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CANONICAL OPERATOR MODELS OVER REINHARDT DOMAINS

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Abstract. Through the works of many, now it is well known that the existence of dilation to the direct sum of a weighted shift is guaranteed by a positivity condition and a convergence condition. But the uniqueness of dilation is still not well understood.

In this paper we show that a theory of canonical operator models, not just the existence of dilation, can be achieved for general weighted shifts, with a level of sophistication at least close to that for the d-shifts on the symmetric Fock spaces.

Most techniques used in the paper are known, but we do have new ingredients which give a clearer picture even on the symmetric Fock space, where a nice theory has been developed by Arveson.

0. Introduction

A classical result in the Sz. Nagy-Foias dilation theory [20] states that purely contractive operators can be dilated to, or modeled on, the Hardy space over the unit disc $H^2(\mathbb{D})$.

Theorem 1. *An operator $T \in B(H)$ acting on a Hilbert space H can be dilated to a vector-valued Hardy space $H^2(\mathbb{D}) \otimes E$ for some Hilbert space E if and only if T is purely contractive.*

Here by "being purely contractive" we mean the $C_{\cdot 0}$ class, or that T satisfies the following *positivity condition*

$$I - TT^* \geq 0 \tag{0.1}$$

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and *convergence condition*

$$T^{*k} \to 0, \quad s.o.t. \quad as\ k \to \infty. \tag{0.2}$$

By "dilated to the Hardy space $H^2(\mathbb{D}) \otimes E$" we mean that, if let M_z denote the multiplication by the coordinate function z on the E-valued Hardy space $H^2(\mathbb{D}) \otimes E$ over the unit disc, then there exists an invariant subspace $\mathcal{M} \subset H^2(\mathbb{D}) \otimes E$ with respect to M_z such that T is unitarily equivalent to the compression of M_z to the corresponding coinvariant subspace $\mathcal{M}^\perp = (H^2(\mathbb{D}) \otimes E) \ominus \mathcal{M}$, denoted by

$$T \cong Pr_{\mathcal{M}^\perp}(M_z) = P_{\mathcal{M}^\perp} M_z|_{\mathcal{M}^\perp}. \tag{0.3}$$

We shall use Pr and P to denote compression and projection, respectively.

There has been numerous generalizations of Theorem 1 to different settings. An extensive program exists for noncommutative tuples which, however, will not be our focus. As for commutative tuples previous work includes, but is not limited to, [1-3, 5-12, 15-19, 21-23], $\cdots$.

In this paper we are interested in replacing the unilateral shift M_z on $H^2(\mathbb{D}) \otimes E$ by multidimensional weighted shifts. Along this direction now it has been largely understood, say through the above cited papers, that to guarantee the existence of dilation with respect to a weighted shift, what one needs is *a positivity condition* corresponding to (0.1) and *a convergence condition* corresponding to (0.2). We shall follow the same pattern in Section 1. So, as far as the existence of dilation is concerned, our contribution in the paper (Theorem 3) is more of the style of presentation than creativity.

However, Theorem 1 only represents a part of the Sz. Nagy-Foias dilation theory. Another important aspect in [20] is the uniqueness of dilation. In other words, given the existence of dilation, there is *a canonical model* for any purely contractive operator. Then a detailed study was devoted to the structure of the canonical models in [20].

The study of canonical models will be our focus in the paper since in several variables, results concerning uniqueness of models, as well as the detailed structure of canonical models, are scarce.

Along this direction the most complete result is on the symmetric Fock space H_n^2 over the unit ball in $\mathbb{C}^n$, where Arveson developed a satisfactory dilation theory [5]. (Note that [5] not only deals with the shift part, but also discusses a spherical isometry part.) Arveson's arguments make use of the fact that the tuple $M_z = (M_{z_1}, \cdots, M_{z_n})$ on H_n^2 is essentially normal, hence the C^*-algebra it generates has a simple structure.

In the case of the (scalar-valued) Hardy space $H^2(\mathbb{D}^n)$ over the polydisc $\mathbb{D}^n \subset \mathbb{C}^n$, the following uniqueness result is obtained by Douglas-Foias [10].

Theorem 2. *If $\mathcal{M}_i \subset H^2(\mathbb{D}^n)$ $(i = 1, 2)$ are two invariant subspaces such that the compression of M_z onto $\mathcal{M}_i^\perp$ $(i = 1, 2)$ are unitarily equivalent*

$$Pr_{\mathcal{M}_1^\perp}(M_z) \cong Pr_{\mathcal{M}_2^\perp}(M_z),$$

then $\mathcal{M}_1 = \mathcal{M}_2$.

Note that we shall always use M_z to denote the tuple of multiplication by coordinate functions on a Hilbert space of holomorphic functions, either scalar-valued or vector-valued. Also, invariant subspaces are always with respect to M_z unless otherwise specified.

In this case, the fact that M_z on $H^2(\mathbb{D}^n)$ is a tuple of isometries plays an important role in [10]. The authors also raised the question of how to obtain uniqueness result for more general spaces. In [24] Yang gave a different proof of Theorem 2, which generalizes to many other spaces. However, it seems that Yang's proof only works for the scalar-valued case, and, at least, we tried, but failed to make his arguments work for vector-valued cases.

The purpose of this paper is to show that in fact almost all spaces related to weighted shifts enjoy a quite satisfactory theory of canonical models.

But our discussion does not include a part that corresponds to the isometry in Sz. Nagy-Foias theory, or the spherical isometry in Arveson's theory on the symmetric Fock space.

