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ON THE RELATIONS BETWEEN THE PARAMETERS OF GRAPHS

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Abstract. Let $G(V, E)$ be a graph of order p. Denote by $\sigma(G)$, $\sigma_1(G)$, $\alpha_T(G)$ and $\beta_T(G)$ the dominating number, the edge dominating number, the total covering number and the total independence number of $G(V, E)$, respectively. Let $\overline{G}$ denote the complement graph of G. This paper establishes some relations among $\sigma(G)$, $\beta_T(G)$, $\alpha_T(G)$, $\sigma_1(G)$, $\sigma(\overline{G})$ and $\sigma_1(\overline{G})$.

1. Notation

Let $G(V, E)$ be a graph. For $u \in V(G)$, let $N_G(u) = \{v|v \in V(G) \text{ and } uv \in E(G)\}$. For a subset $V_1 \subset V(G)$, let $N_G(V_1) = \bigcup_{u \in V_1} N_G(u)$. Denote by $\delta(G)$ the minimum degree of $G(V, E)$, and by $G[S]$ the subgraph of G induced by S.

A subset S of $V(G)$ is called a dominating set of $G(V, E)$ if $S \bigcup N_G(S) = V(G)$. The *dominating number* of $G(V, E)$ is $\sigma(G) = \min\{|S||S \subset V, S \bigcup N_G(S) = V\}$. A set $T \subset E(G)$ is called an *edge dominating set* of $G(V, E)$ if for all $e \in E(G)$, e is adjacent to at least one edge $e' \in T$ or $e \in T$. The *edge dominating number* of $G(V, E)$ is $\sigma_1(G) = \min\{|T||T$ is an edge dominating set of $G\}$. If $S \bigcup N_G(S) = V$ and $|S| = \sigma(G)$, then S is called a minimum dominating set of $G(V, E)$. Similarly, we define a minimum edge dominating set.

Let $a, b \in V(G) \bigcup E(G)$, and $C \subset V(G) \bigcup E(G)$. The elements a and b are called dependent if a and b are adjacent or incident or $a = b$. Otherwise a and b are called independent. A element a is dependent upon C if a is dependent upon at least one element of C. Otherwise, a and C are called independent. A subset $A \subset V(G) \bigcup E(G)$ is called a total covering of $G(V, E)$ if for $\forall a \in V(G) \bigcup E(G)$, a and A are dependent. The quantity $\alpha_T(G) = \min\{|A||A$ is a total covering of $G\}$ is called the total covering number of $G(V, E)$. A subset $B \subset V(G) \bigcup E(G)$ is called a total independent set of $G(V, E)$ if the elements of B are mutually

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independent. The quantity $\beta_T(G) = \max\{|B||B$ is a total independent set of $G\}$ is called the total independent number of $G(V,E)$ [6,7].

2. The Main Results

Theorem 1. *Let $G(V,E)$ be a graph with $\delta(G) > 0$ and $|V| = p$, then*

$$\sigma(G) + \beta_T(G) \le p + \lfloor \frac{p}{4} \rfloor.$$

Moreover, the upper bound is sharp.

Proof. Let $B = V_T \bigcup E_T$ be a maximum total independent set of $G(V,E)$ such that $|E_T|$ is maximum.

If $V_T = \emptyset$, then B is a 1-factor of $G(V,E)$ and $\sigma(G) \le \frac{p}{2}$. Hence

$$\sigma(G) + \beta_T(G) \le \frac{p}{2} + \frac{p}{2}.$$

If $V_T \ne \emptyset$, then the selection of B implies that $V(E_T) \bigcup V_T = V(G)$. For otherwise, there exists $u \in V \setminus (V(E_T) \bigcup V_T)$ and $v \in V_T$ such that $uv \in E$. Let $B' = (B \setminus \{v\}) \bigcup \{uv\}$. Then $|B'| = \beta_T(G)$, B' is a total independent set of $G(V,E)$, and $|B' \bigcap E| > |E_T|$. This contradicts the selection of B. Since $\delta(G) > 0$ and $V(E_T) \bigcup V_T = V$, it is clear that $E_T \ne \emptyset$. For $e \in E(G) \setminus E_T$, we have $\sigma(G - e) \ge \sigma(G)$ and $\beta_T(G - e) \ge \beta_T(G)$. For each $u \in V_T$, select a vertex $u_0 \in N_G(u)$. Let $E' = E_T \cup \{uu_0 : u \in V_T\}$. Then E' induces a spanning subgraph G' of G, which is a forest and each component of G' is a tree with the diameter at most 3. Obviously $\sigma(G') \ge \sigma(G)$ and $\beta_T(G') \ge \beta_T(G)$. Let T be any tree with diameter at most 3. Then

$$\sigma(T) = 2 \text{ if } d(T) = 3 \text{ and } \sigma(T) = 1 \text{ otherwise.}$$

$$\beta_T(T) = 1 \text{ if } d(T) = 1 \text{ and } \beta_T(T) = |V(T)| - 1 \text{ otherwise.}$$

Hence $\sigma(G') + \beta_T(G') = p + m$, where m is the number of the components of G_1 with diameter exactly 3. Since each component of G_1 with diameter 3 contains at least 4 vertices, $m \le \lfloor \frac{p}{4} \rfloor$ and equality hold if each component of G_1 with diameter 3 contains 4 vertices. Therefore

$$\sigma(G) + \beta_T(G) \le \sigma(G') + \beta_T(G') \le p + \lfloor \frac{p}{4} \rfloor,$$

and the upper bound is sharp for any number p.

