TAIWANESE JOURNAL OF MATHEMATICS
Vol. 11, No. 1, pp. 15-26, March 2007
This paper is available online at http://www.math.nthu.edu.tw/tjm/

COMPACTIFICATIONS OF METRIC SPACES

M. Koçak and İ. Akça

Abstract. If X is a discrete topological space, the points of its Stone-Cech compactification βX can be regarded as ultrafilters on X, and this fact is a useful tool in analysing the properties of βX. The purpose of this paper is to describe the compactification $\widetilde{X}$ of a metric space in terms of the concept of near ultafilters. We describe the topological space $\tilde{X}$ and we investigate conditions under which $\tilde{S}$ will be a semigroup compactification if S is a semigroup which has a metric. These conditions will always hold if the topology of S is defined by an invariant metric, and in this case our compactification $\tilde{S}$ coincides with S^{LUC}.

0. Introduction

The purpose of this paper is to describe the compactification of a metric space in terms of the concept of near ultafilters. If X is a discrete topological space, the points of its Stone-Cech compactification βX can be regarded as ultrafilters on X, and this fact is a useful tool in analysing the properties of βX. An analogous concept of "near ultrafilter" is used to describe the points of an arbitrary compactification of a topological group in [5]. We were motivated by this in defining the analogous concept of "near ultrafilter" to describe the points of an arbitrary compactification of a metric space. A metric space X has a compactification $\tilde{X}$ with the property that $C(\tilde{X})$ is isomorphic to the algebra of bounded real-valued uniformly continuous functions defined on X. We believe that near ultrafilters provide a natural and useful method for describing $\tilde{X}$.

In §2 we describe the topological space $\tilde{X}$. In §3 we assume that we have a semigroup S which has a metric and investigate conditions under which $\tilde{S}$ will be a semigroup compactification of S. These conditions will always hold if the topology

Received October 6, 2004, revised October 14, 2005.
Communicated by Ber-Luh Liu.
2000 *Mathematics Subject Classification*: 54D35, 54D60, 32J05.
Key words and phrases: Near ultrafilters, Compactifications, Semigroup compactifications.

of S is defined by an invariant metric, and in this case our compactification $\tilde{S}$ coincides with S^{LUC}.

Our results are not all new. For example, Theorems 4.7 and 4.8 are known for S^{LUC} [1]. We include these theorems, however, because the proofs that we give are a natural application of our construction of $\tilde{S}$.

1. Preliminaries

We first remind the reader of some basic definitions.

Metric Spaces. Let X be a set and $d : X \times X \to \mathbb{R}$ be a function. We say that d is a *metric* on X if the followings are satisfied:

(M-1) For all $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$.

(M-2) For all $x, y \in X$, $d(x, y) = d(y, x)$.

(M-3) For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

If d is a metric on X, the ordered pair (X, d) is called a *metric space*. Suppose that X is also a semigroup then d is called an *invariant metric* if $d(ax, ay) = d(xa, ya) = d(x, y)$ for all $x, y, a \in X$

For each $\varepsilon > 0$ and each $Y \subseteq X$, $B(Y, \varepsilon)$ will denote $\{z \in X | d(y, z) < \varepsilon$ for some $y \in Y\}$. In the case of a singleton set $\{y\}$, we may use $B(y, \varepsilon)$ instead of the cumbersome $B(\{y\}, \varepsilon)$.

A metric d on a set X will generate a topology on X for which the neighbourhoods of each point $x \in X$ are the sets of the form $B(x, \varepsilon)$, where $\varepsilon > 0$. If X has this topology, X is called a *metrisable* space. With this topology X is always Hausdorff.

Suppose that (X, d_1) and (Y, d_2) are metric spaces. A function $f : X \to Y$ is said to be *uniformly continuous* if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d_2(f(x_1), f(x_2)) < \varepsilon$ whenever $d_1(x_1, x_2) < \delta$.

Compactifications. Let X be a topological space. By a *compactification* of X we shall mean a pair (C, e), where C is a compact Hausdorff space, $e : X \to C$ is an embedding and $e[X]$ is dense in C. In this case, we may simply refer to C as being a compactification of X. Two compactifications (C, e) and (C', e') are regarded as equivalent if there is a homeomorphism $h : C \to C'$ for which $e = e'h$.

Semigroups. Let S be a semigroup. For each $s \in S$, we shall use λ_s and ρ_s to denote the mappings from S to itself for which $\lambda_s(t) = st$ and $\rho_s(t) = ts$.

Suppose that S is also a topological space. S will be called a *topological semigroup* if the mapping $(s, t) \mapsto st$ is a continuous mapping from $S \times S$ to S. It will be called a *semitopological semigroup* if, for every $s \in S$, λ_s and ρ_s are both continuous. It will be called a *right topological semigroup* if, for every $s \in S$, ρ_s is continuous.

If S is a right topological semigroup, $\{s \in S | \lambda_s : S \to S \text{ is continuous}\}$ will be called the *topological centre of S.*

Suppose that S is a semitopological semigroup and that (C, e) is a compactification of S. We shall say that (C, e) is a *semigroup compactification* of S if C is a right topological semigroup, e is a homomorphism and $e[S]$ is contained in the topological centre of C.

Notation. We shall use $\mathbb{N}$ to denote the set of positive integers, $\mathbb{Z}$ to denote the set of all integers and $\mathbb{R}$ to denote the set of real numbers.

If X is a topological space, $C(X)$ will denote the set of continuous bounded real-valued functions defined on X, and βX will denote the Stone-Čech compactification of X.

