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CONVERGENCE OF A PERTURBED THREE-STEP ITERATIVE ALGORITHM WITH ERRORS FOR COMPLETELY GENERALIZED NONLINEAR MIXED QUASI-VARIATIONAL INEQUALITIES

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Abstract. In this paper, we introduce and study a new class of completely generalized nonlinear mixed quasi-variational inequalities. Using the resolvent operator technique for maximal monotone operators, we construct a perturbed three-step iterative algorithm with errors for solving this kind of completely generalized nonlinear mixed quasi-variational inequalities. Furthermore, we establish a few existence and uniqueness results of solutions for the completely generalized nonlinear mixed quasi-variational inequality involving relaxed Lipschitz, generalized pseudo-contractive and strongly monotone mappings and prove some convergence results of the iterative sequence generated by the perturbed three-step iterative algorithm with errors.

1. Introduction

Since its birth in the mid-1960's, the area of variational inequalities has experienced a phenomenal growth. It is now considered a field in its own right. It is well known that variational inequality theory provides a convenient mathematical framework for discussing a number of interesting problems in the field of optimization, equilibrium, elasticity and structural analysis, etc. For details we refer to [1-4] and [6-23]. An important generalization of variational inequalities is a variational inequality containing a nonlinear term. Due to the presence of the nonlinear term, the projection method cannot be used to study the existence of solutions for the variational inequalities. In 1994, Hassouni and Moudafi [2] used the resolvent

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operator technique for maximal monotone operator to study a new class of mixed variational inequalities. In 2000, Huang, Bai, Cho and Kang [4] extended this technique for a new class of general mixed variational inequalities. In 2003, Liu, Debnath, Kang and Ume [7] modified this technique for another new class of generalized nonlinear quasi-variational inclusions. In 2004, Liu and Kang [16] used the resolvent technique to establish the equivalence between the completely generalized nonlinear quasi-variational inequality and the fixed point problem. What's more, they also suggested a perturbed three-step iterative algorithm for solving the class of completely generalized nonlinear quasi-variational inequalities.

Inspired and motivated by recent research works [1-4] and [6-23], in this paper, we introduce and study a new class of completely generalized nonlinear mixed quasi-variational inequalities, and construct a perturbed three-step iterative algorithm with errors for finding the approximate solutions of the completely generalized nonlinear mixed quasi-variational inequality involving strongly monotone, relaxed Lipschitz and generalized pseudo-contractive mappings. Under certain conditions, we obtain a few existence and uniqueness of solutions for the completely generalized nonlinear mixed quasi-variational inequality. Furthermore, some convergence results of the iterative sequence generated by the perturbed three-step iterative algorithm with errors are presented in this paper. Our results improve, extend and unify the corresponding results in [2, 4] and [21].

2. Preliminaries

Let H be a Hilbert space endowed with a norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, respectively, and I be the identity mapping on H. 2^H and $CC(H)$ denote the families of all the nonempty subsets of H, and all the nonempty closed convex subsets of H, respectively. Suppose that $W : H \times H \to 2^H$ is a multi-valued mapping such that for each fixed $t \in H$, $W(\cdot, t) : H \to 2^H$ is maximal monotone.

Given mappings $g, a, b, c, d : H \to H$, $N : H \times H \times H \to H$ and $f \in H$, we consider the following problem:

Find $u \in H$ such that

$$f \in g(u) - N(a(u), b(u), c(u)) + W(g(u), d(u)), \tag{2.1}$$

which is called a *completely generalized nonlinear mixed quasi-variational inequality*.

Special cases of problem (2.1) are as follows:

(a) If $f = 0$, $d = I$ and $N(a(u), b(u), c(u)) = -N(a(u), b(u)) + g(u)$ for all $u \in H$, problem (2.1) is equivalent to finding $u \in H$ such that

$$0 \in N(a(u), b(u)) + W(g(u), u), \tag{2.2}$$

which is called the *generalized nonlinear mixed quasi-variational inequality* introduced and studied by Huang et al. [4].

(b) If $f = 0$, $N(a(u), b(u), c(u)) = b(u) - a(u) + g(u)$ and $W(u, v) = \partial\varphi(u)$ for all $u, v \in H$, where $\partial\varphi$ stands for the sub-differential of a proper, convex and lower semi-continuous functional $\varphi : H \to R \cup \{+\infty\}$, then problem (2.1) is equivalent to the following problem studied in [2]:

Find $u \in H$ such that

$$\langle a(u) - b(u), v - g(u)\rangle \geq \varphi(g(u)) - \varphi(v), \quad \forall v \in H. \tag{2.3}$$

(c) If $f = 0$, $g = I$, $N(u, v, w) = w$, $W(u, v) = \partial\varphi(u)$ and $\varphi(u) = I_K(u)$ for all $u, v, w \in H$, where $K \in CC(H)$ and I_K is the indicator function on K defined by

$$I_K(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{if } u \notin K, \end{cases}$$

then problem (2.1) collapses to finding $u \in K$ such that

$$\langle u - c(u), v - u\rangle \geq 0, \quad v \in K, \tag{2.4}$$

which is called *nonlinear variational inequality* and introduced in [21].

