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SOME HADAMARD'S INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS IN A RECTANGLE FROM THE PLANE

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Abstract. A monotonic nondecreasing mapping connected with Hadamard type inequalities in two variables is given and some Hadamard type inequalities for Lipschizian mapping in two variables are established.

1. INTRODUCTION

Let $f : I \subseteq R \to R$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as Hadamard's inequality for convex mappings. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f.

For refinements, counterparts, generalizations and new Hadarmard-type inequalities, see the papers [1-10, 12-13] and [17-21].

Let us consider the bidimensional interval $\Delta := [a, b] \times [c, d]$ in R^2 with $a < b$ and $c < d$. Recall that the mapping $f : \Delta \to R$ is convex in Δ if

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x, y) + (1-\lambda)f(z, w)$$

holds for all (x, y),$(z, w) \in \Delta$ and $\lambda \in [0, 1]$; A function $f : \Delta \to R$ is called co-ordinated convex on Δ if the partial mappings $f_y : [a, b] \to R$, $f_y(u) := f(u, y)$ and $f_x : [c, d] \to R$, $f_x(v) := f(x, v)$, are convex for all $y \in [c, d]$ and $x \in [a, b]$.

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Note that every convex mapping $f : \Delta \to R$ is co-ordinated convex but the converse is not generally true.

Recently, in [11], Dragomir established the following similar inequality of Hadamard's type for co-ordinated convex mapping on a retangle from the plane R^2.

Theorem A. *Suppose that $f : \Delta = [a, b] \times [c, d] \to R$ is co-ordinated convex on Δ. Then one has the inequalities*

$$
\begin{aligned}
f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx \\
&\leq \frac{f(a,b)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned} \tag{1.2}
$$

The above inequalities are sharp.

The following inequality is considered the mapping connected with the first inequality of (1.2).

Theorem B. *Suppose that $f : \triangle \subseteq R^2 \to R$ is co-ordinated convex on $\triangle$ and the mapping $H : [0,1]^2 \to R$ is defined by*

$$
\begin{aligned}
H(t,s) := \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d & f(tx + (1-t)(\frac{a+b}{2}), sy \\
&+ (1-s)(\frac{c+d}{2}))dy\ dx
\end{aligned} \tag{1.3}
$$

Then

(i) *The mapping H is co-ordinated convex on $[0,1]^2$.*

(ii) *The mapping H is co-ordinated monotonic nondecreasing.*

(iii) $\sup_{(t,s)\in[0,1]^2} H(t,s) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx = H(0,0)$,

and $\inf_{(t,s)\in[0,1]^2} H(t,s) = f(\frac{a+b}{2}, \frac{c+d}{2}) = H(1,1)$.

The authors in [14] and [22] have proved the following inequalities of Hadamard's type for Lipschitizan mapping.

Theorem C. *Let $f : I \subseteq R \to R$ be L -Lipschitzian on I and $a, b \in I$ with $a < b$. Then we have*

$$
\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{L(b-a)}{3} \tag{1.4}
$$

and

$$|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx| \le \frac{L(b-a)}{4} \tag{1.5}$$

Later in $[15]$, M. Matic and J. Pecarić improve the bound of (1.4) by $\frac{L(b-a)}{4}$.

In the following section, we will establish a monotonic nondecreasing mapping connected with the secondary inequality of (1.2). In section 3, we will prove some Hadamard type inequalities for Lipschitzian mapping in two variables.

2. Adamard's Inequality

In order to establish the main theorem, we need the following lemma.

