TAIWANESE JOURNAL OF MATHEMATICS
Vol. 10, No. 6, pp. 1671-1683, December 2006
This paper is available online at http://www.math.nthu.edu.tw/tjm/

FRACTIONAL CALCULUS AND SOME PROPERTIES OF k-UNIFORM CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

H. Özlem Güney*, S. Sümer Eker and Shigeyoshi Owa

Abstract. In this paper, we introduce a class of functions $(k, A, B, \alpha) - UCV$ which is convex in the unit disk. We give some results for the class $(k, A, B, \alpha) - UCV$, integral operators and radius of k-uniform convexity. Further, the proofs of distortion theorems for fractional calculus for functions $(k, A, B, \alpha) - UCV$ is given.

1. INTRODUCTION

Let H denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and let S denote the class of functions (1.1), *analytic* and *univalent* in $\mathbb{U}$. By CV, we denote the subclass of convex and univalent functions defined by the condition

$$CV = \left\{ f \in S : Re\{1 + \frac{zf''}{f'}\} > 0 \quad , z \in \mathbb{U} \right\}. \tag{1.2}$$

In 1991, Goodman in [3] gave the following definition and theorem for the class UCV.

Definition A. A function $f \in H$ is said to be $uniformly\ convex$ in $\mathbb{U}$, if it is convex in $\mathbb{U}$, and has the property that for every circular $arc\,\gamma$, contained in $\mathbb{U}$, with center ζ, also in $\mathbb{U}$, $arcf(\gamma)$ is convex.

Received May 11, 2005, revised October 7, 2005.
Communicated by H. M. Srivastava.
2000 *Mathematics Subject Classification*: Primary 30C45, Secondary 26A33.
Key words and phrases: Integral operators, Subordination, Starlike, Convex, k-uniform, Fractional calculus.
*E-mail: ozlemg@dicle.edu.tr

For $\gamma = 0$, we obtain the class CV, and also that if γ is a complete circle contained in $\mathbb{U}$, it is well known that $f(\gamma)$ is a convex curve also for $f \in CV$.

Theorem A. *Let $f \in H$. Then $f \in UCV$ if and only if*

$$Re\left\{1 + \frac{(z-\zeta)f''}{f'}\right\} \geq 0$$

for $(z, \zeta) \in \mathbb{U} \times \mathbb{U}$.

Also,in 1999, Kanas et al. in [4] gave the following definition and theorem.

Definition B. Let $0 \leq k < \infty$.A function $f \in S$ said to be k-uniformly convex in $\mathbb{U}$, if the image of every circular $arc\,\gamma$, contained in $\mathbb{U}$, with center ζ,where $|\zeta| \leq k$, is convex.

For fixed k, the class of all k-uniformly convex functions is denoted by $k - UCV$. Note that $0 - UCV = CV$ and $1 - UCV = UCV$ in [3].

Theorem B. *L et $f \in H$ and $0 \leq k < \infty$. Then $f \in k - UCV$ if and only if*

$$Re\left\{1 + \frac{(z-\zeta)f''}{f'}\right\} \geq 0 \tag{1.3}$$

for $z \in \mathbb{U}$ and $|\zeta| \leq k$.

Let T denote the subclass of S whose elements can be expressed in the form ,

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n > 0. \tag{1.4}$$

A function $f \in T$ is said to be in the class $(k, A, B, \alpha) - UCV$ if it satisfies the inequality

$$Re\left\{1 + \frac{(z-\zeta)f''}{f'}\right\} \geq \alpha \tag{1.5}$$

for $|\zeta| \leq k, \quad \alpha(0 \leq \alpha < 1)$ and all $z \in \mathbb{U}$.