For a suitable choice of the mappings g, a, b, c, d, N, the element f, and the space H, one can obtain a number of known and new classes of variational inequalities from the generalized nonlinear mixed quasi-variational inequality (2.1). Furthermore, these types of variational inequalities enable us to study many problems arising in mathematical, regional, physical and engineering sciences in a general and unified framework.

Definition 2.1. Let $N : H \times H \times H \to H$ and $g : H \to H$ be mappings.

(1) g is said to be *r-strongly monotone* if there exists a constant $r > 0$ such that

$$\langle g(u) - g(v), u - v\rangle \geq r \parallel u - v \parallel^2, \quad \forall u, v \in H;$$

(2) g is said to be *s-Lipschitz continuous* if there exists a constant $s > 0$ such that

$$\|g(x) - g(y)\| \leq s \parallel x - y \parallel, \quad \forall x, y \in H;$$

(3) g is said to be *r-generalized pseudo-contractive* with respect to the first argument of N if there exists a constant $r > 0$ such that

$$\langle N(g(u), x, y) - N(g(v), x, y), u - v\rangle \leq r\|u - v\|^2, \quad \forall u, v, x, y \in H;$$

(4) g is said to be r-*relaxed Lipschitz* with respect to the second argument of N, if there exists a constant $r > 0$ such that

$$\langle N(x, g(u), y) - N(x, g(v), y), u - v\rangle \leq -r \parallel u - v \parallel^2, \quad \forall u, v, x, y \in H.$$

(5) g is said to be r-*strongly monotone* with respect to the third argument of N if there exists a constant $r > 0$ such that

$$\langle N(x, y, g(u)) - N(x, y, g(v)), u - v\rangle \geq r \parallel u - v \parallel^2, \quad \forall u, v, x, y \in H.$$

Definition 2.2. A multi-valued operator $W : H \to 2^H$ is said to be

(1) *monotone* if

$$\langle x - y, u - v\rangle \geq 0, \quad \forall u, v \in H,\ x \in Wu,\ y \in Wv;$$

(2) *maximal monotone* if W is monotone and $(I + \lambda W)(H) = H$ for all $\lambda > 0$.

Remark 2.1. It is well known that an operator W is maximal monotone if and only if it is monotone and there is no other monotone operator whose graph contains strictly the graph $Graph(W)$ of W, where $Graph(W) = \{(u, x) \in H \times H : x \in Wu\}$.

Definition 2.3. Let H be a Hilbert space and $W : H \to 2^H$ be a maximal monotone operator. For any fixed $\rho > 0$, the mapping $J_\rho^W(x) : H \to H$ defined by

$$J_\rho^W(x) = (I + \rho W)^{-1}(x), \quad \forall x \in H,$$

is said to be *resolvent operator* of W.

It is well-known that the resolvent operator J_ρ^W is single valued and non-expansive.

Definition 2.4. Let $W, W_n : H \to 2^H$ be maximal monotone operators for $n \geq 0$. The sequence $\{W_n\}_{n\geq 0}$ is said to be *graph-convergence* to W (write $W_n \xrightarrow{G} W$ as $n \to \infty$) if for each $(x, y) \in Graph(W)$, there exists a sequence $\{(x_n, y_n)\}_{n\geq 0}$ such that $(x_n, y_n) \in Graph(W_n)$, $\forall n \geq 0$, $x_n \to x$ and $y_n \to y$ in H as $n \to \infty$.

Lemma 2.1. ([5]) *Let* $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ *and* $\{\gamma_n\}_{n\geq 0}$ *be nonnegative sequences satisfying*

$$\alpha_{n+1} \leq (1 - \delta_n)\alpha_n + \delta_n\beta_n + \gamma_n, \quad \forall n \geq 0,$$

where $\{\delta_n\}_{n\geq 0} \subset [0,1]$, $\sum\limits_{n=0}^{\infty} \delta_n = \infty$, $\lim\limits_{n\to\infty} \beta_n = 0$ *and* $\sum\limits_{n=0}^{\infty} \gamma_n < \infty$. *Then* $\lim\limits_{n\to\infty} \alpha_n = 0$.

Lemma 2.2. ([1]) *Let* W_n *and* W *be maximal monotone operators for* $n \geq 0$. *Then* $W_n \stackrel{G}{\to} W$ *as* $n \to \infty$ *if and only if* $J_\rho^{W_n}(x) \to J_\rho^{W}(x)$ *as* $n \to \infty$ *for every* $x \in H$ *and* $\rho > 0$.

3. A Perturbed Three-step Iterative Algorithm with Errors

We first transfer the completely generalized nonlinear mixed quasi-variational inequality (2.1) into a fixed point problem.

Lemma 3.1. *Let* ρ *and* t *be positive parameters. Then the following conditions are equivalent.*

(a) *the completely generalized nonlinear mixed quasi-variational inequality* (2.1) *has a solution* $u \in H$;

(b) *there exists* $u \in H$ *satisfying*

$$g(u) = J_\rho^{W(\cdot,d(u))}((1-\rho)g(u) + \rho N(a(u), b(u), c(u)) + \rho f), \tag{3.1}$$

where $J_\rho^{W(\cdot,d(u))}$ *is the resolvent operator;*

(c) *the mapping* $F : H \longrightarrow H$ *defined by*

$$\begin{aligned} Fx = (1-t)x + t(x - g(x) + J_\rho^{W(\cdot,d(x))}&((1-\rho)g(x) \\ &+ \rho N(a(x), b(x), c(x)) + \rho f)), \quad \forall x \in H \end{aligned} \tag{3.2}$$

has a fixed point $u \in H$.