Lemma. *Let $f:[a,b]\to R$ be a convex function and let $a \le y_1 \le x_1 \le x_2 \le y_2 \le b$ and $x_1+x_2 = y_1+y_2$. Then*

$$f(x_1)+f(x_2) \le f(y_1)+f(y_2).$$

Proof. For $y_1 = y_2$, the result is obvious. Write

$$x_1 = \frac{y_2-x_1}{y_2-y_1}y_1 + \frac{x_1-y_1}{y_2-y_1}y_2 \quad \text{and} \quad x_2 = \frac{y_2-x_2}{y_2-y_1}y_1 + \frac{x_2-y_1}{y_2-y_1}y_2,$$

Since f is convex, we have

$$\begin{aligned} f(x_1)+f(x_2) &\le \frac{y_2-x_1}{y_2-y_1}f(y_1) + \frac{x_1-y_1}{y_2-y_1}f(y_2) + \frac{y_2-x_2}{y_2-y_1}f(y_1) + \frac{x_2-y_1}{y_2-y_1}f(y_2) \\ &= \frac{2y_2-(x_1+x_2)}{y_2-y_1}f(y_1) + \frac{(x_1+x_2)-2y_1}{y_2-y_1}f(y_2) \\ &= f(y_1)+f(y_2). \end{aligned}$$

This completes the proof.

Theorem 1. *Suppose that $f : \triangle \subseteq R^2 \to R$ is co-ordinated convex on $\Delta := [a,b]\times[c,d]$ and the mapping $F:[0,1]^2 \to R$ is defined by*

$$\begin{aligned} F(t,s) = {} & \frac{1}{4(b-a)(d-c)} \\ & \times \int_a^b\int_c^d \Bigg[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x, \left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right) \\ & + f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x, \left(\frac{1+s}{2}\right)d+\left(\frac{1-s}{2}\right)y\right) \end{aligned} \tag{2.1}$$

$$+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x,\left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2})y\right)\right.$$
$$\left.+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x,\left(\frac{1+s}{2}\right)d+\left(\frac{1-s}{2}\right)y\right)\right]dydx.$$

Then

(i) *The mapping* F *is a co-ordinated convex on* $[0,1]^2$,

(ii) *The mapping* F *is a co-ordinated monotonic nondecreasing on* $[0,1]^2$.

(iii) *We have the bounds*

$$\inf_{(t,s)\in[0,1]^2} F(t,s) = \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dxdy = F(0,0)$$
$$\sup_{(t,s)\in[0,1]^2} F(t,s) = \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} = F(1,1).$$

Proof. (i) Fix $s\in[0,1]$. Then for all $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ and $t_1,t_2\in[0,1]$, we have :

$$\begin{aligned}
&F(\alpha t_1+\beta t_2,s)\\
=\ &\frac{1}{4(b-a)(d-c)}\\
&\times\int_a^b\int_c^d\left[f\left(\frac{1+(\alpha t_1+\beta t_2)}{2}a+\frac{1-(\alpha t_1+\beta t_2)}{2}x,\frac{(1+s)}{2}c+\frac{(1+s)}{2}y\right)\right.\\
&+f\left(\frac{1+(\alpha t_1+\beta t_2)}{2}a+\frac{1-(\alpha t_1+\beta t_2)}{2}x,\frac{(1+s)}{2}d+\frac{(1-s)}{2}y\right)\\
&+f\left(\frac{1+(\alpha t_1+\beta t_2)}{2}b+\frac{1-(\alpha t_1+\beta t_2)}{2}x,\frac{(1+s)}{2}c+\frac{(1-s)}{2}y\right)\\
&\left.+f\left(\frac{1+(\alpha t_1+\beta t_2)}{2}b+\frac{1-(\alpha t_1+\beta t_2)}{2}x,\frac{(1+s)}{2}d+\frac{(1-s)}{2}y\right)\right]dydx\\
=\ &\frac{1}{4(b-a)(d-c)}\times\\
&\int_a^b\int_c^d\left[f\left(\alpha\frac{(1+t_1)a+(1-t_1)x}{2}+\beta\frac{(1+t_2)a+(1-t_2)x}{2},\frac{(1+s)}{2}c+\frac{(1-s)}{2}y\right)\right.\\
&+f\left(\alpha\frac{(1+t_1)a+(1-t_1)x}{2}+\beta\frac{(1+t_2)a+(1-t_2)x}{2},\frac{(1+s)}{2}d+\frac{(1-s)}{2}y\right)
\end{aligned}$$