In other words, a function f belonging to the class T is said to be in the class $(k, A, B, \alpha) - UCV$ iff it satisfies the condition

$$\left|\frac{(z-\zeta)f''(z)}{(A-B)(1-\alpha)f'(z) + B(z-\zeta)f''(z)}\right| < 1 \tag{1.6}$$

where $-1 \leq B < A \leq 1, \quad -1 \leq B < 0, \quad 0 \leq \alpha < 1, \quad |\zeta| \leq k$ and all $z \in \mathbb{U}$.

The class $k - UCV$ was introduced by Kanas et al.[4], where its geometric definition and connections with the conic domains were considered. Kanas and Srivastava [5] studied further developments involving the class $k - UCV$. Also, Gangadharan et al.[2] use linear operator in order to establish a number of connections between the class $k - UCV$ and various other subclasses of H.

The aim of this paper is to give various basic properties of functions belonging to general class $(k, A, B, \alpha) - UCV$, radius of k-uniform convexity. We also prove several distortion theorems in fractional calculus for functions in the class $(k, A, B, \alpha) - UCV$.

2. Some Results for the Class $(k, A, B, \alpha) - UCV$

Theorem 2.1. *A function $f \in T$ is in the class $(k, A, B, \alpha) - UCV$ iff*

$$\sum_{n=2}^{\infty}[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]na_n \leq (A-B)(1-\alpha). \tag{2.1}$$

The result is sharp.

Proof. Suppose that $f \in (k, A, B, \alpha) - UCV$. Then we have from (1.6) that

$$\left|\frac{(z-\zeta)f''(z)}{(A-B)(1-\alpha)f'(z)+B(z-\zeta)f''(z)}\right|$$

$$= \left|\frac{(z-\zeta)\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}}{(A-B)(1-\alpha)(1-\sum_{n=2}^{\infty}na_n z^{n-1})+B(z-\zeta)\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}}\right| < 1.$$

Since $Re(z) \leq |z|$ for all $z \in \mathbb{U}$.

$$Re\left\{\frac{(z-\zeta)\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}}{(A-B)(1-\alpha)(1-\sum_{n=2}^{\infty}na_n z^{n-1})+B(z-\zeta)\sum_{n=2}^{\infty}n(n-1)a_n z^{n-2}}\right\} < 1.$$

If we choose z and ζ real and letting $z \to 1^-$ and $\zeta \to -k^+$, we have

$$\sum_{n=2}^{\infty}[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]na_n \leq (A-B)(1-\alpha)$$

which is equivalent to (2.1). Conversely, assume that (2.1) is true and $|z| = 1$ and $|\zeta| \le k$. Then we have

$$\begin{aligned} &|(z-\zeta)f''(z)| - |(A-B)(1-\alpha)f'(z) + B(z-\zeta)f''(z)| \\ \le &\sum_{n=2}^{\infty}[(1-B)(1+|\zeta|)(n-1) + (A-B)(1-\alpha)]na_n - (A-B)(1-\alpha) \\ \le &\sum_{n=2}^{\infty}[(1-B)(1+k)(n-1) + (A-B)(1-\alpha)]na_n - (A-B)(1-\alpha) \le 0 \end{aligned}$$

by hypothesis. This implies that $f \in (k, A, B, \alpha) - UCV$.

The result (2.1) is sharp for the function

$$f(z) = z - \frac{(A-B)(1-\alpha)}{[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]n} z^n, \ n \in \mathbb{N}, \ 0 \le k < \infty. \tag{2.2}$$

Remark. We note that $(0, 1, -1, \alpha) - UCV \equiv C(\alpha)$. Therefore, our class $(k, A, B, \alpha) - UCV$ is the generalization of $C(\alpha)$ by Silverman [8].