Proof. Note that (3.1) holds if and only if

$$(1-\rho)g(u) + \rho N(a(u), b(u), c(u)) + \rho f \in g(u) + \rho W(g(u), d(u)),$$

which is equivalent to

$$f \in g(u) - N(a(u), b(u), c(u)) + W(g(u), d(u)).$$

On the other hand, F has a fixed point $u \in H$ if and only if

$$\begin{aligned} u = F(u) + (1-t)u + t(u - g(u) + J_\rho^{W(\cdot,d(u))}&((1-\rho)g(u) \\ &+ \rho N(a(u), b(u), c(u)) + \rho f)), \end{aligned}$$

which is equivalent to (3.1). This completes the proof.

Based on Lemma 3.1, we suggest the following perturbed three-step iterative algorithm with errors for the completely generalized nonlinear mixed quasi-variational inequality (2.1).

Algorithm 3.1. Let $g, a, b, c, d\colon H \to H$, $N : H \times H \times H \to H$ be mappings and $f \in H$. Let

$$Ex = (1-\rho)g(x) + \rho N(a(x), b(x), c(x)) + \rho f, \quad \forall x \in H. \tag{3.3}$$

Given $u_0 \in H$, the iterative sequence $\{u_n\}_{n\geq 0}$ is defined by

$$\begin{aligned} u_{n+1} &= (1-a_n-b_n)u_n + a_n(v_n - g(v_n) + J_\rho^{W_n(\cdot,d(v_n))}(E(v_n))) + b_n p_n, \\ v_n &= (1-a'_n-b'_n)u_n + a'_n(w_n - g(w_n) + J_\rho^{W_n(\cdot,d(w_n))}(E(w_n))) + b'_n q_n, \\ w_n &= (1-a''_n-b''_n)u_n + a''_n(u_n - g(u_n) + J_\rho^{Wn(\cdot,d(u_n))}(E(u_n))) + b''_n r_n \end{aligned}$$

for all $n \geq 0$, where each $W_n : H \times H \to 2^H$ is a multi-valued mapping such that for each $y \in H$, $W_n(\cdot, y) : H \to 2^H$ is maximal monotone, ρ is a positive constant, $\{p_n\}_{n\geq 0}$, $\{q_n\}_{n\geq 0}$ and $\{r_n\}_{n\geq 0}$ are bounded sequences in H introduced to take into account possible in inexact computation and the sequences $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, $\{a'_n\}_{n\geq 0}$, $\{b'_n\}_{n\geq 0}$, $\{a''_n\}_{n\geq 0}$ and $\{b''_n\}_{n\geq 0}$ are in $[0, 1]$ and satisfying

$$\max\{a_n + b_n, a'_n + b'_n, a''_n + b''_n\} \leq 1, \quad \forall\, n \geq 0; \tag{3.4}$$

$$\sum_{n=0}^{\infty} a_n = \infty, \quad \lim_{n\to\infty} a'_n b''_n = \lim_{n\to\infty} b'_n = 0 \tag{3.5}$$

and one of the following conditions:

$$\sum_{n=0}^{\infty} b_n < \infty; \tag{3.6}$$

(3.7) there exists a nonnegative sequence $\{h_n\}_{n\geq 0}$ with $\lim_{n\to\infty} h_n = 0$ and $b_n = a_n h_n, \quad \forall n \geq 0.$

Remark 3.1. If $a''_n = b''_n = 0$ for all $n \geq 0$, then the perturbed three-step iterative algorithm with errors reduces to the Ishikawa type perturbed iterative algorithm with errors. Furthermore, if $a'_n = b'_n = 0$ for all $n \geq 0$, then the Ishikawa type perturbed iterative algorithm with errors reduces to the Mann type perturbed iterative algorithm with errors.

4. Existence and Convergence

In this section, we discuss those conditions under which the approximate solutions u_n obtained from the perturbed three-step iterative algorithm with errors converge strongly to the exact solution $u \in H$ of the completely generalized nonlinear mixed quasi-variational inequality (2.1).

Theorem 4.1. *$a, b, c, d : H \to H$ be p-Lipschitz continuous, q-Lipschitz continuous, r-Lipschitz continuous, s-Lipschitz continuous, respectively, $g : H \to H$ satisfy that $I - g$ is l-Lipschitz continuous. Let $N : H \times H \times H \to H$ be α-Lipschitz continuous in the first argument, β-Lipschitz continuous in the second argument, h-Lipschitz continuous in the third argument, a be ξ-generalized pseudo-contractive with respect to the first argument of N, b be ζ-relaxed Lipschitz with respect to the second argument of N, c be η-strongly monotone with respect to the third argument of N. Suppose that $W_n, W : H \times H \to 2^H$ are such that, for each $y \in H$ and $n \geq 0$, $W_n(\cdot, y), W(\cdot, y) : H \to 2^H$ are maximal monotone, $W_n(\cdot, y) \xrightarrow{G} W(\cdot, y)$ as $n \to \infty$, and*