Next, we fix the notations. For any sequence of multi-indexed positive numbers

$$k = \{\alpha_I\}_{I \in \mathbb{Z}_+^n},$$

we complete the polynomial ring $\mathbb{C}[z_1, \cdots, z_n]$ with respect to an inner product to obtain a Hilbert space H_k^2 such that

$$||z^I|| = ||z_1^{i_1} \cdots z_n^{i_n}|| = \alpha_I,$$

for any $I = (i_1, \cdots, i_n) \in \mathbb{Z}_+^n$, and if $I \neq J \in \mathbb{Z}_+^n$, then

$$z^I \perp z^J.$$

We always choose $\alpha_{(0,\cdots,0)} = 1$ so that $||1|| = 1$, and assume

$$sup_{I \in \mathbb{Z}_+^n} \frac{\alpha_{I+e_k}}{\alpha_I} < \infty$$

for each $k = 1, \cdots, n$, so that the multiplication M_{z_k} by the kth coordinate function is a bounded operator on H_k^2. Here $e_1 = (1, 0, \cdots, 0)$, $\cdots$, $e_n = (0, \cdots, 0, 1)$.

Technical Assumptions on the Kernel

So far we consider arbitrary weighted shifts. For most natural examples, H_k^2 is a Hilbert space of holomorphic functions over a Reinhardt domain $\Omega \subset \mathbb{C}^n$, which is the convergence domain of the series $\sum_{I\in\mathbb{Z}_+^n} \alpha_I^{-2} z^{2I}$. In this case H_k^2 admits a reproducing kernel which, by abusing the notation a little bit, is still denoted by k

$$k(z,w) = \sum_{I\in\mathbb{Z}_+^n} \alpha_I^{-2} z^I \bar{w}^I. \tag{0.4}$$

Our first assumption on the kernel k is that the above Reinhardt domain Ω is indeed a nonempty domain around the origin. Then, for each $x \in \Omega$, the function $k_x(\cdot) = k(\cdot, x)$ is in H_k^2. In fact, H_k^2 can be obtained as the closure of the linear span of these k_x.

We move on to the second assumption. Let $\sum_{I\in\mathbb{Z}_+^n} \beta_I z^I \bar{w}^I$ be the power series determined by the following equation of power series

$$k(z,w) = \sum_{I\in\mathbb{Z}_+^n} \alpha_I^{-2} z^I \bar{w}^I = \frac{1}{\sum_{I\in\mathbb{Z}_+^n} \beta_I z^I \bar{w}^I}. \tag{0.5}$$

Now, we make a very mild restriction on the kernel k. For $M_z = (M_{z_1}, \cdots, M_{z_n})$ acting on the standard space H_k^2 by the multiplications of coordinate functions, we define the so-called hereditary calculus [1] by

$$\Delta(M_z, M_z^*) = \sum_{I\in\mathbb{Z}_+^n} \beta_I M_z^I M_z^{*I}. \tag{0.6}$$

This brings up the question of the convergence of (0.6). *Our second assumption on the kernel* k is that $\Delta(M_z, M_z^*)$ is convergent in the strong operator topology. In fact, at the formal level, $\Delta(M_z, M_z^*)$ always exists, and is equal to P_0, the orthogonal projection onto the constant terms (Lemma 2). Since we are unable to verify the convergence for general k, we incorporate the convergence which we shall use into our assumption.

If $\sum_{I\in\mathbb{Z}_+^n} \beta_I z^I \bar{w}^I$ is a polynomial, such as for the Hardy or Bergman spaces over the unit ball, then the convergence of $\Delta(M_z, M_z^*)$ poses no restriction at all.

If the kernel $k(z,w)$ is a Nevanlinna-Pick kernel, then it is not hard to verify directly that $\Delta(M_z, M_z^*)$ exists, see Lemma 1.4 in [14], or Lemma 2 in Section 1 for more details.

We do not know any examples of k such that $\Delta(M_z, M_z^*)$ does not exist.

Remark. We suggest the readers to keep a few examples in mind as we move along with the above general notations. When $H_k^2 = H^2(\mathbb{D})$, the Hardy space over the unit disc, Equation (0.5) is just the familiar formula

$$k(z,w) = 1 + z\bar{w} + z^2\bar{w}^2 + \cdots = \frac{1}{1 - z\bar{w}}.$$

For the Hardy space over the polydisc $H^2(\mathbb{D}^n)$,

$$\begin{aligned} k(z,w) = \sum_{I \in \mathbb{Z}_+^n} z^I \bar{w}^I &= \frac{1}{1-(z_1\bar{w}_1+\cdots+z_n\bar{w}_n)+\cdots+(-1)^n z_1\cdots z_n\bar{w}_1\cdots\bar{w}_n} \\ &= \frac{1}{(1-z_1\bar{w}_1)\cdots(1-z_n\bar{w}_n)}. \end{aligned}$$

1. Existence of Models

Since we shall deal with compressions onto coinvariant subspaces constantly, we begin with a simple lemma which is reminiscent of Sarason's lemma on semi-invariant subspaces.

Lemma 1. *For any operator $S \in B(K)$ acting on a Hilbert space K, we compress it to a subspace $H \subset K$ so that we have $T = Pr_H(S) = P_H S|_H$.*

Then, the subspace H is S-coinvariant if and only if

$$T^i T^{*j} = Pr_H(S^i S^{*j})$$

for all $i, j \geq 0$.

Similarly, H is S-invariant if and only if

$$T^{*i} T^j = Pr_H(S^{*i} S^j)$$

for all $i, j \geq 0$.

The proof involves only fairly simple matrix calculations with respect to the decomposition $K = H \oplus H^\perp$, and is skipped. As a consequence, when one considers operator models on coinvariant subspaces, it can be expected that one has to deal with operators of the type $T \cdot T^*$. We shall apply this observation several times.