Theorem 2.2. *Let the function f and g be in the class $(k, A, B, \alpha) - UCV$. Then for $\lambda \in [0, 1]$, the function $h(z) = (1-\lambda)f(z) + \lambda g(z) = z - \sum_{n=2}^{\infty} c_n z^n$ is in the class $(k, A, B, \alpha) - UCV$.*

Proof. Since the function f and g be in the class $(k, A, B, \alpha) - UCV$, they satisfy the inequality (2.1). Therefore, if we define the function $h(z)$ by

$$h(z) = (1-\lambda)f(z) + \lambda g(z) = z - \sum_{n=2}^{\infty} c_n z^n, \qquad c_n = (1-\lambda)a_n + \lambda b_n > 0$$

be in the class T, we can get the result.

Theorem 2.3. *Let $f_1(z) = z$ and $f_n(z) = z - \frac{(A-B)(1-\alpha)}{[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]n} z^n$ for $0 \le \alpha < 1, \quad 0 \le k < \infty$ and $n \in \mathbb{N}$. Then $f \in (k, A, B, \alpha) - UCV$ iff it can be expressed in the form*

$$f(z) = \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z), \tag{2.3}$$

where $\lambda_n \ge 0$ and $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$.

Proof. Suppose that

$$f(z) = \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z)$$
$$= z - \sum_{n=2}^{\infty} \frac{(A-B)(1-\alpha)}{[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]n} \lambda_n z^n.$$

Then from Theorem 2.1, we have

$$\sum_{n=2}^{\infty} [(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]n \frac{(A-B)(1-\alpha)}{[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]n} \lambda_n \leq (A-B)(1-\alpha).$$

Hence $f \in (k, A, B, \alpha) - UCV$. Conversely, let $f \in (k, A, B, \alpha) - UCV$. Then

$$a_n \leq \frac{(A-B)(1-\alpha)}{[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]n}.$$

Setting $\lambda_n = \frac{[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]n}{(A-B)(1-\alpha)} a_n$ and $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$, we see that $f(z)$ can be expressed in the form (2.3).

Corollary 2.1. *The extreme points of the class* $(k, A, B, \alpha) - UCV$ *are*

$$f_1(z) = z \text{ and } f_n(z) = z - \frac{(A-B)(1-\alpha)}{[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]n} z^n, n \in \mathbb{N}.$$

Definition 2.1. For the functions

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n > 0) \quad and \quad g(z) = z - \sum_{n=2}^{\infty} b_n z^n, \quad (b_n > 0),$$

the $modified$ Hadamard product is denoted by

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n.$$

We now prove the following.

Theorem 2.4. *If* $f, g \in (k, A, B, \alpha) - UCV$*, then* $(f*g) \in (k, A, B, \beta) - UCV$ *where*

$$\beta = 1 - \frac{(A-B)(1-\alpha)^2(1-B)(1+k)}{[(1-B)(1+k)+(A-B)(1-\alpha)]^2 + [(1-B)(1+k)+2(A-B)(1-\alpha)](1-B)(1+k)}.$$

The result is sharp for the functions $f(z)$ and $g(z)$ given by

$$f(z) = g(z) = z - \frac{(A-B)(1-\alpha)}{2[(1-B)(1+k)+(A-B)(1-\alpha)]}z^2$$

where $0 \leq \alpha < 1$ *and* $0 \leq k < \infty$.

Proof. From Theorem 2.1, we have

$$\sum_{n=2}^{\infty} \frac{[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]n}{(A-B)(1-\alpha)}a_n \leq 1. \tag{2.4}$$

and

$$\sum_{n=2}^{\infty} \frac{[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]n}{(A-B)(1-\alpha)}b_n \leq 1. \tag{2.5}$$

We have to find the largest β such that

$$\sum_{n=2}^{\infty} \frac{[(1-B)(1+k)(n-1)+(A-B)(1-\beta)]n}{(A-B)(1-\beta)}a_n b_n \leq 1. \tag{2.6}$$

From (2.4) and (2.5), we find,by means of Cauchy-Schwarz inequality, that

$$\sum_{n=2}^{\infty} \frac{[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]n}{(A-B)(1-\alpha)}\sqrt{a_n b_n} \leq 1. \tag{2.7}$$