$$+f\left(\alpha\frac{(1+t_1)b+(1-t_1)x}{2}+\beta\frac{(1+t_2)b+(1-t_1)x}{2},\frac{(1+s)}{2}c+\frac{(1-s)}{2}y\right)$$
$$+f\left(\alpha\frac{(1+t_1)b+(1-t_1)x}{2}+\beta\frac{(1+t_2)b+(1-t_2)x}{2},\frac{(1+s)}{2}d+\frac{(1-s)}{2}y\right)\Bigg]dydx$$
$$\leq \frac{1}{4(b-a)(d-c)}\int_c^d\Bigg\{\alpha\int_a^b\Bigg[f\left(\frac{(1+t_1)}{2}a+\frac{(1-t_1)}{2}x,\frac{(1+s)}{2}c+\frac{(1-s)}{2}y\right)$$
$$+f\left(\frac{(1+t_1)}{2}a+\frac{(1-t_1)}{2}x,\frac{(1+s)}{2}d+\frac{(1-s)}{2}y\right)$$
$$+f\left(\frac{(1+t_1)}{2}b+\frac{(1-t_1)}{2}x,\frac{(1+s)}{2}c+\frac{(1-s)}{2}y\right)$$
$$+f\left(\frac{(1+t_1)}{2}b+\frac{(1-t_1)}{2}x,\frac{(1+s)}{2}d+\frac{(1-s)}{2}y\right)\Bigg]dx$$
$$+\beta\int_a^b\Bigg[f\left(\frac{(1+t_2)}{2}a+\frac{(1-t_2)}{2}x,\frac{(1+s)}{2}c+\frac{(1-s)}{2}y\right)$$
$$+f\left(\frac{(1+t_2)}{2}a+\frac{(1-t_1)}{2}x,\frac{(1+s)}{2}d+\frac{(1-s)}{2}y\right)$$
$$+f\left(\frac{(1+t_2)}{b}b+\frac{(1-t_1)}{2}x,\frac{(1+s)}{2}c+\frac{(1-s)}{2}y\right)$$
$$+f\left(\frac{(1+t_2)}{2}b+\frac{(1-t_1)}{2}x,\frac{(1+s)}{2}d+\frac{(1-s)}{2}y\right)\Bigg]dx\Bigg\}dy$$
$$=\alpha F(t_1,s)+\beta F(t_2,s).$$

If $t\in[0,1]$ is fixed, then for all $s_1,s_2\in[0,1]$ and $\alpha,\beta\geq 0$ with $\alpha+\beta=1$, we also have:

$$F(t,\alpha s_1+\beta s_2)\leq \alpha F(t,s_1)+\beta F(t,s_2),$$

so that F is co-ordinated convex on $[0,1]^2$.

(ii) Fix $s\in[0,1]$. Let $0\leq t_1\leq t_2\leq 1$, $a\leq x\leq b$.
Since

$$\int_a^b\int_c^d\left[f\left(\left(\frac{1+t_1}{2}\right)b+\left(\frac{1-t_1}{2}\right)x,\left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right)\right]dydx$$
$$=\int_a^b\int_c^d\left[f\left(\left(\frac{1+t_1}{2}\right)b+\left(\frac{1-t_1}{2}\right)(b+a-x)\right),\left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right]dydx,$$

we have

$$F(t_1,s)=\frac{1}{4(b-a)(d-c)}\int_a^b\int_d^c\Big[f((\frac{1+t_1}{2})a+(\frac{1-t_1}{2})x,(\frac{1+s}{2})c$$
$$+(\frac{1-s}{2})y)+f((\frac{1+t_1}{2})a+(\frac{1-t_1}{2})x,(\frac{1+s}{2})d+(\frac{1-s}{2})y)$$
$$+f((\frac{1+t_1}{2})b+(\frac{1-t_1}{2})(b+a-x),(\frac{1+s}{2})c+(\frac{1-s}{2})y)$$
$$+f((\frac{1+t_1}{2})b+(\frac{1-t_1}{2})(b+a-x),(\frac{1+s}{2})d+(\frac{1-s}{2})y)\Big]dydx$$