2. The Topological Space $\tilde{X}$

Definition 2.1. Suppose that (X, d) is a metric space and that $\mathcal{G} \subseteq \mathcal{P}(X)$. We shall say that $\mathcal{G}$ has the *near finite intersection property* if $\mathcal{G}$ is non-empty and if, for every finite subset $\mathcal{F}$ of $\mathcal{G}$ and every $\varepsilon > 0$, $\bigcap_{Y \in \mathcal{F}} B(Y, \varepsilon) \neq \emptyset$.

Definition 2.2. Let $\xi \subseteq \mathcal{P}(X)$. We shall say that ξ is a *near ultrafilter on* (X, d) if ξ is maximal subject to being a subset of $\mathcal{P}(X)$ with the near finite intersection property.

In this case, we may simply refer to ξ as being a near ultrafilter if it is clear which metric space (X, d) is being referred to.

Notation. We shall use $(\widetilde{X}, d)$ to denote the set of all near ultrafilters on (X, d). We may simply denote this set by $\tilde{X}$ if there is no ambiguity about which metric structure is being used.

Remark 2.3 It is immediate from Zorn's Lemma that every subset of $\mathcal{P}(X)$ with the near finite intersection property is contained in a near ultrafilter. It is also clear that, if $\xi \in (\widetilde{X}, d)$ and if $Y \subseteq X$, $Y \in \xi$ if and only if $B(Y, \varepsilon) \cap \bigcap_{Z \in \mathcal{F}} B(Z, \varepsilon) \neq \emptyset$ for every finite subset $\mathcal{F}$ of ξ and every $\varepsilon > 0$.

We observe that the concept of a near ultrafilter generalises the concept of an ultrafilter. If d denotes the discrete metric on a set X, a near ultrafilter on (X, d) is simply an ultrafilter on X.

Throughout this section, we shall assume that (X, d) denotes a given metric space.

Lemma 2.4. *Let $\xi \in \tilde{X}$. For every finite subset $\mathcal{F}$ of ξ and every $\varepsilon > 0$, $\bigcap_{Y\in\mathcal{F}} B(Y,\varepsilon) \in \xi$.*

Proof. If $\bigcap_{Y\in\mathcal{F}} B(Y,\varepsilon) \notin \xi$, there will be a finite subset $\mathcal{F}'$ of ξ and a $\delta > 0$ for which $B(\bigcap_{Y\in\mathcal{F}} B(Y,\varepsilon),\delta) \cap \bigcap_{Y\in\mathcal{F}'} B(Y,\delta) = \emptyset$. We can choose $\sigma > 0$ satisfying $2\sigma \leq \min\{\varepsilon,\delta\}$. This will imply that $\bigcap_{Y\in\mathcal{F}\cup\mathcal{F}'} B(Y,\sigma) = \emptyset$ - contradictiong our assumption that ξ has the near finite intersection property. ■

Lemma 2.5. *Let $\xi \in \tilde{X}$ and let $Y \subseteq X$. The following statements are equivalent:*

(i) $Y \in \xi$;

(ii) *For every $\varepsilon > 0$ and every $Z \in \xi$, $B(Y,\varepsilon) \cap Z \neq \emptyset$;*

(iii) *For every $\varepsilon > 0$ and every $Z \in \xi$, $Y \cap \hat{U}(Z) \neq \emptyset$;*

Proof. **(i)** $\Leftrightarrow$ **(ii)** If $Y \notin \xi$ there will be a finite subset $\mathcal{F}$ of ξ and an $\varepsilon > 0$ such that $B(Y,\varepsilon) \cap \bigcap_{Y'\in\mathcal{F}} B(Y',\varepsilon) = \emptyset$. If Z denotes $\bigcap_{Y'\in\mathcal{F}} B(Y',\varepsilon)$, then $Z \in \xi$ by Lemma 2.4 and $B(Y,\varepsilon) \cap Z = \emptyset$.

Conversely, suppose that $B(Y,\varepsilon) \cap Z = \emptyset$ for some $Z \in \xi$ and some $\varepsilon > 0$. We can choose a $\delta > 0$ satisfying $2\delta \leq \varepsilon$. We claim that $B(Y,\delta) \cap B(Z,\delta) = \emptyset$. To see this, assume that there is a point $x \in B(Y,\delta) \cap B(Z,\delta)$. Since $d(x,y) < \delta$ for some $y \in Y$ and $d(x,z) < \delta$ for some $z \in Z$, it follows that $d(y,z) < 2\delta \leq \varepsilon$. Thus $z \in B(Y,\varepsilon) \cap Z$ - contradiction. This shows that $B(Y,\delta) \cap B(Z,\delta) = \emptyset$ and hence that $Y \notin \xi$.

(ii) $\Leftrightarrow$ **(iii)** For every $\varepsilon > 0$ and every $Y, Z \subseteq X$, $B(Y,\varepsilon) \cap Z \neq \emptyset \Leftrightarrow Y \cap B(Z,\varepsilon) \neq \emptyset$. ■

Lemma 2.6. *Let $\xi \in \tilde{X}$ and let $Y \subseteq X$. Then $Y \in \xi$ if and only if $B(Y,\varepsilon) \in \xi$ for every $\varepsilon > 0$. Furthermore, this is the case if and only if $\overline{Y} \in \xi$.*

Proof. Clearly, if $Y \in \xi$, then $B(Y,\varepsilon) \in \xi$ for every $\varepsilon > 0$, because $Y \subseteq B(Y,\varepsilon)$.