Therefore (2.6) holds true if

$$\begin{aligned} &\frac{[(1-B)(1+k)(n-1)+(A-B)(1-\beta)]n}{(A-B)(1-\beta)}a_n b_n \\ \leq\ &\frac{[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]n}{(A-B)(1-\alpha)}\sqrt{a_n b_n} \end{aligned}$$

or

$$\sqrt{a_n b_n} \leq \frac{(1-\beta)}{(1-\alpha)}\frac{[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]}{[(1-B)(1+k)(n-1)+(A-B)(1-\beta)]}.$$

Note that from (2.7)

$$\sqrt{a_n b_n} \leq \frac{(A-B)(1-\alpha)}{[(1-B)(1+k)(n-1)+(A-B)(1-\alpha))]n}.$$

Thus if

$$\frac{(A-B)(1-\alpha)}{[(1-B)(1+k)(n-1)+(A-B)(1-\alpha))]n} \leq \frac{(1-\beta)}{(1-\alpha)} \frac{[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]}{[(1-B)(1+k)(n-1)+(A-B)(1-\beta)]}.$$

or, equivalently, if

$$\beta \leq 1-\frac{(A-B)(1-\alpha)^2(1-B)(1+k)}{(1-B)^2(1+k)^2n(n-1)+2n(A-B)(1-\alpha)(1-B)(1+k)+(A-B)^2(1-\alpha)^2}.$$

Defining the function $\Theta(n)$ by

$$\Theta(n)=1-\frac{(A-B)(1-\alpha)^2(1-B)(1+k)}{(1-B)^2(1+k)^2n(n-1)+2n(A-B)(1-\alpha)(1-B)(1+k)+(A-B)^2(1-\alpha)^2},$$

we can see that $\Theta(n)$ is an increasing function of n. Therefore,

$$\beta \leq \Theta(2)=1-\frac{(A-B)(1-\alpha)^2(1-B)(1+k)}{[(1-B)(1+k)+(A-B)(1-\alpha)]^2+[(1-B)(1+k)+2(A-B)(1-\alpha)](1-B)(1+k)}$$

which completes the assertion of theorem.

3. Integral Operators

Theorem 3.1. *Let c be real number such that $c>-1$. If $f \in (k,A,B,\alpha)-UCV$, then the function F defined by*

$$f(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \tag{3.1}$$

also belongs to $(k,A,B,\alpha)-UCV$.

Proof. Let $f(z)=z-\sum_{n=2}^{\infty}a_nz^n$. Then from representation of F, it follows that

$$F(z)=z-\sum_{n=2}^{\infty}b_nz^n \quad where \quad b_n=\left(\frac{c+1}{c+n}\right)a_n.$$

Therefore using Theorem 2.1 for the coefficients of F, we obtain $F \in (k,A,B,\alpha)-UCV$.

Theorem 3.2. *Let c be real number such that $c>-1$. If $F \in (k,A,B,\alpha)-UCV$, then the function f defined by (3.1) is univalent in $|z|<R^*$, where*

$$R^* = inf_n \left\{ \left[\frac{[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]}{(A-B)(1-\alpha)} \left(\frac{c+1}{c+n} \right) \right]^{\frac{1}{n-1}} \right\}.$$

The result is sharp. The sharpness follows if we take

$$f(z) = z - \left(\frac{c+n}{c+1} \right) \frac{(A-B)(1-\alpha)}{n[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)]} z^n.$$

4. Radius of k-Uniform Convexity

The following known result for the class $k - UCV$ will be required in our investigation.

Lemma A. (See [1]) *Let* $f \in H$ *and* $0 \leq k < \infty$. *Then* $f \in k - UCV$ *iff*

$$Re\left\{1 + \frac{zf''}{f'}\right\} > k\left|\frac{zf''}{f'}\right| + \alpha$$

where $0 \leq \alpha < 1$ *and* $z \in \mathbb{U}$.