Since $(\frac{1+t_2}{2})a+(\frac{1-t_2}{2})x\le(\frac{1+t_1}{2})a+(\frac{1-t_1}{2})x$

$$\le (\frac{1+t_1}{2})b+(\frac{1-t_1}{2})(b+a-x)$$
$$\le (\frac{1+t_2}{2})b+(\frac{1-t_2}{2})(b+a-x);$$
$$\Big[(\frac{1+t_1}{2})a+(\frac{1+t_1}{2})x\Big]+\Big[(\frac{1+t_1}{2})b+(\frac{1-t_1}{2})(b+a-x)\Big]$$
$$=\Big[(\frac{1+t_2}{2})a+(\frac{1+t_2}{2})x\Big]+\Big[(\frac{1+t_2}{2})b+(\frac{1-t_2}{2})(b+a-x)\Big]$$

and f is co-ordinated convex on Δ, by Lemma, we have

$$F(t_1,s)\le\frac{1}{4(b-a)(d-c)}$$
$$\times\int_a^b\int_c^d\Big[f((\frac{1+t_2}{2})a+(\frac{1-t_2}{2})x,(\frac{1+s}{2})c+(\frac{1-s}{2})y)$$
$$+f((\frac{1+t_2}{2})b+(\frac{1-t_2}{2})(b+a-x),(\frac{1+s}{2})c+(\frac{1-s}{2})y)$$
$$+f((\frac{1+t_2}{2})a+(\frac{1-t_2}{2})x,(\frac{1+s}{2})d+(\frac{1-s}{2})y)$$
$$+f((\frac{1+t_2}{2})b+(\frac{1-t_2}{2})(b+a-x),(\frac{1+s}{2})d+(\frac{1-s}{2})y)]dydx.$$

$$=\frac{1}{4(b-a)(d-c)}\quad\int_a^b\int_c^d\Big[f((\frac{1+t_2}{2})a+(\frac{1-t_2}{2})x,(\frac{1+s}{2})c+(\frac{1-s}{2})y$$
$$+f((\frac{1+t_2}{2})b+(\frac{1-t_2}{2})x,(\frac{1+s}{2})c+(\frac{1-s}{2})y)$$
$$+f((\frac{1+t_2}{2})a+(\frac{1-t_2}{2})x,(\frac{1+s}{2})d+(\frac{1-s}{2})y)$$
$$+f((\frac{1+t_2}{2})b+(\frac{1-t_2}{2})x,(\frac{1+s}{2})d+(\frac{1-s}{2})y)]dydx$$
$$=F(t_2,s).$$

This shows that $F(t,s)$ is co-ordinated nondecreasing for all $t \in [0,1]$. If $t \in [0,1]$ is fixed, then for all $s \in [0,1]$, we also have $F[t,s]$ is co-ordinated nondecreasing for all $s \in [0,1]$. Thus the mapping F is co-ordinated monotonic nondecreasing on $[0,1]^2$. <iii> It follows from <ii> that, for all $(t,s) \in [0,1]^2$,

$$\begin{aligned} F(t,s) \;&\geq\; F(0,s) ge F(0,0) \\ &= \frac{1}{4(b-a)(d-c)} \int_a^b \int_c^d \left[f\left(\frac{a+x}{2}, \frac{c+y}{2}\right) + f\left(\frac{a+x}{2}, \frac{d+y}{2}\right) \right. \\ &\quad \left. + f\left(\frac{b+x}{2}, \frac{c+y}{2}\right) + f\left(\frac{b+x}{2}, \frac{d+y}{2}\right)\right] dydx \\ &= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx, \end{aligned}$$

and
$$\begin{aligned} F(t,s) &\leq F(t,1) \leq F(1,1) \\ &= \frac{1}{4(b-a)(d-c)} \int_a^b \int_c^d [f(a,c)+f(a,d)+f(b,c)+f(b,d)]dydx \\ &= \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

This completes the proof.