Conversely, if $Y \notin \xi$, then $B(Y,\varepsilon) \cap Z = \emptyset$ for some $\varepsilon > 0$ and some $Z \in \xi$ (by Lemma 2.5). Let $\delta > 0$ satisfying $2\delta \leq \varepsilon$. Then $B(Y,2\delta) \subseteq B(Y,\varepsilon)$ and so $B(Y,2\delta) \cap Z = \emptyset$ and $B(Y,\delta) \notin \xi$.

Now, for every $\varepsilon > 0$, $Y \subseteq \overline{Y} \subseteq B(Y,\varepsilon)$. It follows that $Y \in \xi$ if and only if $\overline{Y} \in \xi$. ■

Lemma 2.7. *Let $\xi \in \tilde{X}$. For any $Y_1, Y_2 \subseteq X$, $Y_1 \cup Y_2 \in \xi$ implies that $Y_1 \in \xi$ or $Y_2 \in \xi$.*

Proof. If $Y_1, Y_2 \notin \xi$, there will be sets $Z_1, Z_2 \in \xi$ and $\varepsilon_1, \varepsilon_2 > 0$ for which $Y_1 \cap B(Z_1,\varepsilon_1) = Y_2 \cap B(Z_2,\varepsilon_2) = \emptyset$ (by Lemma 2.5). We choose $\varepsilon > 0$ satisfying

$2\varepsilon \le \min\{\varepsilon_1, \varepsilon_2\}$, and claim that $B(Y_1,\varepsilon) \cap B(Z_1,\varepsilon) = B(Y_2,\varepsilon) \cup B(Z_2,\varepsilon) = \emptyset$. To see this, suppose that $x \in B(Y_i,\varepsilon) \cap B(Z_i,\varepsilon)$, where $i \in \{1,2\}$. Then there will be points $y \in Y_i, z \in Z_i$ for which $d(x,y) < \varepsilon, d(x,z) < \varepsilon$. This implies that $d(y,z) < 2\varepsilon \le \varepsilon_i$ and hence that $y \in Y_i \cap B(Z_i, \varepsilon_i)$ - contradiction.

Since $B(Y_1 \cup Y_2, \varepsilon) = B(Y_1,\varepsilon) \cup B(Y_2,\varepsilon)$, we have shown that $B(Y_1 \cup Y_2,\varepsilon) \cap B(Z_1,\varepsilon) \cap B(Z_2,\varepsilon) = \emptyset$ and hence that $Y_1 \cup Y_2 \notin \xi$. ∎

3. The Topological Space $\tilde{X}$

Definition 3.1. For each $Y \subseteq X$, we put $\mathcal{C}_Y = \{\xi \in \tilde{X} | Y \in \xi\}$.

Lemma 3.2. *For every $Y_1, Y_2 \subseteq X$, $\mathcal{C}_{Y_1 \cup Y_2} = \mathcal{C}_{Y_1} \cup \mathcal{C}_{Y_2}$. Furthermore, $\mathcal{C}_\emptyset = \emptyset$ and $\mathcal{C}_X = \tilde{X}$.*

Proof. The first statement follows from Lemma 2.7, and the second is immediate from the definition. ∎

Definition 3.3. We define the topology of $\tilde{X}$ by choosing the sets of the form $\mathcal{C}_Y$, where $Y \in \mathcal{P}(X)$, as a base for the closed sets.

Theorem 3.4. *$\tilde{X}$ is a compact Hausdorff space.*

Proof. Let $(\mathcal{C}_{Y_\alpha})_{\alpha\in A}$ be a family of basic closed subsets of $\tilde{X}$ with the finite intersection property. We shall show that $\bigcap_{\alpha\in A} \mathcal{C}_{Y_\alpha} \ne \emptyset$. It will follow that $\tilde{X}$ is compact.

For any finite $F \subseteq A$ and any $\varepsilon > 0$, there will be a near ultrafilter $\xi_F \in \bigcap_{\alpha\in F} \mathcal{C}_{Y_\alpha}$ and so, since $Y_\alpha \in \xi_F$ for every $\alpha \in F$, $\bigcap_{\alpha\in A} B(Y_\alpha,\varepsilon) \ne \emptyset$. This shows that the family $(Y_\alpha)_{\alpha\in A}$ has the near finite intersection property and hence that it is contained in a near ultrafilter ξ. Since $\xi \in \bigcap_{\alpha\in A} \mathcal{C}_{Y_\alpha}$, it follows that $\bigcap_{\alpha\in A} \mathcal{C}_{Y_\alpha} \ne \emptyset$.

To see that $\tilde{X}$ is Hausdorff, suppose that ξ_1, ξ_2 are distinct elements of $\tilde{X}$. Choose any $Y_1 \in \xi_1 \setminus \xi_2$. There will be a set $Y_2 \in \xi_2$ and a $\varepsilon > 0$ for which $Y_1 \cap B(Y_2,\varepsilon) = \emptyset$ (by Lemma 2.5). We choose a $\delta > 0$ satisfying $2\delta \le \varepsilon$ and put $Z = \tilde{X} \setminus B(Y_2,\delta)$. It is easy to check that $Y_1 \cap B(Y_2, 2\delta) = \emptyset$ and hence that $\xi_1 \in \tilde{X} \setminus \mathcal{C}_{B(Y_2,\delta)}$ (by Lemma 2.5). Also, since $Z \cap B(Y_2,\delta) = \emptyset$, $\xi_2 \in \tilde{X} \setminus \mathcal{C}_Z$. Now $\mathcal{C}_{B(Y_2,\delta)} \cup \mathcal{C}_Z = \tilde{X}$, by Lemma 2.7, and so $(\tilde{X} \setminus \mathcal{C}_{B(Y_2,\delta)}) \cap (\tilde{X} \setminus \mathcal{C}_Z) = \emptyset$. Thus $\tilde{X}$ is indeed Hausdorff. ∎

Definition 3.5. We define a mapping e on X by stating that, for each $x \in X$, $e(x) = \{Y \in \mathcal{P}(X) | x \in \overline{Y}\}$.