Theorem 4.1. *Let the function* f *be defined by (1.4)be in the class* $(k, A, B, \alpha)-UCV$ *of order* $\delta(0 \leq \delta < 1), \quad 0 \leq \alpha + \delta < 1$. *Then* f *is* k*-uniform convex in* $|z| < R(k, A, B, \alpha, \delta)$, *where*

$$\begin{aligned} & |z| < R(k, A, B, \alpha, \delta) \\ (4.1) \qquad & = \inf_n \left\{ \frac{[(1-B)(1+k)(n-1)+(A-B)(1-\alpha)](1-\delta-\alpha)}{[k(n-1)+(1-\delta-\alpha)](A-B)(1-\alpha)} \right\}^{\frac{1}{n-1}}. \end{aligned}$$

The result is sharp.

Proof. In order to establish the required result in Theorem 4.1, it is sufficient to show that

$$k\left|\frac{zf''}{f'}\right| + \alpha \leq 1 - \delta \quad for \quad |z| < R(k, A, B, \alpha, \delta).$$

5. Distortion Theorems Involving Fractional Calculus

In this section, we shall prove several distortion theorems for functions to general class $(k, A, B, \alpha) - UCV$. Each of these theorems would involve certain operators of fractional calculus which are defined as follows [6,7,9,10].

Definition 5.1. The fractional integral of order λ is defined, for a function f , by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi \quad ; (\lambda > 0) \tag{5.1}$$

where f is an analytic function in a simply - connected region of the z-plane containing the origin, and the multiplicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 5.2. The fractional derivative of order λ is defined, for a function f , by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi; (0 \leq \lambda < 1) \tag{5.2}$$

where f is constrained, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed, as in Definition 5.1.

Definition 5.3. Under the hypotheses of Definition 5.2, the fractional derivative of order $(n+\lambda)$ is defined by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) \tag{5.3}$$

where $0 \leq \lambda < 1$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. From Definition 5.2, we have

$$D_z^0 f(z) = f(z) \tag{5.4}$$

which, in view of Definition 5.3 yields,

$$D_z^{n+0} f(z) = \frac{d^n}{dz^n} D_z^0 f(z) = f^n(z). \tag{5.5}$$

Thus, it follows from (5.4) and (5.5) that

$$\lim_{\lambda \to 0} D_z^{-\lambda} f(z) = f(z) \quad and \quad \lim_{\lambda \to 0} D_z^{1-\lambda} f(z) = f'(z).$$

Theorem 5.1. *Let $f \in (k, A, B, \alpha) - UCV$. Then we have*

$$\begin{aligned} &\left| D_z^{-\lambda} f(z) \right| \\ &\leq |z|^{1+\lambda} \left\{ \frac{1}{\Gamma(\lambda+2)} + \frac{(A-B)(1-\alpha)}{\Gamma(\lambda+3)[(1-B)(1+k)+(A-B)(1-\alpha)]} |z| \right\} \end{aligned} \tag{5.6}$$

and

$$(5.7)\quad \left|D_z^{-\lambda} f(z)\right| \geq |z|^{1+\lambda} \left\{ \frac{1}{\Gamma(\lambda+2)} - \frac{(A-B)(1-\alpha)}{\Gamma(\lambda+3)[(1-B)(1+k)+(A-B)(1-\alpha)]}|z| \right\}$$

for $z \in \mathbb{U}$ *and* $\lambda > 0$. *The inequalities in* (5.6) *and* (5.7) *are attained for the function*

$$(5.8)\qquad f(z) = z - \frac{(A-B)(1-\alpha)}{2[(1-B)(1+k)+(A-B)(1-\alpha)]} z^2.$$

Proof. Using Theorem 2.1, we have

$$(5.9)\qquad \sum_{n=2}^{\infty} a_n \leq \frac{(A-B)(1-\alpha)}{2[(1-B)(1+k)+(A-B)(1-\alpha)]}.$$