Lemma 2. [Projection formula] *Let P_0 denote the projection onto the constant terms in H_k^2, then*

$$\Delta(M_z, M_z^*) = P_0.$$

Proof. For any $x, y \in \Omega$, let $k_x(\cdot) = k(\cdot, x)$, then

$$\langle \Delta(M_z, M_z^*) k_x, k_y \rangle = \langle \sum_{I \in \mathbb{Z}_+^n} \beta_I M_z^I M_z^{*I} k_x, k_y \rangle$$

$$
\begin{aligned}
&= \sum_{I\in\mathbb{Z}_+^n} \beta_I \langle M_z^{*I} k_x, M_z^{*I} k_y\rangle \\
&= \sum_{I\in\mathbb{Z}_+^n} \beta_I \overline{x}^I y^I \langle k_x, k_y\rangle \\
&= k^{-1}(y,x)k(y,x) \\
&= 1.
\end{aligned}
$$

But $k_x(0) = k_y(0) = 1$, our lemma follows. ■

Various equivalent forms of Lemma 2 has been known. The first place we know of where it shows up in a general form is Lemma 1.4 in [14].

This projection formula also shows that the C^*-algebra generated by the identity I and $M_z = (M_{z_1}, \cdots, M_{z_n})$ on H_k^2 contains at least one compact operator. A little more effort will show that this C^*-algebra is irreducible. Hence, by a standard result in operator algebra, it contains all compact operators acting on H_k^2. This fact plays a role in Arveson's proof on the symmetric Fock space [5]. In this paper we shall have more direct approach to get around these C^*-arguments.

Putting Lemma 1 and 2 together, we have the first necessary condition for the existence of operator models: *the positivity condition*. If a tuple $T = (T_1, \cdots, T_n)$ is the compression of $M_z = (M_{z_1}, \cdots, M_{z_n})$ on $H_k^2 \otimes E$ for some Hilbert space E to a coinvariant subspace H, then

$$
\Delta(T, T^*) = Pr_H(\Delta(M_z, M_z^*)) = Pr_H(P_0) \tag{1.1}
$$

exists, and is a positive operator, because the operation of compression is continuous in strong operator topology.

Recall that we use M_z to denote the multiplication tuple on both H_k^2 and $H_k^2 \otimes E$, and use P_0 to denote the projection onto constant terms on both H_k^2 and $H_k^2 \otimes E$. Later on, since we are interested in the dimension of E, we shall instead write $H_k^2 \otimes \mathbb{C}^N$, here $N = 1, 2, \cdots, \infty$, with $\mathbb{C}^\infty$ being understood as l^2.

From Lemma 2 it can be verified directly that

$$
\sum_{I\in\mathbb{Z}_+^n} \alpha_I^{-2} M_z^I \Delta(M_z, M_z^*) M_z^{*I} = I_{H_k^2\otimes\mathbb{C}^N} \quad (s.o.t.). \tag{1.2}
$$

Now, if T is the compression of M_z onto a coinvariant subspace H, then by the same reasoning as in (1.1), we have *a convergence condition* for dilation

$$
\sum_{I\in\mathbb{Z}_+^n} \alpha_I^{-2} T^I \Delta(T, T^*) T^{*I} = I_H \quad (s.o.t.). \tag{1.3}
$$

Next, we show that these two necessary conditions put together are in fact sufficient for the existence of dilation. Our arguments rely on an elementary, but useful reformulation of dilation as an intertwining relationship (equation (1.4)). Before getting to this, we give a definition for convenience.

Definition 3. For any given kernel k, we call a tuple $T = (T_1, \cdots, T_n)$ acting on a Hilbert space H to be k**-contractive** if

1. $\Delta(T, T^*)$ exists strongly, and is a positive operator;
2. Equation (1.3) holds.

Remark. The convergence condition given by Equation (1.3) awaits more explanation since it is not transparent at all, although it is parallel to several existing conditions in literature. If one multiplies everything out, then Equation (1.3) is always *formally* true. The nontrivial part is, after multiplying out, whether the tail term converges to zero strongly. Examples in the case of the Hardy and Bergman space over the unit disc, and the Hardy space over the bidisc are discussed at the end of the paper in Section 4.

Next, we give the promised general point of view for dilation. Let $T = Pr_H(S)$ be the compression of $S \in B(K)$ onto an S-coinvariant subspace $H \subset K$. If we let $U : H \to K$ denote the natural isometric embedding map, then the compression can be written as

$$T = P_H S|_H = U^* SU.$$

By applying U from the left we obtain

$$UT^* = UU^* S^* U = P_H S^* U.$$

Since H is S^*-invariant, one has

$$UT^* = S^* U. \tag{1.4}$$

We claim that the above equation characterizes the dilation relationship completely. That is, conversely, if $T \in B(H)$ and $S \in B(K)$ satisfy (1.4) for an isometric embedding $U : H \to K$, then

$$TU^* = U^* S,$$

hence

$$T = TU^* U = U^* SU \tag{1.5}$$

which implies that T is equivalent to the compression of S onto $U(H)$. Finally, we point out that $U(H)$ is automatically S-coinvariant since

$$S^* U(H) = UT^*(H) \subset U(H).$$

Armed with the above reformulation (1.4), we now take a closer look at the intertwining relation (1.4) on $H^2_k \otimes l^2$.