$$\begin{aligned} &\sup\{\|J_\rho^{W(\cdot,x)}(z) - J_\rho^{W(\cdot,y)}(z)\|, \|J_\rho^{W_n(\cdot,x)}(z) - J_\rho^{W_n(\cdot,y)}(z)\| : n \geq 0\} \\ &\leq \mu\|x - y\|, \quad \forall x, y, z \in H, \end{aligned} \tag{4.1}$$

where μ is a constant. Let

$$\begin{aligned} k &= l + \mu s, \quad j = \sqrt{1 - 2\zeta + \beta^2 q^2} + \sqrt{1 - 2\eta + h^2 r^2}, \\ A &= 1 + \alpha^2 p^2 - 2\xi - (j - l)^2, \quad B = 1 - \xi - (1 - k - l)(j - l), \\ C &= 1 - (1 - k - l)^2, \quad A' = 1 + \alpha^2 p^2 - 2\xi - (j + l)^2, \\ B' &= 1 - \xi - (1 - k + l)(j + l), \quad C' = 1 - (1 - k + l)^2. \end{aligned}$$

If there exists a constant $\rho > 0$ satisfying

$$k + \rho j + |1 - \rho| l < 1 \tag{4.2}$$

and one of the following conditions:

$$\begin{aligned} &A > 0, \quad B > \sqrt{AC}, \\ &\rho \in \left(\frac{C}{B + \sqrt{B^2 - AC}}, \frac{C}{B - \sqrt{B^2 - AC}}\right) \cap (0, 1]; \end{aligned} \tag{4.3}$$

$$A < 0, \quad \rho \in \left(\frac{C}{B + \sqrt{B^2 - AC}}, +\infty\right) \cap (0, 1]; \tag{4.4}$$

$$A' > 0, \quad B' > \sqrt{A'C'},$$
$$\rho \in \left(\frac{C'}{B' + \sqrt{B'^2 - A'C'}}, \frac{C'}{B' - \sqrt{B'^2 - A'C'}} \right) \cap (1, +\infty); \tag{4.5}$$

$$A' < 0, \quad \rho \in \left(\frac{C'}{B' + \sqrt{B'^2 - A'C'}}, +\infty \right) \cap (1, +\infty), \tag{4.6}$$

then the completely generalized nonlinear mixed quasi-variational inequality (2.1) *has a unique solution* $u \in H$ *and the sequence* $\{u_n\}_{n \geq 0}$ *defined by Algorithm 3.1 converges strongly to u.*

Proof. First we show that the completely generalized nonlinear mixed quasi-variational inequality (2.1) has a unique solution $u \in H$. According to Lemma 3.1, it is enough to prove that the mapping $F : H \to H$ defined by (3.2) has a unique fixed point $u \in H$, where $t \in (0, 1]$ is a parameter. Let x, y be arbitrary elements in H. Using the conditions of Theorem 4.1 and the non-expansivity of $J_\rho^{W(\cdot,x)}$, we infer that

$$\begin{aligned}
&\|F(x) - F(y)\| \\
&\leq (1-t)\|x-y\| + t\|x - y - (g(x) - g(y))\| \\
&\quad + t\|J_\rho^{W(\cdot,d(x))}(E(x)) - J_\rho^{W(\cdot,d(x))}(E(y))\| \\
&\quad + t\|J_\rho^{W(\cdot,d(x))}(E(y)) - J_\rho^{W(\cdot,d(y))}(E(y))\| \\
&\leq (1 - t + tk)\|x-y\| + t\|E(x) - E(y)\| \\
&\leq (1 - t + tk)\|x-y\| + t|1-\rho|\|g(x) - g(y) - (x-y)\| \\
&\quad + t\|(1-\rho)(x-y) + \rho(N(a(x), b(x), c(x)) - N(a(y), b(x), c(x)))\| \\
&\quad + t\rho\|x - y + N(a(y), b(x), b(x)) - N(a(y), b(y), c(x))\| \\
&\quad + t\rho\|N(a(y), b(y), c(x)) - N(a(y), b(y), c(y)) - (x-y)\| \\
&\leq [1 - t + t(k + |1-\rho|l)]\|x-y\| \\
&\quad + t[\sqrt{(1-\rho)^2 + 2\xi\rho(1-\rho) + \rho^2\alpha^2p^2} \\
&\quad + \rho(\sqrt{1 - 2\zeta + \beta^2q^2} + \sqrt{1 - 2\eta + h^2r^2})]\|x-y\| \\
&= [1 - (1-\theta)t]\|x-y\|,
\end{aligned} \tag{4.7}$$

where

$$\theta = k + \rho j + |1-\rho|l + \sqrt{(1-\rho)^2 + 2\xi\rho(1-\rho) + \rho^2\alpha^2p^2}. \tag{4.8}$$

In view of (4.2) and (4.8), we have

$$\theta < 1 \Longleftrightarrow \begin{cases} A\rho^2 - 2B\rho < -C, & \text{if } 0 < \rho \le 1, \\ A'\rho^2 - 2B'\rho < -C', & \text{if } \rho > 1. \end{cases} \tag{4.9}$$

It follows from (4.8) and one of (4.3)-(4.6) that $\theta < 1$. Since $t \in (0, 1]$, F is a contraction mapping. Hence it has a unique fixed point $u \in H$, which is a unique solution of the completely generalized nonlinear mixed quasi-variational inequality (2.1).