It is easy to verify that $e(x) \in \tilde{X}$.

Theorem 3.6. *The mapping e embeds X as a dense subspace in $\tilde{X}$.*

Proof. We first remark that e is injective. To see this, suppose that x_1, x_2 are distinct points of X. Then $\{x_1\} \in e(x_1) \setminus e(x_2)$ and so $e(x_1) \neq e(x_2)$.

Now, for any $Y \subseteq X$ and any $x \in X$,

$$x \in \overline{Y} \Leftrightarrow Y \in e(x) \Leftrightarrow e(x) \in \mathcal{C}_Y.$$

This shows that $e^{-1}(\mathcal{C}_Y) = \overline{Y}$ and hence that e is continuous.

It also shows that, for any closed subset Y of X, $e[Y] = \mathcal{C}_Y \cap e[X]$. Since this is a closed subset of $e[X]$, e is a closed mapping from X to $e[X]$ and therefore defines a homeomorphism from X to $e[X]$.

Finally, suppose that $\mathcal{C}_Y \neq \tilde{X}$. If $\xi \in \tilde{X} \setminus \mathcal{C}_Y$, then $Y \cap B(Z, \varepsilon) = \emptyset$ for some $Z \in \xi$ and some $\varepsilon > 0$. This implies that $B(Y, \varepsilon) \cap Z = \emptyset$ and hence that $\overline{Y} \neq X$, because $\overline{Y} \subseteq B(Y, \varepsilon)$. Thus we can choose $x \in X \setminus \overline{Y}$. This implies that $e(x) \in \tilde{X} \setminus \mathcal{C}_Y$ and shows that $e[X]$ is dense in $\tilde{X}$, because every non-empty open subset of $\tilde{X}$ will contain a non-empty set of the form $\tilde{X} \setminus \mathcal{C}_Y$. ∎

Theorem 3.7. *Suppose that (X, d_1) and (Y, d_2) are metric spaces and that $f : X \to Y$ is uniformly continuous. Then there is a continuous function $\tilde{f} : \tilde{X} \to \tilde{Y}$ which is an extension of f in the sense that $\tilde{f} e_X = e_Y f$, where e_X, e_Y denote the natural embeddings of X, Y in $\tilde{X}, \tilde{Y}$ respectively.*

Proof. Given $\xi \in \tilde{X}$, we define $\eta = \{T \in \mathcal{P}(Y) | f^{-1}(B(Y, \delta)) \in \xi$ for every $\delta > 0\}$. We shall show that $\eta \in \tilde{Y}$.

We first show that η has the near finite intersection property. To see this, suppose that $\mathcal{F}$ is a finite subset of η and that $\sigma > 0$. We choose $\delta > 0$ satisfying $2\delta \leq \sigma$. Then, there is $\varepsilon > 0$ such that $d_1(x_1, x_2) < \varepsilon$ implies that $d_2(f(x_1), f(x_2) < \delta$. It follows that $\bigcap_{T \in \mathcal{F}} B(f^{-1}(B(T, \delta)), \varepsilon) \neq \emptyset$. If x is in this set, then, for each $T \in \mathcal{F}$, there will be a point $x_T \in f^{-1}(B(T, \delta))$ for which $d_1(x, x_T) < \varepsilon$. This implies that $d_2(f(x), f(x_T)) < \delta$ and hence, since $f(x_T) \in B(T, \delta)$, that $f(x) \in B(T, 2\delta) \subseteq B(T, \sigma)$. Thus $\bigcap_{T \in \mathcal{F}} B(T, \sigma) \neq \emptyset$ and η does have the near finite intersection property.

We now show that η is a near ultrafilter. If $T \notin \eta$, $f^{-1}(B(T, \delta)) \notin \xi$ for some $\delta > 0$. This implies that $f^{-1}(B(T, \delta)) \cap S = \emptyset$ for some $S \in \xi$, and hence that $B(T, \delta) \cap f[S] = \emptyset$. Now $f[S] \in \eta$, because, for every $\sigma > 0$, $f^{-1}(B(f[S], \sigma)) \supseteq f^{-1}(f[S]) \supseteq S$. It follows that η is maximal subject to having the near finite intersection property.

We can thus define a mapping $\tilde{f} : \tilde{X} \to \tilde{Y}$ by stating that $f(\xi) = \eta$. It is immediate that $\tilde{f}$ is continuous, because, if $T \subseteq Y$, $(\tilde{f})^{-1}(\mathcal{C}_T) = \bigcap_{\delta > 0} \mathcal{C}_{f^{-1}(B(T, \delta))}$.