Lemma 4. *Let $T = (T_1, \cdots, T_n)$ be a tuple of commuting operators on a Hilbert space H. A bounded map $U : H \to H^2_k \otimes l^2$ intertwines T and M_z as $UT_i^* = M_{z_i}^* U$ for all i if and only if there exists a bounded map $\mathbf{A} : H \to l^2$ such that*

$$U(h) = \sum_{I \in \mathbb{Z}^n_+} \alpha_I^{-2} \mathbf{A} T^{*I} h \cdot z^I, \quad \forall\, h \in H. \tag{1.6}$$

To be precise, we should write $z^I \otimes \mathbf{A} T^{*I} h$, but we want to make it look like a power series.

Proof of Lemm 4. For any $h \in H$, we expand $U(h)$ as

$$U(h) = \sum_{I \in \mathbb{Z}^n_+} h^{(I)} \cdot z^I.$$

If we use the map

$$P_I : H^2_k \otimes l^2 \to \{z^I \cdot \mathbb{C}\} \otimes l^2$$

to denote the orthogonal projection from $H^2_k \otimes l^2$ to $\{z^I \cdot \mathbb{C}\} \otimes l^2$, then

$$P_I \cdot U(h) = h^{(I)} \cdot z^I.$$

It follows that the map $F_I : H \to l^2$ defined by $F_I(h) = h^{(I)}$, $h \in H$ is a bounded map. For any $h \in H$,

$$U(h) = \sum_{I \in \mathbb{Z}^n_+} F_I(h) \cdot z^I.$$

Apply it to $UT_k^*(h)$ and $M_{z_k}^* U(h)$,

$$\begin{aligned} UT_k^*(h) &= \sum_{I \in \mathbb{Z}^n_+} F_I T_k^*(h) \cdot z^I, \\ M_{z_k}^* U(h) &= \sum_{I \geq e_k} \frac{\alpha_I^2}{\alpha_{I-e_k}^2} F_I(h) \cdot z^{I-e_k}. \end{aligned}$$

So, by comparing the last two equations, $UT_k^* = M_{z_k}^* U$ if and only if

$$F_I T_k^*(h) = \frac{\alpha_{I+e_k}^2}{\alpha_I^2} F_{I+e_k}(h), \qquad \forall\, h \in H.$$

Multiplying the denominator α_I^2 to the left, one has an expression which is suitable for iteration

$$\begin{array}{rcll} (\alpha_I^2 F_I)(h) & = & (\alpha_{I-e_k}^2 F_{I-e_k})T_k^*(h), & \forall\, I \geq e_k, \\ \cdots & & \cdots\cdots\cdots\cdots & \\ & = & (\alpha_0^2 F_0)T^{*I}(h). & \end{array}$$

It follows that for any $I \in \mathbb{Z}_+^n$,

$$F_I = \alpha_I^{-2} F_0 T^{*I},$$

the desired form. So the proof is completed. ∎

Lastly, we verify that the two conditions in Definition 3 of being k-contractive allow us to construct a map

$$U : H \to H_k^2 \otimes \overline{\Delta^{1/2} H} \tag{1.7}$$

which is in fact an isometry, and intertwines T and M_z as equation (1.6) does. By the first positivity condition we can form $\Delta^{1/2} = \Delta(T, T^*)^{1/2}$. Then define

$$U(h) = \sum_{I \in \mathbb{Z}_+^n} \alpha_I^{-2} \Delta^{1/2} T^{*I} h \cdot z^I, \qquad \forall h \in H. \tag{1.8}$$

It follows from the definition of U in (1.8) that U intertwines T and M_z. The fact that U is an isometry, that is $U^*U = I$, is actually the same as the second condition of being k-contractive. So, we have proved

Theorem 3. [Existence of operator models]. *A tuple $T = (T_1, \cdots, T_n)$ of commuting operators can be dilated to $H_k^2 \otimes E$ for some Hilbert space E if and only if T is k-contractive.*

Again, we emphasize that various results similar to Theorem 3 have appeared in literature [1-3, 5-12, 15-19, 21-23], $\cdots$.

2. Structure of Models

In this and the next sections we study the structure of those tuples which are already known to be representable on coinvariant subspaces. Let $T = (T_1, \cdots, T_n)$ be such a tuple acting on H with a representation $T \cong Pr_{\mathcal{M}^\perp}(M_z)$ as the compression of $M_z = (M_{z_1}, \cdots, M_{z_n})$ on $H_k^2 \otimes \mathbb{C}^N$ to a coinvariant subspace $\mathcal{M}^\perp = H_k^2 \otimes \mathbb{C}^N \ominus \mathcal{M}$. We aim at answering the following questions:

(1) what is smallest number of copies of H_k^2 needed for a representation of T?

We denote this number by $d(T) = d_{H_k^2}(T) = inf\ N$. By the construction in Section 1, one has $d(T) \leq rank\ \Delta(T,T^*) = rank\ \Delta(T,T^*)^{\frac{1}{2}}$.

(2) if T is represented in such a way that $N > d(T)$, then can we eliminate extra copies of H_k^2 from the representation?

It is clear that, at least, any copy of H_k^2 contained in $\mathcal{M}$ should be eliminated from the representation.

(3) how to tell when a representation is minimal, or satisfies $N = d(T)$?