Now we show that $\lim_{n\to\infty} u_n = u$. Notice that

$$\begin{aligned} u &= (1a_n - b_n)u + a_n(u - g(u) + J_\rho^{W(\cdot,d(u))}(E(u))) + b_n u \\ &= (1 - a'_n - b'_n)u + a'_n(u - g(u) + J_\rho^{W(\cdot,d(u))}(E(u))) + b'_n u \\ &= (1 - a''_n - b''_n)u + a''_n(u - g(u) + J_\rho^{W(\cdot,d(u))}(E(u))) + b''_n u, \ \forall\ n \ge 0. \end{aligned} \tag{4.10}$$

Put $d_n = \|J_\rho^{W_n(\cdot,d(u))}(E(u)) - J_\rho^{W(\cdot,d(u))}(E(u))\|$ for all $n \ge 0$ and $L = \sup\{\|p_n - u\|, \|q_n - u\|, \|r_n - u\| : n \ge 0\}$. Lemma 2.2 ensures that

$$\lim_{n\to\infty} d_n = 0. \tag{4.11}$$

Using (3.3), (3.4) and (4.10), we know that

$$\begin{aligned} &\|u_{n+1} - u\| \\ &\le (1 - a_n - b_n)\|u_n - u\| + a_n\|v_n - u - (g(v_n) - g(u))\| \\ &\quad + a_n\|J_\rho^{W_n(\cdot,d(v_n))}(E(v_n)) - J_\rho^{W_n(\cdot,d(v_n))}(E(u))\| \\ &\quad + a_n\|J_\rho^{W_n(\cdot,d(v_n))}(E(u)) - J_\rho^{W_n(\cdot,d(u))}(E(u))\| \\ &\quad + a_n\|J_\rho^{W_n(\cdot,d(u))}(E(u)) - J_\rho^{W(\cdot,d(u))}(E(u))\| + b_n\|p_n - u\| \\ &\le (1 - a_n - b_n)\|u_n - u\| + a_n l\|v_n - u\| + a_n\|E(v_n) - E(u)\| \\ &\quad + a_n\mu s\|v_n - u\| + a_n d_n + b_n L \\ &\le (1 - a_n - b_n)\|u_n - u\| + a_n k\|v_n - u\| \\ &\quad + a_n(|1 - \rho| l + \rho j)\|v_n - u\| \\ &\quad + a_n\sqrt{(1-\rho)^2 + 2\xi\rho(1-\rho) + \rho^2\alpha^2 p^2}\|v_n - u\| + a_n d_n + b_n L \\ &= (1 - a_n - b_n)\|u_n - u\| + a_n\theta\|v_n - u\| + a_n d_n + b_n L \end{aligned} \tag{4.12}$$

for all $n \ge 0$. Similarly, we have

$$\begin{aligned} \|v_n - u\| &\le (1 - a'_n - b'_n)\|u_n - u\| + a'_n\theta\|w_n - u\| + a'_n d_n + b'_n L, \\ \|w_n - u\| &\le (1 - a''_n - b''_n)\|u_n - u\| + a''_n\theta\|u_n - u\| + a''_n d_n + b''_n L \end{aligned} \tag{4.13}$$

for all $n \geq 0$. Substituting (4.13) into (4.12), we get that

$$
\begin{aligned}
&\|u_{n+1} - u\| \\
&\leq [1 - a_n - b_n + a_n\theta(1 - a'_n - b'_n + a'_n\theta(1 - a''_n - b''_n + a''_n\theta))]\|u_n - u\| \\
&\quad + a_n[\theta a'_n(\theta a''_n d_n + \theta L b''_n + d_n) + \theta L b'_n + d_n] + b_n L \\
&\leq (1 - (1-\theta)a_n)\|u_n - u\| + a_n(3d_n + a'_n b''_n L + L b'_n) + b_n L
\end{aligned} \tag{4.14}
$$

for all $n \geq 0$. It follows from Lemma 2.1, (3.5), (4.11), (4.14) and one of (3.6) and (3.7) that $\lim_{n\to\infty} u_n = u$. This completes the proof.

Theorem 4.2. *Let k, a, b, d, g, N, W, $\{W_n\}_{n\geq 0}$, C and C' be as in Theorem 4.1. Suppose that $c : H \to H$ is r-Lipschitz continuous. Let*

$$
\begin{aligned}
&\sigma = \xi - \zeta,\ \tau = (\alpha p + \beta q)^2, \\
&A = 1 - 2\sigma + \tau - (hr - l)^2, \quad B = 1 - \sigma - (hr - l)(1 - k - l), \\
&A' = 1 - 2\sigma + \tau - (hr + l)^2, \quad B' = 1 - \sigma - (hr + l)(1 - k + l).
\end{aligned}
$$

