_1 and g_1. Therefore, there is a point $\bar{x} \in X$ such that $0 \leq g_1(\bar{x}, y) \leq e^c$ for all $y \in Y$, i.e. $g(\bar{x}, y) \leq c$ for all $y \in Y$.

Now, from Theorem 2.6 we give a new answer to problem (3.1).

Theorem 3.5. *Let X be a topological space, Z be a Hausdorff topological space and $T \in \mathfrak{A}_c^{\kappa}(X, Z)$. Assume that $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$, f, g_1 and g_2 are real bifunctions on $Z \times Z$ satisfying the following conditions:*

(1) *$g_1(z,z) \geq \alpha$ and $g_2(z,z) \leq \beta$, for all $z \in Z$,*

(2) *for each $y \in Z$, the set $\{z \in Z : \alpha \leq f(z,y) \leq \beta\}$ is compactly closed,*

(3) *there exist compact subset K of Z and $M \in \mathcal{F}(Z)$ such that the set $\{y \in M : f(z,y) < \alpha \text{ or } f(z,y) > \beta\}$ is nonempty for each $z \in Z \setminus K$,*

(4) *for every $N \in \mathcal{F}(Z)$ with $|N| = n+1$, there exists a continuous mapping $\varphi_N : \Delta_n \to X$ such that any $z \in Z$ and subset $J \subseteq N \bigcap \{y \in Z : f(z,y) < \alpha \text{ or } f(z,y) > \beta\}$ with $|J| = k+1$, $T(\varphi_N(\Delta_k)) \subseteq \{y \in Z : g_1(z,y) < \alpha \text{ or } g_2(z,y) > \beta\}$.*

Then there exists $\bar{z} \in K$ such that $\alpha \leq f(\bar{z}, y) \leq \beta$ for each $y \in Z$.

Proof. Let $F : Z \to 2^Z$ be defined by

$$F(y) = \{z \in Z : \alpha \leq f(z,y) \leq \beta\}.$$

If F is not a generalized R-KKM mapping with respect to T, then there exist a finite subset $N = \{y_0, ..., y_n\} \subseteq Z$ and $e_{i_0}, e_{i_1}, ..., e_{i_k}$ such that $T(\varphi_N(\Delta_k)) \not\subseteq \bigcup_{j=0}^{k} F(y_{i_j})$, where φ_N is the continuous mapping in touch with N in condition

(4). Therefore, there exists an $z \in T(\varphi_N(\Delta_k))$ such that for all $j \in \{0, ..., k\}$, either $f(z, y_{i_j}) < \alpha$ or $f(z, y_{i_j}) > \beta$. Thus, by condition (4), $g_1(z, z) < \alpha$ or $g_2(z, z) > \beta$ which contradicts condition (1). Thus, F is a generalized R-KKM mapping with respect to T. Condition (3) implies that $\bigcap_{y \in M} F(y) \subseteq K$, hence by Theorem 2.6, $(\bigcap_{y \in Z} F(y)) \cap K \neq \emptyset$.

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