Finally, let $x \in X$. It is obvious that $\{f(x)\} \in \tilde{f}(e_X(x))$ and hence that $\tilde{f}(e_X(x)) = e_Y(f(x))$. ∎

Lemma 3.8. *Let $\xi \in \tilde{X}$ and let $Y \subseteq X$. Then $\xi \in \mathbf{cl}_{\tilde{X}} e[Y]$ if and only if $Y \in \xi$.*

Proof. Clearly, $\mathrm{cl}_{\tilde{X}} e[Y] = \bigcap\{\mathcal{C}_Z | \mathcal{C}_Z \supseteq e[Y]\}$. Now $y \in Y \Rightarrow Y \in e(y) \Rightarrow e(y) \in \mathcal{C}_Y$. So $\mathcal{C}_Y \supseteq e[Y]$. On the other hand, suppose that $Z \in \mathcal{P}(X)$ satisfies $\mathcal{C}_Z \supseteq e[Y]$. Then $y \in Y \Rightarrow e(y) \in \mathcal{C}_Z \Rightarrow Z \in e(y) \Rightarrow y \in \mathrm{cl}_X Z$. So $Y \subseteq \overline{Z}$ and hence $\mathcal{C}_Y \subseteq \mathcal{C}_{\overline{Z}} = \mathcal{C}_Z$ (by Lemma 2.6). Thus $\mathrm{cl}_{\tilde{X}} e[Y] = \mathcal{C}_Y$. ∎

Corollary 3.9. *For any $Y_1, Y_2 \in \mathcal{P}(X)$, $\mathrm{cl}_{\tilde{X}}(Y_1) \cap \mathrm{cl}_{\tilde{X}}(Y_2) \neq \emptyset$ if and only if $B(Y_1, \varepsilon) \cap B(Y_2, \varepsilon) \neq \emptyset$ for every $\varepsilon > 0$.*

Proof. The condition that $B(Y_1, \varepsilon) \cap B(Y_2, \varepsilon) \neq \emptyset$ for every $\varepsilon > 0$ is equivalent to the condition that $\mathcal{C}_{Y_1} \cap \mathcal{C}_{Y_2} \neq \emptyset$. ∎

Remark 3.10. We shall henceforward regard X as being a subspace of $\tilde{X}$ by identifying the point $x \in X$ with the point $e(x) \in \tilde{X}$.

The following Lemma is elementary and obviously well-known. We include it for the sake of completeness.

Lemma 3.11. *Let (f_n) be a sequence of uniformly continuous real-valued functions defined on a metric space (X, d). If (f_n) converges uniformly on X to a function f, then f is uniformly continuous.*

Proof. Let $\epsilon > 0$. We can choose $n \in \mathbb{N}$ so that $|f(x) - f_n(x)| < \frac{\epsilon}{3}$ for every $x \in X$. We can then choose $\delta > 0$ so that $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$ whenever $d(x, y) < \delta$. It follows that $|f(x) - f(y)| < \epsilon$ whenever $d(x, y) < \delta$. ∎

Theorem 3.12. *A bounded continuous function $f : X \to \mathbb{R}$ has a continuous extension $\tilde{f} : \tilde{X} \to \mathbb{R}$ if and only if it is uniformly continuous.*

Proof. Let $C(\tilde{X})$ denote the set of all continuous real-valued functions defined on $\tilde{X}$. We know from Theorem 3.7 that a bounded uniformly continuous bounded functions $f : X \to \mathbb{R}$ does have a continuous extension $\tilde{f} : \tilde{X} \to \mathbb{R}$. The set of all functions $\tilde{f}$ which arise in this way will be a uniformly closed subalgebra of $C(\tilde{X})$ (by Lemma 3.11) and will contain the constant functions. By the Stone-Weierstrass Theorem, it will be the whole of $C(\tilde{X})$ if it separates the points of $\tilde{X}$.

To see that it does, let ξ_1, ξ_2 be distinct points of $\tilde{X}$. By Lemma 2.5, we can choose $Y_1 \in \xi_1$, $Y_2 \in \xi_2$ and $\varepsilon > 0$ for which $B(Y_1, \varepsilon) \cap Y_2 = \emptyset$. There will be a uniformly continuous function $f : X \to [0, 1]$ for which $f[Y_1] = \{0\}$ and $f[Y_2] = \{1\}$ (cf. [7]). Since $\xi_1 \in \mathrm{cl}_{\tilde{X}} Y_1$ and $\xi_2 \in \mathrm{cl}_{\tilde{X}} Y_2$ (by Lemma 3.8), it follows that $\tilde{f}(\xi_1) = 0$ and $\tilde{f}(\xi_2) = 1$. Thus the functions of the form $\tilde{f}$ do separate the points of $\tilde{X}$. ∎

Corollary 3.13. *$C(\tilde{X})$ can be identified with the algebra of uniformly continuous bounded real-valued functions defined on X.*

Theorem 3.14. *Suppose that the metric space (X, d) is not totally bounded. Then $\tilde{X}$ contains a topological copy of $\beta\mathbb{N}$.*

Proof. We can choose a symmetric vicinity $\varepsilon > 0$ for which the covering $\{B(x, \varepsilon) | x \in X\}$ of X has no finite subcovering. We can then choose a sequence $(x_n) \subseteq X$ with the property that, for each $n \in \mathbb{N}$, $x_n \notin \bigcup_1^{n-1} B(x_m, \varepsilon)$. We do this inductively, first choosing x_1 to be any element of X. We then assume that x_m has been chosen for each $m = 1, 2, \ldots, n-1$ and choose x_n to be any element of $X \setminus \bigcup_1^{n-1} B(x_m, \varepsilon)$.

We then choose $\delta > 0$ satisfying $2\delta \leq \varepsilon$. This implies that the sets $B(x_n, \delta)$ will be pairwise disjoint.