If there exists a constant $\rho > 0$ satisfying

$$
k + \rho hr + |1 - \rho| l < 1 \tag{4.15}
$$

and one of (4.3)-(4.6)*, then the completely generalized nonlinear mixed quasi-variational inequality* (2.1) *has a unique solution $u \in H$ and the sequence $\{u_n\}_{\geq 0}$ defined by Algorithm 3.1 converges strongly to u.*

Proof. Using the conditions in Theorem 4.3, we infer that

$$
\begin{aligned}
&\|(1-\rho)(x-y) + \rho[N(a(x), b(x), c(x)) - N(a(y), b(x), c(x)) \\
&\quad + N(a(y), b(x), c(x)) - N(a(y), b(y), c(x))]\|^2 \\
&= (1-\rho)^2\|x-y\|^2 + 2\rho(1-\rho)\langle x - y, N(a(x), b(x), c(x)) \\
&\quad - N(a(y), b(x), c(x))\rangle + 2\rho(1-\rho)\langle x - y, N(a(y), b(x), c(x)) \\
&\quad - N(a(y), b(y), c(x))\rangle + \rho^2\|N(a(x), b(x), c(x)) - N(a(y), b(x), c(x)) \\
&\quad + N(a(y), b(x), c(x)) - N(a(y), b(y), c(x))\|^2 \\
&\leq [(1-\rho)^2 + 2\sigma\rho(1-\rho)]\|x-y\|^2 + \rho^2[\|N(a(x), b(x), c(x)) \\
&\quad - N(a(y), b(x), c(x))\| + \|N(a(y), b(x), c(x)) - N(a(y), b(y), c(x))\|]^2 \\
&\leq [(1-\rho)^2 + 2\sigma\rho(1-\rho) + (\alpha p + \beta q)^2\rho^2]\|x-y\|^2, \quad \forall x, y \in H.
\end{aligned} \tag{4.16}
$$

Depending on the proof of Theorem 4.1, by (4.16) we know that

$$
\begin{aligned}
&\|F(x)-F(y)\| \\
&\le (1-t+t(l+\mu s))\|x-y\| + t|1-\rho|\|g(x)-g(y)-(x-y)\| \\
&\quad + t\|(1-\rho)(x-y)+\rho[N(a(x),b(x),c(x))-N(a(y),b(x),c(x)) \\
&\quad + N(a(y),b(x),c(x))-N(a(y),b(y),c(x))]\| \\
&\quad + t\rho\|N(a(y),b(y),c(x))-N(a(y),b(y),c(y))\| \\
&\le (1-(1-\theta)t)\|x-y\|, \quad \forall x,y\in H,
\end{aligned}
$$

where

$$
\theta = k+\rho hr+|1-\rho|l+\sqrt{(1-\rho)^2+2\sigma\rho(1-\rho)+(\alpha p+\beta q)^2\rho^2}.
$$

Thus, (4.15) and one of (4.3)-(4.6) ensure that $\theta<1$. That is, F has a unique fixed point $u\in H$, which is a unique solution of the completely generalized nonlinear mixed quasi-variational inequality (2.1).

Similarly, we can show that

$$
\begin{aligned}
\|u_{n+1}-u\| &\le (1-(1-\theta)a_n)\|u_n-u\| \\
&\qquad + a_n(3d_n+a_n'b_n''L+Lb_n')+b_nL, \quad \forall n\ge 0.
\end{aligned}
$$

The rest of the argument follows as in the proof of Theorem 4.1 and is therefore omitted. This completes the proof.

Theorem 4.3. *Let k, a, c, d, g, N, W, $\{W_n\}_{n\ge0}$, C and C' be as in Theorem 4.1. Suppose that $b: H\to H$ is q-Lipschitz continuous. Let*

$$
\begin{aligned}
j &= \beta q+\sqrt{1-2\eta+h^2r^2}, \\
A &= \alpha^2p^2-(l-j)^2, \quad B=(1-k-l)(l-j)-\xi, \\
A' &= \alpha^2p^2-(j+l)^2, \quad B'=-(1-k+l)(l+j)-\xi.
\end{aligned}
$$

If there exists a constant $\rho>0$ satisfying (4.2) and one of (4.3)-(4.6), then the completely generalized nonlinear mixed quasi-variational inequality (2.1) has a unique solution $u\in H$ and the sequence $\{u_n\}_{\ge0}$ defined by Algorithm 3.1 converges strongly to u.

Proof. Depending on the proof of Theorem 4.1, we infer that

$$\begin{aligned}
&\|F(x)-F(y)\| \\
&\le (1-t+tk)\|x-y\| + t\|E(x)-E(y)\| \\
&\le (1-t+tk)\|x-y\| + t|1-\rho|\|g(x)-g(y)-(x-y)\| \\
&\quad + t\|x-y+\rho[N(a(x),b(x),c(x))-N(a(y),b(x),c(x))]\| \\
&\quad + t\rho\|N(a(y),b(x),c(x))-N(a(y),b(x),c(y))-(x-y)\| \\
&\quad + t\rho\|N(a(y),b(x),c(y))-N(a(y),b(y),c(y))\| \\
&\le [1-t+t(k+|1-\rho|l+\rho j+\sqrt{1+2\xi\rho+\alpha^2p^2\rho^2})]\|x-y\| \\
&\le (1-(1-\theta)t)\|x-y\|, \quad \forall x,y\in H,
\end{aligned}$$

where

$$\theta = k+|1-\rho|l+\rho j+\sqrt{1+2\xi\rho+\alpha^2p^2\rho^2}.$$

Thus, (4.2) and one of (4.3)-(4.6) ensure that $\theta<1$. That is, F has a unique fixed point $u\in H$, which is a unique solution of the completely generalized nonlinear mixed quasi-variational inequality (2.1).