Theorem 4. *Assume that a commuting tuple $T = (T_1, \cdots, T_n)$ acting on H can be represented as*

$$T \cong Pr_{\mathcal{M}^\perp}(M_z) \tag{2.1}$$

for some invariant subspace $\mathcal{M} \subset H_k^2 \otimes \mathbb{C}^N$ $(N = 1, 2, \cdots, \infty)$. Then

(1) (`minimal models`) *the smallest possible N in a representation of T, denoted by $d(T)$, is given by*

$$d(T) = rank\ \Delta(T,T^*)$$

(2) (`Eliminating redundancy`) *in a representation with $N > d(T)$, one can find a subspace $\mathcal{E} \subset \mathbb{C}^N$ with codimension $d(T)$ in $\mathbb{C}^N$, such that $H_k^2 \otimes \mathcal{E} \subset \mathcal{M}$.*

(3) (`Characterization of minimal models`) *in a representation such that $N = d(T) < \infty$, one has*

$$P_0(\mathcal{M}^\perp) = \mathbb{C}^N. \tag{2.2}$$

Conversely, for every coinvariant subspace $\mathcal{M}^\perp \subset H_k^2 \otimes \mathbb{C}^N$ $(N \leq \infty)$ such that equation (2.2) holds, one has

$$d(Pr_{\mathcal{M}^\perp}(M_z)) = N. \tag{2.3}$$

Recall that P_0 denotes the projection onto constant terms. A simple useful fact complementary to Equation (2.2) is

Lemma 5. *For an invariant subspace $\mathcal{M} \subset H_k^2 \otimes \mathbb{C}^N$ $(N \leq \infty)$, the following are equivalent*

1. $P_0(\mathcal{M}^\perp) = \mathbb{C}^N$;

2. $\mathcal{M} \cap \mathbb{C}^N = \{0\}$.

Proof of Theorem 4. (1) By the construction in the last section we know that

$$rank\ \Delta(T, T^*)^{\frac{1}{2}}$$

many copies of H_k^2 are sufficient for the representation of T on H. It suffices to show that if $T \cong Pr_{\mathcal{M}^\perp}(M_z)$ is a representation with $\mathcal{M}^\perp \subset H_k^2 \otimes \mathbb{C}^N$, then

$$\begin{aligned} N &\geq rank\ \Delta(T, T^*) \\ &= rank\ \Delta(T, T^*)^{\frac{1}{2}}. \end{aligned}$$

By considering Sarason-type compression (Lemma 1) and the projection formula (Lemma 2), one has

$$\begin{aligned} \Delta(T, T^*) &= Pr_H(\Delta(M_z, M_z^*)) \\ &= Pr_H(P_0). \end{aligned}$$

Hence

$$rank\ \Delta(T, T^*) \leq rank\ P_0 = N.$$

(2) Firstly, we remark that since $N > d(T)$, N can be ∞, but $d(T)$ cannot in this case. This small imperfection will be discussed after the proof of Theorem 4.

Claim. It suffices to prove that, if $N > d(T)$, then

$$\mathcal{M} \cap \mathbb{C}^N \neq \{0\}.$$

Suppose so. If we consider a maximal subspace $\mathcal{E} \subset \mathbb{C}^N$ such that

$$\mathcal{E} \subset \mathcal{M},$$

or equivalently,

$$H_k^2 \otimes \mathcal{E} \subset \mathcal{M},$$

then T can be represented on

$$H_k^2 \otimes (\mathbb{C}^N \ominus \mathcal{E})$$

since $\mathcal{M}^\perp$ is not effected when restricted to $H_k^2 \otimes (\mathbb{C}^N \ominus \mathcal{E})$. Now, if the codimension of $\mathcal{E}$ is bigger than $d(T)$

$$dim(\mathbb{C}^N \ominus \mathcal{E}) > d(T),$$

then we apply our claim to $\mathbb{C}^N \ominus \mathcal{E}$, so we can find a bigger $\mathcal{E}' \supset \mathcal{E}$ such that $\mathcal{E}' \subset \mathcal{M}$, which contradicts the maximality of $\mathcal{E}$.

Next, we prove the claim. Suppose that $N > d(T)$, but $\mathcal{M} \cap \mathbb{C}^N = \{0\}$. Then $P_0(\mathcal{M}^\perp) = \mathbb{C}^N$ (Lemma 5). Apply Sarason-type compression (Lemma 1) and the projection formula (Lemma 2) again,

$$\begin{aligned} d(T) &= rank\ \Delta(T, T^*) \\ &= rank\ Pr_{\mathcal{M}^\perp}(P_0) \\ &= dim\ P_{\mathcal{M}^\perp}(P_0(\mathcal{M}^\perp)), \end{aligned} \tag{2.4}$$

which is

$$dim\ P_{\mathcal{M}^\perp}(\mathbb{C}^N)$$

by our assumption. Since we know $N > d(T)$, $P_{\mathcal{M}^\perp}$ restricted to $\mathbb{C}^N$ must have a nontrivial kernel, say

$$P_{\mathcal{M}^\perp}(e) = 0,\ for\ some\ e\ (\neq 0) \in \mathbb{C}^N.$$

That is, e is orthogonal to $\mathcal{M}^\perp$, hence $e \in \mathcal{M}$. Contradiction. The claim is proved.

(3) We avoid the case of $N = d(T) = \infty$. Otherwise, $\mathcal{M}$ may contain copies of H_k^2 in a trivial way: for any given representation, we can just change the ambient space to $(H_k^2 \otimes \mathbb{C}^N) \oplus H_k^2$, and change the invariant subspace to $\mathcal{M} \oplus H_k^2$. This issue is fixed after the proof of Theorem 4.

Now, since $N = d(T) < \infty$ is the smallest possible number for a representation, $\mathcal{M}$ cannot contain any copy of H_k^2. That is, $\mathcal{M} \cap \mathbb{C}^N = \{0\}$, hence $P_0(\mathcal{M}^\perp) = \mathbb{C}^N$ (Lemma 5).