Let D denote the discrete subspace $\{x_n | n \in \mathbb{N}\}$ of X. We shall show that $\mathrm{cl}_{\tilde{X}} D \simeq \beta\mathbb{N}$.

The mapping $f : \mathbb{N} \to \tilde{X}$, defined by stating that $f(n) = x_n$, has a continuous extension $f^\beta : \beta\mathbb{N} \to \tilde{X}$. It will be sufficient to show that f^β is injective. Suppose then that μ_1 and μ_2 are distinct elements of $\beta\mathbb{N}$, and that G_1 and G_2 are disjoint open subsets of $\beta\mathbb{N}$ containing μ_1 and μ_2 respectively. Let $M_i = \mathbb{N} \cap G_i$ $(i = 1, 2)$. Since $B(f[M_1], \delta) \cap B(f[M_2], \delta) = \emptyset$, $\mathrm{cl}_{\tilde{X}}(f[M_1]) \cap \mathrm{cl}_{\tilde{X}}(f[M_2]) = \emptyset$, by the Corollary to Lemma 3.8. Now $f^\beta(\mu_i) \in \mathrm{cl}_{\tilde{X}}(f[M_i])$ for $i = 1, 2$, and so $f^\beta(\mu_1) \neq f^\beta(\mu_2)$. ■

Remark 3.15. It follows from Theorem 3.14 that $\tilde{X}$ has at least 2^{c} points if (X, d) is not totally bounded, because it is well known that $|\beta\mathbb{N}| = 2^{\mathsf{c}}$ (cf. [9]). However, if X is a non-compact totally bounded space, $\tilde{X}$ need not be as vast as this. For example, let X denote the subspace $\{\frac{1}{n} | n \in \mathbb{N}\}$ of $\mathbb{R}$, with its standard metric. Then $\tilde{X}$ is the countable subspace $X \cup \{0\}$ of $\mathbb{R}$, because the functions in $C(X)$ which have continuous extensions to $X \cup$

Definition 3.16. Suppose that Y is a subspace of a metric space (X, d). Then Y is also a metric space with the induced metric $d_Y : Y \times Y \to$

Theorem 3.17. *Suppose that Y is a subspace of a metric space (X, d) and that Y has the induced metric d_Y. Then $\tilde{Y} \simeq \mathrm{cl}_{\tilde{X}} Y$.*

Proof. The inclusion map $i : Y \to X$ is uniformly continuous and therefore has a continuous extension $\tilde{\imath} : \tilde{Y} \to \tilde{X}$ (by Theorem 3.7). We shall show that $\tilde{\imath}$ is injective.

Suppose that μ_1, μ_2 are distinct points in $\tilde{Y}$. There will then be sets $Z_1, Z_2 \subseteq Y$ and a $\varepsilon > 0$ for which $B_Y(Z_1, \varepsilon) \cap Z_2 = \emptyset$, where $B_Y(Z_1, \varepsilon)$ denotes $B(Z_1, \varepsilon) \cap Y$.

Now $B_Y(Z_1, \varepsilon) \cap Z_2 = \emptyset$ implies that $B_Y(Z_1, \varepsilon) \cap Z_2 = \emptyset$ and hence that $\mathrm{cl}_{\tilde{X}}(Z_1) \cap \mathrm{cl}_{\tilde{X}}(Z_2) = \emptyset$, by the Corollary to Lemma 3.8. Since $\tilde{\imath}(\mu_i) \in \mathrm{cl}_{\tilde{X}}(Z_i)$ for $i = 1, 2$, it follows that $\tilde{\imath}(\mu_1) \neq \tilde{\imath}(\mu_2)$. ∎

4. The Compactification of a Semigroup

We shall now suppose that (S, d) is a metric space and that S is a semigroup. We shall give conditions under which the semigroup operation on S can be extended to $\tilde{S}$, giving $\tilde{S}$ the structure of a compact right topological semigroup.

Notation. For each $s \in S$, we define $\lambda_s : S \to S$ and $\rho_s : S \to S$ by stating that $\lambda_s(t) = st$ and $\rho_s(t) = ts$.

Theorem 4.1. *Suppose that the two following conditions are satisfied:*

(i) *For every $s \in S$, the mapping $\lambda_s : S \to S$ is uniformly continuous;*

(ii) *For every $\varepsilon > 0$, there exists a $\delta > 0$ with the property that*

$$d(s_1, s_2) < \delta \Rightarrow d(s_1 t, s_2 t) < \varepsilon (\forall t \in S).$$

Then the semigroup operation defined on S can be extended to $\tilde{S}$ in such a way that $\tilde{S}$ becomes a semigroup compactification of S.

Proof. For each $s \in S$, the uniformly continuous mapping λ_s can be extended to a continuous mapping $\tilde{\lambda}_s : \tilde{S} \to \tilde{S}$, by Theorem 3.7. If $\eta \in \tilde{S}$, we shall denote $\tilde{\lambda}_s(\eta)$ by $s\eta$.

We shall show that, for each $\eta \in \tilde{S}$, the mapping $s \mapsto s\eta$ from S to $\tilde{S}$ is uniformly continuous.