Similarly, we can show that

$$\begin{aligned}
\|u_{n+1}-u\| \le {} & (1-(1-\theta)a_n)\|u_n-u\| \\
& + a_n(3d_n+a'_nb''_nL+Lb'_n)+b_nL, \quad \forall n\ge 0.
\end{aligned}$$

The rest of the argument follows as in the proof of Theorem 4.1 and is therefore omitted. This completes the proof.

Replacing the Lipschitz continuity of $I-g$ by the Lipschitz continuity and the strong monotonicity of g in Theorems 4.1, 4.2, and 4.3, respectively, we have the following results.

Theorem 4.4. *Let j, a, b, c, d, A, B, C, A', B', C', N, W and $\{W_n\}_{n\ge0}$ be as in Theorem 4.1. Let $g:H\to H$ be l-Lipschitz continuous and m-strongly monotone and*

$$k=\sqrt{1-2m+l}+\mu s.$$

If there exists a constant $\rho>0$ satisfying (4.2) and one of (4.3)-(4.6), then the completely generalized nonlinear mixed quasi-variational inequality (2.1) has a unique solution $u\in H$ and the sequence $\{u_n\}_{n\ge0}$ defined by Algorithm 3.1 converges strongly to u.

Proof. Since g is l-Lipschitz continuous and m-strong monotone, it follows that

$$\begin{aligned}&\|x-y-(g(x)-g(y))\|^2\\&=\|x-y\|^2-2\langle x-y,g(x)-g(y)\rangle+\|g(x)-g(y)\|^2\\&\leq(1-2m+l^2)\|x-y\|^2,\quad\forall x,y\in H.\end{aligned}$$

By a similar argument used in the proof of Theorem 4.1, the result follows. This completes the proof.

Theorem 4.5. *Let* σ, τ, a, b, c, d, A, B, C, A', B', C', N, W *and* $\{W_n\}_{n\geq 0}$ *be as in Theorem 4.2. Let* g *and* k *be as in the Theorem 4.4. If there exists a constant* $\rho>0$ *satisfying* (4.15) *and one of* (4.3)-(4.6), *then the completely generalized nonlinear mixed quasi-variational inequality* (2.1) *has a unique solution* $u\in H$ *and the sequence* $\{u_n\}_{n\geq 0}$ *defined by Algorithm 3.1 converges strongly to* u.

Theorem 4.6. *Let* j, a, b, c, d, A, B, C, A', B', C', N, W *and* $\{W_n\}_{n\geq 0}$ *be as in Theorem 4.3. Let* g *and* k *be as in the Theorem 4.4. If there exists a constant* $\rho>0$ *satisfying* (4.2) *and one of* (4.3)-(4.6), *then the completely generalized nonlinear mixed quasi-variational inequality* (2.1) *has a unique solution* $u\in H$ *and the sequence* $\{u_n\}_{n\geq 0}$ *defined by Algorithm 3.1 converges strongly to* u.

Remark 4.1. Theorems 4.1-4.6 improve, extend and unify Theorems 2.1 in [2], Theorem 4.1 in [4] and Theorem 2.2 in [21].

5. Examples

In this section, we construct two examples to explain our results.

Example 5.1. Let $H=(-\infty,+\infty)$ with the usual metric $|\cdot|$. Define $W,W_n: H\to 2^H$ by

$$W(x)=\begin{cases}\{0\} & \text{for}\quad x<0,\\ [0,1] & \text{for}\quad x=0,\\ \{1+x\} & \text{for}\quad x>0,\end{cases}$$

and

$$W_n(x)=\begin{cases}\{0\} & \text{for}\quad x<0,\\ \left[0,\dfrac{n+1}{n+2}\right] & \text{for}\quad x=0,\\ \left\{1+\dfrac{n+1}{n+2}x\right\} & \text{for}\quad x>0,\end{cases}$$

where $n\in\{0,1,2,\cdots\}$.

Let $a, b, c, g : H \to H$ and $N : H \times H \times H \to H$ be mappings such that

$$a(x) = \sin(x-3), \quad b(x) = x, \quad c(x) = -x+2, \quad g(x) = \frac{3}{4}x + 1, \quad \forall x \in H$$

and

$$N(x, y, z) = 2|x| - y + \frac{1}{2}z - 1, \quad \forall x, y, z \in H.$$

It is easy to verify that W and each W_n are maximal monotone and $W_n \xrightarrow{G} W$ as $n \to \infty$. Take $p = q = r = \beta = h = 1$, $l = \frac{1}{4}$, $\alpha = \xi = 2$, $\zeta = -1$, $\eta = \frac{1}{2}$, $\mu = s = 0$, then $j = 3$, $A = -\frac{105}{16}$, $B = -\frac{19}{8}$, $C = \frac{3}{4}$ and $\frac{C}{B+\sqrt{B^2-AC}} = \frac{6}{7}$. Obviously, the conditions of Theorem 4.1 are all satisfied for $W(x, y) = W(x)$, $W_n(x, y) = W_n(x)$, $\forall x, y \in H$, $n \geq 0$. It follows from Theorem 4.1 that for $\rho \in (\frac{6}{7}, 1]$, the following quasi-variational inequality:

$$f \in g(u) - N(a(u), b(u), c(u)) + W(g(u)) \tag{5.1}$$

has a unique solution $u \in H$ and the sequence $\{u_n\}_{n\geq 0}$ defined by Algorithm 3.1 with $W_n(x, y) = W_n(x)$, $\forall x, y \in H$, $n \geq 0$ converges strongly to u.