On the other hand, by the calculation in part (2) (Equation (2.4)), we always have

$$d(Pr_{\mathcal{M}^\perp}(M_z)) = dim\ P_{\mathcal{M}^\perp} P_0(\mathcal{M}^\perp). \tag{2.5}$$

But it is elementary to see that

$$dim\ P_{\mathcal{M}^\perp} P_0(\mathcal{M}^\perp) = dim\ P_0(\mathcal{M}^\perp), \tag{2.6}$$

which is $dim(\mathbb{C}^N) = N$ by our assumption. ∎

We have pointed out during the proof of Theorem 4 that there are some small imperfections in Part (2) and (3) of Theorem 4, since if we allow $d(T) = \infty$, then some nuisances can happen when we compare infinity with infinity. This can be nicely fixed by the following theorem which allows us to consider not only the dimensions, but also their underlying spaces.

Theorem 5. *The invariant subspace generated by a coinvariant subspace is reducing.*

More precisely, for any coinvariant subspace $\mathcal{M}^\perp \subset H^2_k \otimes \mathbb{C}^N$ $(N \leq \infty)$,

$$\overline{span}\{M_z^I \mathcal{M}^\perp : I \in \mathbb{Z}_+^n\} = H_k^2 \otimes P_0(\mathcal{M}^\perp), \tag{2.7}$$

here $P_0 : H^2_k \otimes \mathbb{C}^N \to \mathbb{C}^N$ is the projection onto constants.

Remark. Simple as it is, it can in fact clarify a number of issues concerning dilation. In particular, a basic problem in dilation theory is that, if a tuple $T = (T_1, \cdots, T_n)$ acting on H is represented as the compression of M_z on $H^2_k \otimes \mathbb{C}^N$ $(N \leq \infty)$ to a coinvariant subspace $\mathcal{M}^\perp$, then one often seeks to find the minimal dilation of the tuple T. Naturally, one may be led to consider the reducing subspace generated by $H = \mathcal{M}^\perp$ in $H^2_k \otimes \mathbb{C}^N$, that is,

$$\overline{C^*\{I, M_z\} \cdot \mathcal{M}^\perp} \tag{2.8}$$

where $C^*\{I, M_z\}$ is the C^*-algebra generated by $I, M_{z_1}, \cdots, M_{z_n}$. For instance, this strategy was used in [5]. However, by Theorem 5, we know that this smallest reducing subspace is actually obtained by, instead of looking at the C^*-algebra, looking at the span of $M_z^I \cdot \mathcal{M}^\perp$.

It is well known that compressions onto coinvariant subspaces are suitable for operator models, while invariant subspaces are not. But (2.8) is not able to tell the difference. In fact, the conjugate term of (2.7)

$$\overline{span}\{M_z^{*I} \mathcal{M} : I \in \mathbb{Z}_+^n\} \tag{2.9}$$

is, in general, not reducing, while its C^* version certainly is.

Proof of Theorem 5. For any nonzero $f = \sum_{I \in \mathbb{Z}_+^n} a_I z^I \in \mathcal{M}^\perp$ with $a_I \in \mathbb{C}^N$,

$$\begin{aligned} M_z^{*I}(f) &= \alpha \cdot a_I + \cdots && \in \mathcal{M}^\perp, \\ P_0 \cdot M_z^{*I}(f) &= \alpha \cdot a_I && \in P_0(\mathcal{M}^\perp), \\ M_z^I \cdot P_0 \cdot M_z^{*I}(f) &= \alpha \cdot a_I z^I && \in H^2_k \otimes P_0(\mathcal{M}^\perp), \end{aligned}$$

here α is a nonzero constant depending on I. It implies

$$f \in H^2_k \otimes P_0(\mathcal{M}^\perp).$$

Hence

$$\overline{span}\ \{M_z^I \mathcal{M}^\perp : I \in \mathbb{Z}_+^n\} \subset H^2_k \otimes P_0(\mathcal{M}^\perp).$$

For the other inclusion, it suffices to show

$$P_0(\mathcal{M}^\perp) \subset \overline{span}\ \{M_z^I \cdot \mathcal{M}^\perp :\ I \in \mathbb{Z}_+^n\}$$

since the latter space is invariant. Now, if $a_0 \in P_0(\mathcal{M}^\perp)$, then there exists $f = a_0 + \tilde{f} \ \in \mathcal{M}^\perp$, where $\tilde{f}(0) = 0$. Observe that

$$a_0 = P_0 f = \Delta(M_z, M_z^*) f \in \overline{span}\ \{M_z^I \mathcal{M}^\perp : I \in \mathbb{Z}_+^n\}.$$

Our theorem is proved. ■

Remark. By Theorem 5, now it is clear that even if we let $d(T) = \infty$ in part (2) of Theorem 4, we can still eliminate the redundancy part, which is

$$H_k^2 \otimes (\mathbb{C}^N \ominus P_0(\mathcal{M}^\perp)).$$

As for the first half of part (3) in Theorem 4, if $d(T) = \infty$, then one only needs to replace the condition

$$N = d(T) < \infty,$$

which is a numerical way of saying that the representation is minimal, directly by the condition that

"*the representation is minimal*",

in the sense that there is no subspace E of $\mathbb{C}^N$ such that $\mathcal{M}^\perp \subset H_k^2 \otimes E$. It follows from Theorem 5 that the first half of part (3) in Theorem 4 still holds.