Let $\phi : \tilde{S} \to \mathbb{R}$ be continuous. Then, by Theorem 3.12, $\phi_{|S}$ is uniformly continuous. Thus, if $\epsilon > 0$, there will be a $\delta > 0$ such that $|\phi(s) - \phi(s')| < \epsilon$ if $d(s, s') < \delta$. By condition ii), there will be a $\sigma > 0$ such that, whenever $d(s, s') < \sigma$, $d(st, s't) < \delta$ for every $t \in S$. So, if $d(s, s') < \sigma$, $|\phi(st) - \phi(s't)| < \epsilon$ for every $t \in S$. Now $|\phi(s\eta) - \phi(s'\eta)| = \lim_{t \to \eta} |\phi(st) - \phi(s't)|$, and so $|\phi(s\eta) - \phi(s'\eta)| \leq \epsilon$ if $d(s, s') < \sigma$. Using the fact that the unique metric structure on $\tilde{S}$ can be defined by the functions in $C(\tilde{S})$, we have shown that the mapping $s \mapsto s\eta$ from S to $\tilde{S}$ is uniformly continuous.

It now follows from Theorem 3.7 that the mapping $s \mapsto s\eta$ can be extended to a continuous mapping from $\tilde{S}$ to itself. The image of the element $\xi \in \tilde{S}$ under this extension will be denoted by $\xi\eta$.

Thus we have defined a binary operation on $\tilde{S}$ by a double limit process. If $\xi, \eta \in \tilde{S}$,

$$\xi\eta = \lim_{s \to \xi} \lim_{t \to \eta} st.$$

We observe that our definitions ensure that, for each $s \in S$, the mapping $\eta \mapsto s\eta$ is a continuous mapping from $\tilde{S}$ to itself. Furthermore, for each $\eta \in \tilde{S}$, the mapping $\xi \mapsto \xi\eta$ is also a continuous mapping from $\tilde{S}$ to itself.

The associativity of the operation defined on $\tilde{S}$ is immediate from the following equations: For every $\xi, \eta, \zeta \in \tilde{S}$,

$$\xi(\eta\zeta) = \lim_{s\to\xi}\lim_{t\to\eta}\lim_{u\to\zeta} s(tu);$$
$$(\xi\eta)\zeta = \lim_{s\to\xi}\lim_{t\to\eta}\lim_{u\to\zeta}(st)u.$$ ■

Remark 4.2. The conditions used in Theorem 4.1 are satisfied by any semigroup S whose topology is defined by an invariant metric.

We shall henceforward assume that S is a semitopological semigroup for which the conditions of Theorem 4.1 are satisfied, and that $\tilde{S}$ has the semigroup structure defined in this theorem.

Remark 4.3. Suppose that T is a subsemigroup of S. We have seen in Theorem 3.17 that $\tilde{T}$ can be regarded as topologically embedded in $\tilde{S}$, if T is assumed to have the metric induced by that of S. The embedding is also algebraic, because the inclusion map $i : T \to S$ has an extension $\tilde{\imath} : \tilde{T} \to \tilde{S}$ which is readily seen to be a homomorphism. Thus $\tilde{T}$ can be regarded as a subsemigroup of $\tilde{S}$.

Lemma 4.4. *Let $s \in S$ and $\xi \in \tilde{S}$. Then, if $Y \in \xi$, $sY \in s\xi$.*

Proof. This follows from Lemma 3.8, since the mapping $\lambda_s : \tilde{S} \to \tilde{S}$ is continuous. So, if $\xi \in \mathrm{cl}_{\tilde{S}}Y$, $s\xi \in \mathrm{cl}_{\tilde{S}}sY$. ■

Lemma 4.5. *If S is a group, then, for each $s \in S$ and each $\xi \in \tilde{S}$, $s\xi = \{sY | Y \in \xi\}$.*

Proof. This is immediate from Lemma 4.4. ■

Lemma 4.6. *Let (X, d) be a metric space and let $\xi \in \tilde{X}$. For each $Y \in \xi$ and each $\varepsilon > 0$, $\mathcal{C}_{B(Y,\varepsilon)}$ is a neighbourhood of ξ in $\tilde{X}$. Furthermore, the sets of this form provide a basis for the neighbourhoods of ξ in $\tilde{X}$.*

Proof. Since $\xi \in \tilde{X} \setminus \mathcal{C}_{X\setminus B(Y,\varepsilon)} \subseteq \mathcal{C}_{B(Y,\varepsilon)}$, $\mathcal{C}_{B(Y,\varepsilon)}$ is a neighbourhood of ξ.

On the otherhand, suppose that $T \subseteq X$ and that $\xi \in \tilde{X} \setminus \mathcal{C}_T$. Then $T \notin \xi$ and so $T \cap B(Y, \delta) = \emptyset$ for some $Y \in \xi$ and some $\delta > 0$ (by Lemma 2.5). Let $\varepsilon > 0$ be satisfying $2\varepsilon \leq \delta$. Then $\xi \in \mathcal{C}_{B(Y,\varepsilon)}$ and $\mathcal{C}_{B(Y,\varepsilon)} \subseteq \tilde{X} \setminus \mathcal{C}_T$ because $B(Y, \varepsilon) \cap B(T, \delta) = \emptyset$. Thus the sets of the form $\mathcal{C}_{B(Y,\varepsilon)}$ do provide a basis for the neighbourhoods of ξ. ■

Theorem 4.7. *Suppose that S is a topological group. Then the mapping $(s,\xi) \mapsto s\xi$ is a continuous mapping from $S \times \tilde{S}$ to $\tilde{S}$.*