Remark 1. We note that the proofs of *Theorem 1* for mapping in one variable is simpler than the one given by Yan, Homg and Wang in [19-21].

3. Lipschitzian Mappings

In what follows we recall the following definition, see [16, P. 305].

Definition. Consider a function $f : V \to R$ defined on a subset V of $R^m, m \in N$. Let $L = (L_1, L_2, \ldots, L_m)$ where $L_i \geq 0$, $i = 1, \ldots, m$. we say that f is L-Lipschitian function if

$$|f(x) - f(y)| \leq \sum_{i=1}^{m} L|x_i - y_i|$$

for all $x, y \in V$. For the functions F and H, defined by (2.1) and (1,3), we have the following theorem:

Theorem 2. *Let $f : \Delta \to R$ satisfy Lipschitzian conditions. That is, for (t_1, s_1) and (t_2, s_2) belong to $\Delta = [a,b] \times [c,d]$, we have*

$$|f(t_1, s_1) - f(t_2, s_2)| \leq L_1|t_1 - t_2| + L_2|s_1 - s_2|$$

where L_1 and L_2 are positive constants. Then

$$|F(t_1,s_1)-F(t_2,s_2)| \le \frac{L_1|t_1-t_2|(b-a)+L_2|s_1-s_2|(d-c)}{4} \tag{3.1}$$

and

$$|H(t_1,s_1)-H(t_2,s_2)| \le \frac{L_1|t_1-t_2|(b-a)+L_2|s_1-s_2|(d-c)}{4} \tag{3.2}$$

Proof. For (t_1,s_1) and (t_2,s_2) belong to $\triangle=[a,b]\times[c,d]$, we have

$$\begin{aligned}
&|F(t_1,s_1)-F(t_2,s_2)| \le \frac{1}{4(b-a)(d-c)} \\
&\times \int_a^b\int_c^d \Bigg[\Bigg|f\Bigg(\Big(\frac{1+t_1}{2}\Big)a+\Big(\frac{1-t_1}{2}\Big)x,\Big(\frac{1+s_1}{2}\Big)c+\Big(\frac{1-s_1}{2}\Big)y\Bigg) \\
&-f\Bigg(\Big(\frac{1+t_2}{2}\Big)a+\Big(\frac{1-t_2}{2}\Big)x,\Big(\frac{1+s_2}{2}\Big)c+\Big(\frac{1-s_2}{2}\Big)y\Bigg| \\
&+\Bigg|f\Bigg(\Big(\frac{1+t_1}{2}\Big)a+\Big(\frac{1-t_1}{2}\Big)x,\Big(\frac{1+s_1}{2}\Big)d+\Big(\frac{1-s_1}{2}\Big)y\Bigg) \\
&-f\Bigg(\Big(\frac{1+t_2}{2}\Big)a+\Big(\frac{1-t_2}{2}\Big)x,\Big(\frac{1+s_2}{2}\Big)d+\Big(\frac{1-s_2}{2}\Big)y\Bigg)\Bigg| \\
&+\Bigg|f\Bigg(\Big(\frac{1+t_1}{2}\Big)b+\Big(\frac{1-t_1}{2}\Big)x,\Big(\frac{1+s_1}{2}\Big)c+\Big(\frac{1-s_1}{2}\Big)y\Bigg) \\
&-f\Bigg(\Big(\frac{1+t_2}{2}\Big)b+\Big(\frac{1-t_2}{2}\Big)x,\Big(\frac{1+s_2}{2}\Big)c+\Big(\frac{1-s_2}{2}\Big)y\Bigg)\Bigg| \\
&+|f\Bigg(\Big(\frac{1+t_1}{2}\Big)b+\Big(\frac{1-t_1}{2}\Big)x,\Big(\frac{1+s_1}{2}\Big)d+\Big(\frac{1-s_1}{2}\Big)y\Bigg) \\
&-f\Bigg(\Big(\frac{1+t_2}{2}\Big)b+\Big(\frac{1-t_2}{2}\Big)x,\Big(\frac{1+s_2}{2}\Big)d+\Big(\frac{1-s_2}{2}\Big)y\Bigg)\Bigg|\Bigg]dydx \\
&\le \frac{1}{4(b-a)(d-c)} \\
&\times \int_a^b\int_c^d \Bigg[L_1\Bigg|\Big(\frac{t_1-t_2}{2}\Big)a+\Big(\frac{t_2-t_1}{2}\Big)x\Bigg|+L_2|\Big(\frac{s_1-s_2}{2}\Big)c+\Big(\frac{s_2-s_1}{2}\Big)y\Bigg| \\
&+L_1\Bigg|\Big(\frac{t_1-t_2}{2}\Big)a+\Big(\frac{t_2-t_1}{2}\Big)x\Bigg|+L_2\Bigg|\Big(\frac{s_1-s_2}{2}\Big)d+\Big(\frac{s_2-s_1}{2}\Big)y\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&+L_1\left|\Big(\frac{t_1-t_2}{2}\Big)b+\Big(\frac{t_2-t_1}{2}\Big)x\right|+L_2\left|\Big(\frac{s_1-s_2}{2}\Big)c+\Big(\frac{s_2-s_1}{2}\Big)y\right| \\
&+L_1\left|\Big(\frac{t_1-t_2}{2}\Big)b+\Big(\frac{t_2-t_1}{2}\Big)x\right|+L_2\left|\Big(\frac{s_1-s_2}{2}\Big)d+\Big(\frac{s_2-s_1}{2}\Big)y\right|\Bigg]dydx \\
&=\frac{L_1}{2(b-a)}\int_a^b\left|\Big(\frac{t_1-t_2}{2}\Big)a+\Big(\frac{t_2-t_1}{2}\Big)x\right|+\left|\Big(\frac{t_1-t_2}{2}\Big)b+\Big(\frac{t_2-t_1}{2}\Big)x\right|dx \\
&+\frac{L_2}{2(d-c)}\int_c^d\left|\Big(\frac{s_1-s_2}{2}\Big)c+\Big(\frac{s_2-s_1}{2}\Big)y\right|+\left|\Big(\frac{s_1-s_2}{2}\Big)d+\Big(\frac{s_2-s_1}{2}\Big)y\right|dy \\
&=\frac{L_1|t_2-t_1|(b-a)+L_2|s_2-s_1|(d-c)}{4}
\end{aligned}
$$