3. Uniqueness of Models

Our main result in this section is

Theorem 6. *If $\mathcal{M}_1$ and $\mathcal{M}_2$ are two invariant subspaces of $H_k^2 \otimes \mathbb{C}^N$ $(N = 1, 2, \cdots, \infty)$, such that the compression of $M_z = (M_{z_1}, \cdots, M_{z_n})$ onto the corresponding coinvariant subspaces are unitarily equivalent,*

$$Pr_{\mathcal{M}_1^\perp}(M_z) \cong Pr_{\mathcal{M}_1^\perp}(M_z),$$

and both of them do not contain any copies of H_k^2,

$$\mathcal{M}_i \cap \mathbb{C}^N = \{0\}, \quad i = 1, 2,$$

then there exists an $N \times N$ unitary matrix $W : \mathbb{C}^N \to \mathbb{C}^N$, which induces a unitary operator on $H_k^2 \otimes \mathbb{C}^N$, also denoted by $W : H_k^2 \otimes \mathbb{C}^N \to H_k^2 \otimes \mathbb{C}^N$, such that

$$W(\mathcal{M}_1) = \mathcal{M}_2, \ and \ W(\mathcal{M}_1^\perp) = \mathcal{M}_2^\perp.$$

Since the uniqueness problem has been longstanding, say, current satisfactory results only exist for the symmetric Fock space [5], and we do not even have an extension of Douglas-Foias' Theorem [2] to vector-valued Hardy spaces over the polydisc or the unit ball, and our solution to this problem for weighted shifts is unexpectedly simple, we want to briefly look at some of the methods used to tackle the problem in the past, and to see what are the obstacles.

It will be convenient to have abstract notations here, so we let $T^{(i)} = (T_1^{(i)}, \cdots, T_n^{(i)})$ acting on H_i to denote the compressions $Pr_{\mathcal{M}_i^\perp}(M_z)$ acting on $\mathcal{M}_i^\perp$, and let

$$U_i : H_i = \mathcal{M}_i^\perp \to H_k^2 \otimes \mathbb{C}^N \tag{3.1}$$

denote the isometric embedding. Note that for an abstract T on H an explicit expression for such an isometric embedding is given by Equation (1.6) or (1.7). If $T^{(1)}$ and $T^{(2)}$ are unitarily equavalent, then there exists an unitary map

$$V : H_1 \to H_2 \tag{3.2}$$

such that $VT^{(1)} = T^{(2)}V$. Then the uniqueness problem for operator models amounts to extend V to a unitary on the ambient space

$$\tilde{V} : H_k^2 \otimes \mathbb{C}^N \to H_k^2 \otimes \mathbb{C}^N \tag{3.3}$$

through the above U_i. More informatively, we need to show that the diagram

$$\begin{array}{ccc} H_1 & \xrightarrow{V} & H_2 \\ {\scriptstyle U_1}\Big\downarrow & & {\scriptstyle U_2}\Big\downarrow \\ H_k^2 \otimes \mathbb{C}^N & \xrightarrow{\tilde{V}} & H_k^2 \otimes \mathbb{C}^N. \end{array}$$

commutes, and $\tilde{V} M_z = M_z \tilde{V}$.

If we look at the smallest reducing subspace containing H_1, which is often expressed as $\overline{C^*\{I, M_z\}H_1}$, then it is easy to see how $\tilde{V}$ should be defined on elements of the form

$$S(h) \in H_k^2 \otimes \mathbb{C}^N, \qquad S \in C^*\{I, M_z\},\ h \in H_1 = \mathcal{M}_1^\perp$$

by

$$\tilde{V} S(h) = S(Vh). \tag{3.4}$$

The difficulty with this approach is how to show that $\tilde{V}$ is unitary.

In [5] Arveson took a somehow different approach which in fact generalizes to essentially normal weighted shifts, that is, with compact commutators $[M_{z_i}^*, M_{z_j}]$. In this case, a key fact is that the C^*-algebra generated by M_{z_i} and I has a simple structure: let $\mathcal{A}$ denote the (non-selfajoint) algebra generated by M_{z_i} and I, one has

$$C^*\{I, M_z\} = \overline{\mathcal{A}\mathcal{A}^*}. \tag{3.5}$$

Then, roughly speaking, the key idea in [5] is the following general observation: any k-contractive tuple T can give arise to a representation of $\mathcal{A}$ in a straightforward way, and by equation (3.5), as well as a deep theorem of Arveson himself [4], the representation can automatically extend from $\mathcal{A}$ to the C^*-algebra $C^*\{I, M_z\}$.

In [24] Yang took a completely different approach for the scalar-valued spaces which is based on the fact that the coinvariant subspace $\mathcal{M}_i^\perp$ is completely determined by the diagonal values of the normalized reproducing kernel. Then a calculation shows that these diagonal values can be written in terms of $T^{(i)}$, and only depends on the unitary equivalence class of $T^{(i)}$. This is an interesting approach but seems to be hard to generalize to the general vector-valued cases.

Next we show that a pretty modest trick allows us to get around the above difficulties. Instead of trying to go from H_i (see (3.2)) to $H_k^2 \otimes \mathbb{C}^N$ (see (3.3)) directly, we detour to a map between $\overline{\Delta_i H_i}$, $(i = 1, 2)$.