Proof. We now from Theorem 4.1 that the maps $s \mapsto s\xi$ from S to $\tilde{S}$ are uniformly continuous. Suppose that $\phi : \tilde{S} \to \mathbb{R}$ is continuous and that $\varepsilon < 0$. There is $\delta > 0$ such that $|\phi(s\xi) - \phi(s'\xi)| < \varepsilon$ whenever $d(s,s') < \delta$ and every $\xi \in \tilde{S}$. Let $s \in S$ and $\xi \in \tilde{S}$. Since the map $\lambda_s : \tilde{S} \to \tilde{S}$ is continuous, there is a neighbourhood W of ξ in $\tilde{S}$ such that $|\phi(s\xi) - \phi(s\xi')| < \varepsilon$ whenever $\xi' \in W$. It follows that $|\phi(s\xi) - \phi(s'\xi')| < 2\varepsilon$ whenever $d(s,s') < \delta$ and $\xi' \in W$. ■

In the next theorem, we show that there is a sense in which $\tilde{S}$ is the largest semigroup compactification of S in which the continuity condition of Theorem 3.7 is satisfied.

Theorem 4.8. *Let S be a topological group. Suppose that T is a compact right topological semigroup and that $h : S \to T$ is a continuous homomorphism. Suppose also that the mapping $(s,\eta) \mapsto h(s)\eta$ is a continuous mapping from $S \times T$ to T. Then there is a continuous homomorphism $\tilde{h} : \tilde{S} \to T$ for which $h = \tilde{h}_{|S}$.*

Proof. We shall first show that h is uniformly continuous.

Let $\phi : T \to [0,1]$ be a continuous function and let $\epsilon > 0$. For each $\eta \in T$ there will be a neighbourhood $N(\eta)$ of η in T, and a neigbourhood $U(\eta)$ of the identity in S, for which $|\phi(h(s)\zeta) - \phi(\eta)| < \frac{\epsilon}{2}$ whenever $s \in U(\eta)$ and $\zeta \in N(\eta)$. Now T will be covered by a finite number of neighbourhoods of the form $N(\eta)$, corresponding to points $\eta_1, \eta_2, \ldots, \eta_n$ in T. Let $U = \bigcap_{i=1}^{n} U(\eta_i)$.

Suppose that $s_1, s_2 \in S$ satisfy $s_1 \in Us_2$. If $h(s_2) \in N(\eta_i)$, then

$$|\phi(h(s_1 s_2^{-1})h(s_2)) - \phi(\eta_i)| < \frac{\epsilon}{2}$$

and

$$|\phi(h(s_2)) - \phi(\eta_i)| < \frac{\epsilon}{2},$$

and so $|\phi(h(s_1)) - \phi(h(s_2))| < \epsilon$. Thus h is uniformly continuous.

It follows from Theorem 3.7 that there is a continuous function $\tilde{h} : \tilde{S} \to T$ for which $h = \tilde{h}_{|S}$.

That $\tilde{h}$ is a homomorphism can be seen as follows: For any $\xi_1, \xi_2 \in \tilde{S}$,

$$\begin{aligned}\tilde{h}(\xi_1\xi_2) &= \lim_{s_1\to\xi_1} \lim_{s_2\to\xi_2} h(s_1 s_2) \\ &= \lim_{s_1\to\xi_1} \lim_{s_2\to\xi_2} h(s_1)h(s_2) \\ &= \tilde{h}(\xi_1)\tilde{h}(\xi_2) \blacksquare\end{aligned}$$

Remark 4.9. If S is a semigroup then S^{LUC} compactification of S is defined to be the spectrum of the Banach algebra S^{LUC} of bounded left uniformly continuous functions on S i.e all $f \in CB(S)$ such that the map $s \to {}_sf$, ${}_sf(t) = f(st)$, $s, t \in S$, is continuous when $CB(S)$ has the $\sup$ norm topology. (Cf. [5].)

Corollary 4.10. *If S is a group, $\tilde{S}$ can be identified with the compactification S^{LUC}, since S^{LUC} is known to be the largest semigroup compactification of S in which the continuity condition of*

References

1. J. F. Berglund, H. D. Junghenn and P. Milnes, *Compact right topological semigroups and generalisations of almost periodicity*, Lecture Notes in Math., Springer-Verlag, 1978, 663.
2. M. Filali, The metric compactification of a locally compact abelian group, *Math. Proc. Cam. Phil. Soc.*, (1990), **108** , 527-538.
3. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, 1960.
4. N. Hindman and J. Pym, Free groups and semigroups in $\beta\mathbb{N}$, *Semigroup Forum*, **30** (1984), 177-193.
5. M. Mitchell Topological Semigroups and Fixed Points, *Illinois J. Math.*, **14** (1970), 630-641.
6. M. Koçak and Z. Arvasi, Near Ulrafilters and Compactification of Topological Groups, *Turksh J. Math.*, **21(2)** (1997), 213-225.
7. M. Koçak and D. Strauss, Near Ulrafilters and Compactifications, *Semigroup Forum*, **55** (1997), 94-109.
8. H. J. Kowalsky, *Topological Spaces*, Academic Press, 1964).
9. P. Samuel, Ultrafilters and compactifications of metric spaces, *Trans. Am. Math. Soc.*, **64** (1948), 100-132.
10. J. van Mill, An introduction to $\beta\omega$, Handbook of Set-Theoretic Topology, *North-Holland*, (1984), 503-568.

M. Koçak and İ. Akça
Osmangazi Universitesi,
Fen Edebiyat Fakültesi,
Matematik Bölümü,
26480 Eskişehir, Turkey
E-mails: iakca@ogu.edu.tr & mkocak@ogu.edu.tr