and

$$
\begin{aligned}
&|H(t_1,s_1)-H(t_2,s_2)| \\
\le\ &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d|f(t_1x+(1-t_1)(\frac{a+b}{2}),s_1y+(1-s_1)(\frac{c+d}{2})) \\
&-f(t_2x+(1-t_2)(\frac{a+b}{2}),s_2y+(1-s_2)(\frac{c+d}{2}))|dydx \\
\le\ &\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\Big[L_1|(t_1-t_2)(x-\frac{a+b}{2})| \\
&+L_2|(s_1-s_2)(y-\frac{c+d}{2})|\Big]dydx \\
=\ &\frac{L_1|t_1-t_2|}{(b-a)}\int_a^b|x-\frac{a+b}{2}|dx+\frac{L_2|s_1-s_2|}{(d-c)}\int_c^d|y-\frac{c+d}{2}|dy \\
=\ &\frac{L_1|t_1-t_2|(b-a)+L_2|s_1-s_2|(d-c)}{4}.
\end{aligned}
$$

This completes the proof of Theorem 2.

Remark 2. If we take $t_1=0, t_2=1, s_1=0$, and $s_2=1$ in *Theorem 2.* then (3.1) and (3.2) reduce to

$$
\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right| \\
&\le\frac{L_1(b-a)+L_2(d-c)}{4}
\end{aligned}
\tag{3.3}
$$

and

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right| \\
&\le\frac{L_1(b-a)+L_2(d-c)}{4},
\end{aligned}
\tag{3.4}
$$

respectively. The inequalities (3.3) and (3.4) are the Hadamard type inequalities for Lipschitzian mapping in two variables.

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