Remark 5.1. We use the assumptions (4.2) and (4.4) of Theorem 4.1 to infer the existence, uniqueness and iterative approximation of solutions for the quasi-variational inequality (5.1).

Example 5.2. Let H, W and $\{W_n\}_{n\geq 0}$ be as in Example 5.1. Let $a, b, g : H \to H$ and $N : H \times H \to H$ be mappings such that

$$a(x) = \frac{1}{4}|x| + 1, \quad b(x) = \sin\left(\frac{1}{2}x + 1\right), \quad g(x) = x - \frac{1}{16}\cos(x-1), \quad \forall x \in H$$

and

$$N(x, y) = x - \frac{1}{2}\sqrt{1+y^2}, \quad \forall x, y \in H.$$

Take $p = \beta = \xi = \frac{1}{4}$, $q = \frac{1}{2}$, $l = \frac{1}{16}$, $r = s = \mu = 0$ and $j = \beta q = \frac{1}{8}$. Clearly, $A = \frac{143}{256}, B = \frac{89}{128}, C = \frac{15}{64}$ and $\frac{c}{B+\sqrt{B^2-AC}} = \frac{60}{178+\sqrt{29539}}$. Hence the assumptions of Theorem 4.1 are fulfilled for $N(x, y, z) = N(x, y)$, $W(x, y) = W(x)$, $W_n(x, y) = W_n(x)$, $\forall x, y, z \in H$, $n \geq 0$. It follows from Theorem 4.1 that for $\rho \in (\frac{60}{178+\sqrt{29539}}, 1]$, the following quasi-variational inequality:

$$f \in g(u) - N(a(u), b(u)) + W(g(u)) \tag{5.2}$$

has a unique solution $u \in H$ and the sequence $\{u_n\}_{n\geq 0}$ defined by Algorithm 3.1 with $N(x, y, z) = N(x, y)$, $W_n(x, y) = W_n(x)$, $\forall x, y, z \in H$, $n \geq 0$.

Remark 5.2. We would like to point out that the mapping b is not ζ-relaxed Lipschitz with respect to the second argument of N in Example 5.2. Otherwise, there exists a positive constant ζ satisfying

$$\begin{aligned} &\langle N(t,b(x)) - N(t,b(y)), x-y\rangle \\ &= -\frac{1}{2}(x-y)\left[\sqrt{1+\sin^2\left(\frac{1}{2}x+1\right)} - \sqrt{1+\sin^2\left(\frac{1}{2}y+1\right)}\right] \\ &\le -\zeta\|x-y\|^2, \quad \forall t,x,y \in H. \end{aligned} \tag{5.3}$$

Let $x = (8n+1)\pi - 2$ and $y = 4n\pi - 2$. It follows from (5.3) that

$$\zeta \le \frac{\sqrt{1+\sin^2(\frac{1}{2}x_n+1)} - \sqrt{1+\sin^2(\frac{1}{2}y_n+1)}}{2(x_n-y_n)} = \frac{\sqrt{2}-1}{2(4n+1)\pi} \to 0 \quad \text{as } n\to\infty,$$

that is, $\zeta \le 0$, which is a contradiction. On the other hand, N is only β-Lipschitz continuous in the second argument. As in the proof of Theorem 4.1, in Example 5.2 we infer that

$$\begin{aligned} &|E(x) - E(y)| \\ &\quad = |(1-\rho)(g(x)-g(y)) + \rho[N(a(x),b(x)) - N(a(y),b(y))]| \\ &\quad \le |1-\rho||g(x)-g(y)-(x-y)| \\ &\qquad + |(1-\rho)(x-y) + \rho[N(a(x),b(x)) - N(a(y),b(x))]| \\ &\qquad + \rho|N(a(y),b(x)) - N(a(y),b(y))| \\ &\quad \le [l|1-\rho| + \sqrt{(1-\rho)^2 + 2\xi\rho(1-\rho) + \alpha^2p^2\rho^2} + q\beta\rho]|x-y|, \quad \forall x,y\in H. \end{aligned}$$

In order to apply Theorem 4.1 in Example 5.2, we have to replace the condition $j = \sqrt{1-2\zeta+\beta^2q^2} + \sqrt{1-2\eta+h^2r^2}$ in Theorem 4.1 by $j = q\beta$. Now the conditions (4.2) and (4.3) of Theorem 4.1 ensure the existence, uniqueness and iterative approximation of solutions for the quasi-variational inequality (5.2).

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