Proof of Theorem 6. We still use the above notations. Observe that

$$VT^{(1)} = T^{(2)}V$$

implies

$$VT^{(1)^*} = T^{(2)^*}V$$

by the Fuglede-Putnam theorem, or by considering the adjoints. For convenience, let $\Delta_i = \Delta(T^{(i)}, T^{(i)^*})^{\frac{1}{2}}$, $i = 1, 2$. It follows that

$$V\Delta_1 = \Delta_2 V.$$

Thus, V sends $\overline{\Delta_1 H_1}$ isometrically into $\overline{\Delta_2 H_2}$, and V^* sends $\overline{\Delta_2 H_2}$ isometrically into $\overline{\Delta_1 H_1}$. So, when restricted to $\overline{\Delta_1 H_1}$, V induces a unitary map

$$V_0 : \overline{\Delta_1 H_1} \to \overline{\Delta_2 H_2}, \tag{3.6}$$

which trivially extends to a unitary map

$$I \otimes V_0 : H_k^2 \otimes \overline{\Delta_1 H_1} \to H_k^2 \otimes \overline{\Delta_2 H_2}. \tag{3.7}$$

Now, by the existence of operator models, one has isometric embedding

$$U_i : H_i \to H_k^2 \otimes \overline{\Delta_i H_i} \tag{3.8}$$

for $i = 1, 2$ as given by equations (1.6) or (1.7). What we need to do now is to show that $I \otimes V_0$ is indeed an extension the original unitary map $V : H_1 \to H_2$. It suffices to check that the following diagram commutes

$$\begin{array}{ccc} H_1 & \xrightarrow{V} & H_2 \\ {\scriptstyle U_1}\big\downarrow & & {\scriptstyle U_2}\big\downarrow \\ H_k^2 \otimes \overline{\Delta_1 H_1} & \xrightarrow{I\otimes V_0} & H_k^2 \otimes \overline{\Delta_2 H_2}. \end{array}$$

This can be done by

$$\begin{aligned} (I \otimes V_0)U_1(h) &= (I \otimes V_0) \sum_{I\in\mathbb{Z}_+^n} \alpha_I^{-2} \Delta_1 T^{(1)*^I} h \cdot z^I \\ &= \sum_{I\in\mathbb{Z}_+^n} \alpha_I^{-2}\ V_0 \cdot \Delta_1 T^{(1)*^I} h \cdot z^I \\ &= \sum_{I\in\mathbb{Z}_+^n} \alpha_I^{-2}\ V \cdot \Delta_1 T^{(1)*^I} h \cdot z^I \\ &= \sum_{I\in\mathbb{Z}_+^n} \alpha_I^{-2}\ \Delta_2 T^{(2)*^I} V h \cdot z^I \\ &= U_2(Vh). \end{aligned}$$

Our proof is completed. ■

4. Examples

Since it is not immediately clear what the convergence condition in Equation (1.3) means, we work out some examples. Note that some general formulation of the tail term has been discussed, say, in [2, 12].

For the Hardy space $H^2(\mathbb{D})$ over the unit disc, Equation (1.3) becomes

$$\sum T^i (I - TT^*) T^{*i} = \lim_{i\to\infty} (I - T^i T^{*i}) = I. \tag{4.1}$$

That is, we need the tail term $T^i T^{*i}$ to approach 0 strongly, which is equivalent to $T^{*i} \to 0$ strongly. Hence, it coincides with the definition given at the beginning of the paper.

The above example seems to be the only case for which the convergence condition is checkable in a reasonable sense.

For the Bergman space $H_k^2 = L_a^2(\mathbb{D})$ over the unit disc with kernel function $k(z,w) = \frac{1}{(1-z\bar{w})^2}$, we observe that a power series corresponding to Equation (1.3) is

$$1 = \sum_{i\geq 0}(i+1)z^i(1-2z+z^2) = \lim_{i\to\infty} 1-(i+1)z^i + iz^{i+1}. \tag{4.2}$$

Hence, the convergence condition on T becomes

$$iT^{i+1}T^{*i+1} - (i+1)T^iT^{*i} \to 0, \tag{4.3}$$

which appears to be hard to verify for concrete operators. We can reformulate it in a slightly more inspiring, but somehow artificial way as did in [13]:

> “the sequence $a_i = \frac{T^{*i}}{\sqrt{i}}$ tends to 0 in s.o.t. faster than $b_i = \frac{1}{i}$ tends to 0 in the following sense: for any $\epsilon > 0$ and any $x \in H$, one can find an i_0 such that whenever $i \geq i_0$

$$|||a_i(x)||^2 - ||a_{i+1}(x)||^2| < \epsilon|b_i - b_{i+1}|.\text{”} \tag{4.4}$$

However, we want to call the readers attention a more elegant condition in [1]: one only needs to check that T is a contraction and $T^{*i} \to 0$ strongly. Roughly speaking, this convergence can be obtained as follows: the positivity condition $I - 2TT^* + T^2T^{*2} \geq 0$ guarantees that T can be dilated to the direct sum of infinite many copies of the Bergman shift, plus a part S satisfying the equation

$$I - 2SS^* + S^2S^{*2} = 0,$$

which resembles the isometry part as in dilation on the Hardy space $H^2(\mathbb{D})$. Now, the effect of the convergence condition is just to show that the “*isometry-like*” term is null, which can be achieved by either (4.3) or $T^{*i} \to 0$ strongly.

Lastly, the Hardy space $H^2(\mathbb{D}^2)$ over the bidisc, with kernel $k(z,w) = \frac{1}{(1-z_1\bar{w}_1)(1-z_2\bar{w}_2)}$. In order to get some idea about the tail term, we calculate the corresponding power series for the convergence condition

$$\begin{aligned} 1 &= \sum z^I(1-z_1-z_2+z_1z_2) \\ &= 1 - \lim_{k\to\infty}\Big(\sum_{i_1+i_2=k+1} z_1^{i_1}z_2^{i_2} - z_1z_2\sum_{i_1+i_2=k} z_1^{i_1}z_2^{i_2}\Big). \end{aligned}$$

Similar to the Bergman space case, if one also considers a part called bi-disc isometry when constructing the dilation, one can get a simpler convergence condition $T_i^{*k} \to 0$ as $k \to \infty$, see [8, 9]. It is hard to imagine that one can have an effective isometry-like theory of for general H_k^2. However, interested readers may consult [3] for some recent progress.

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