From Definition 5.1, we obtain

$$(5.10)\quad D_z^{-\lambda} f(z) z^{-\lambda} \Gamma(\lambda+2) = z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(\lambda+2)}{\Gamma(n+\lambda+1)} a_n z^n = z - \sum_{n=2}^{\infty} \psi_n a_n z^n$$

where

$$\psi_n = \frac{\Gamma(n+1)\Gamma(\lambda+2)}{\Gamma(n+\lambda+1)}, \qquad (n \geq 2).$$

Since

$$0 < \psi_n \leq \psi(2) = \frac{2}{2+\lambda},$$

using (5.9) and (5.10), we find that

$$\begin{aligned} |D_z^{-\lambda} f(z) z^{-\lambda} \Gamma(\lambda+2)| &\leq |z| + \psi(2)|z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| + \frac{\Gamma(\lambda+2)(A-B)(1-\alpha)}{\Gamma(\lambda+3)[(1-B)(1+k)+(A-B)(1-\alpha)]}|z|^2 \end{aligned}$$

and

$$\begin{aligned} |D_z^{-\lambda} f(z) z^{-\lambda} \Gamma(\lambda+2)| &\geq |z| - \psi(2)|z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| - \frac{\Gamma(\lambda+2)(A-B)(1-\alpha)}{\Gamma(\lambda+3)[(1-B)(1+k)+(A-B)(1-\alpha)]}|z|^2 \end{aligned}$$

which are equivalent to (5.6) and (5.7), respectively.

Theorem 5.2. *Let $f \in (k, A, B, \alpha) - UCV$. Then we find that*

$$\left|D_z^{\lambda} f(z)\right| \leq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 + \frac{(A-B)(1-\alpha)}{[(1-B)(1+k)+(A-B)(1-\alpha)]}|z|\right\} \tag{5.11}$$

and

$$\left|D_z^{\lambda} f(z)\right| \geq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)} \left\{1 - \frac{(A-B)(1-\alpha)}{[(1-B)(1+k)+(A-B)(1-\alpha)]}|z|\right\} \tag{5.12}$$

for $z \in \mathbb{U}$ and $0 \leq \lambda < 1$. The inequalities in (5.11) and (5.12) are attained for the function f given by (5.8).

Proof. Using similar argument as given by Theorem 5.1, we can get result.

Corollary 5.1. *If $f \in (k, A, B, \alpha) - UCV$, then we find that for $|z| = r < 1$*

$$\begin{aligned} r - \frac{(A-B)(1-\alpha)}{2[(1-B)(1+k)+(A-B)(1-\alpha)]}r^2 &\leq |f(z)| \\ \leq r + \frac{(A-B)(1-\alpha)}{2[(1-B)(1+k)+(A-B)(1-\alpha)]}r^2 \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} 1 - \frac{(A-B)(1-\alpha)}{(1-B)(1+k)+(A-B)(1-\alpha)}r &\leq |f'(z)| \\ \leq 1 + \frac{(A-B)(1-\alpha)}{(1-B)(1+k)+(A-B)(1-\alpha)}r. \end{aligned} \tag{5.14}$$

Proof. From (5.4), letting $\lambda \to 0$ in (5.6)-(5.7) and $\lambda \to 1$ in (5.11)-(5.12), we have (5.13) and (5.14), respectively.

Theorem 5.3. *Let $f \in (k, A, B, \alpha) - UCV$. Then*

$$\begin{aligned} &\left|D_z^{1-\lambda} f(z)\right| \\ &\geq max\left\{0, \frac{1}{\Gamma(\lambda+2)}|z|^{\lambda}\left((1-\lambda) - \frac{[\Gamma(\lambda+3)+\lambda\Gamma(\lambda+2)](A-B)(1-\alpha)}{\Gamma(\lambda+3)[(1-B)(1+k)+(A-B)(1-\alpha)]}|z|\right)\right\} \end{aligned}$$

and

$$\left|D_z^{1-\lambda}f(z)\right|\leq\frac{1}{\Gamma(\lambda+2)}|z|^{\lambda}\left\{(1+\lambda)-\frac{[\Gamma(\lambda+3)+\lambda\Gamma(\lambda+2)](A-B)(1-\alpha)}{\Gamma(\lambda+3)[(1-B)(1+k)+(A-B)(1-\alpha)]}|z|\right\}$$

for $z \in \mathbb{U}$ *and* $\lambda > 0$.

Proof. From (5.14), we find required results.

Corollary 5.3. *Under the hypothesis of Theorem 5.1,* $|D_z^{-\lambda}f(z)|$ *is included in a disk with center at the origin and radius* $R_1^{-\lambda}$ *given by*

$$R_1^{-\lambda}=\frac{\Gamma(\lambda+3)(1-B)(1+k)+[\Gamma(\lambda+2)+\Gamma(\lambda+3)](A-B)(1-\alpha)}{\Gamma(\lambda+2)\Gamma(\lambda+3)[(1-B)(1+k)+(A-B)(1-\alpha)]}.$$

Furthermore $|D_z^{1-\lambda}f(z)|$ *is included in a disk with center at the origin and radius* $R_2^{1-\lambda}$ *given by*

$$R_2^{1-\lambda}=\frac{[\lambda(2+\lambda)\Gamma(\lambda+3)\Gamma(\lambda+2)](A-B)(1-\alpha)+(1+\lambda)\Gamma(\lambda+3)(1-B)(1+k)}{\Gamma(\lambda+2)\Gamma(\lambda+3)[(1-B)(1+k)+(A-B)(1-\alpha)]}.$$

Acknowledgment

The authors are thankful to the referee for his generous help and useful suggestions.

References

1. R. Bharati, P. Parvatham and A. Swaminathan, On subclasses of uniformly convex functions and corresponding class of starlike functions, *Tamkang J. Math.* **28** (1997), 17-32.
2. A. Gangadharan, T. H. Shanmugam and H. M.Srivastava, Generalized hypergeometric functions associated with k-uniformly convex functions, *Comp. Math. Appl.*, **44** (2002), 1515-1526.
3. A. W. Goodman, On uniformly convex functions, *Ann. Polon. Math.*, **56** (1991), 87-92.
4. S. Kanas and A. Wisniowska, Conic regions and k-uniform convexity, *J. Comp.and Math.*, **105** (1999)327-336.
5. S. Kanas and H. M. Srivastava, Linear operators associated with k-uniformly convex functions, *Integral Transform. Spec. Funct.*, **9(2)** (2000), 121-132.

6. S. Owa, On the distortion theorems, I. *Kyunpook Math. J.*, **18** (1978), 53-59.

7. S. Owa and H. M. Srivastava, Univalent and Starlike generalized hypergeometric functions, *Canad. J. Math.*, **39** (1987), 1057-1077.

8. H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.*, **51** (1975), 109-116.

9. H. M. Srivastava and S. Owa, Some Characterization and distortion theorems involving fractional calculus, linear operators and certain subclasses of analytic functions, *Nagoya Math. J.*, **106** (1987), 1-28.

10. H. M. Srivastava, M. Saigo and S. Owa, A class of distortion theorems involving certain operators of fractional calculus, *J. Math. Anal. Appl.*, **131** (1988), 412-420.

H. Özlem Güney, and S. Sümer Eker
University of Dicle,
Faculty of Science and Art,
Department of Mathematics, 21280,
Diyarbakr, Turkey
E-mail: ozlemg@dicle.edu.tr

Shigeyoshi Owa
Department of Mathematics,
Kinki University,
Higashi-Osaka,
Osaka 577-8520, Japan