Lemma 1. *For any graph G, there exists an independent edge dominating set T of G such that $|T| = \sigma_1(G)$.*

The proof of Lemma 1 is easy and omitted.

Theorem 2. *Let $G(V, E)$ be a graph of order p. Then*

$$\sigma_1(G) + \beta_T(G) = p.$$

Proof. Let T be a minimum edge dominating set. Then $V \setminus V(T)$ is an independent set of G such that $T \bigcup (V \setminus V(T))$ is a total independent set of G. And

$$p = |T| + |T \bigcup (V \setminus V(T))| \leq \sigma_1(G) + \beta_T(G).$$

Let $B = V_T \bigcup E_T$ be a maximum total independence set of G such that $|E_T|$ is as large as possible. By the proof of Theorem 1, $V_T \bigcup V(E_T) = V$, and E_T is an edge dominating set of G. So $\sigma_1(G) + \beta_T(G) \leq |E_T| + |B| = p$. Hence

$$\sigma_1(G) + \beta_T(G) = p.$$

Lemma 2. *Let $G(V, E)$ be a graph of order p and with $\delta(G) > 0$. Then there exists a minimum total covering $A = E_T \bigcup V_T$ of G such that $V(E_T) \bigcap V_T = \emptyset$, and E_T is independent. Let*

$$V_0 = V \setminus (V_T \bigcup V(E_T)), V_1' = \{u | u \in V_T, N_G(u) \bigcap V_0 \neq \emptyset\}$$

$$V_2' = V_T \setminus V_1'.$$

Then for $\forall u \in V_1'$, there exist at least two vertices u_1, u_2 of V_0 such that $N_G(u_1) \bigcap V_1' = N_G(u_2) \bigcap V_1' = \{u\}$. And if $uv \in E_T$, $uu_1, vv_1 \in E$, $u_1, v_1 \in V_2'$, then $u_1 = v_1$, V_2' is independent.

Proof. Let $A = V_T \bigcup E_T$ be a minimal total covering of G such that the number of edges in E_T dependent upon V_T is as small as possible. If $u \in V_T \bigcap V(E_T)$, then $uu_1 \in E_T$ and $A' = A \setminus \{uu_1\}$ is not a total covering of G, so that there exists $u_1 v \in E$ and $u_1 v$ is independent to A'. It is easy to see that $A' \bigcup \{u_1 v\}$ is a minimal total covering of G. And $(E_T \setminus \{uu_1\}) \bigcup \{u_1 v\}$ contains less edges dependent upon V_T than E_T does. This is a contradiction. Hence $V_T \bigcap V(E_T) = \emptyset$.

Let $A = E_T \bigcup V_T$ be a minimum total covering of G such that $V_T \bigcap V[E_T] = \emptyset$ and $|V_T|$ is as large as possible. If $d_{G[E_T]}(u) \geq 2$, $uu_1, uu_2 \in E_T$ ($u_1 \neq u_2$), then $A' = A \setminus \{uu_1\}$ is not a total covering of G. Let $A'' = A' \bigcup \{u_1\}$, then A'' is a minimal total covering of G satisfying $(A'' \bigcap V) \bigcap V[A'' \bigcap E] = \emptyset$ and $|A'' \bigcap V| > |V_T|$. This is a contradiction so that E_T is independent.

Let $A = E_T \bigcup V_T$ be a minimum total covering of G such that $V_T \bigcap V[E_T] = \emptyset$, E_T is independent, V_0, V_1', and V_2' as defined in this Lemma and $|E_T|$ as large as possible. Then V_2' is independent. Otherwise, if $u_1 u_2 \in E$, $u_1, u_2 \in V_2'$, then $A' = (A \setminus \{u_1, u_2\}) \bigcup \{u_1 u_2\}$ is a smaller total covering of G than A. If $uv \in E_T$, $uu', vv' \in E$ and $u', v' \in V_2'$, then $u' = v'$. Otherwise $u' \neq v'$, $(A \setminus \{uv, u', v'\}) \bigcup \{uu', vv'\}$ is a smaller total covering of G than A. If $u \in V_1'$ and there exists exactly one vertex $u_1 \in V_0$ such that $N_G(u_1) \bigcap V_1' = \{u\}$, then $A' = (A \setminus \{u\}) \bigcup \{uu_1\}$ is a minimum total covering of G, $(A' \bigcap V) \bigcap V(A' \bigcap E) = \emptyset$, $A' \bigcap E$ is independent and $|A' \bigcap E| > |E_T|$. This is a contradiction so that Lemma 2 is true.

Theorem 3. *Let $G(V, E)$ be a graph with $\delta(G) > 0$ and $|V| = p$, Then*

$$\sigma(G) + \alpha_T(G) \leq p.$$

Moreover, the upper bound is sharp.

Proof. Let $A = E_T \bigcup V_T$ be a minimum total covering of G satisfying the conditions of Lemma 2. Choose a set V_1 of vertices as follows: For $\forall uv \in E_T$, if u is adjacent to a vertex of V_2', choose u, otherwise choose v. Then $V_1 \bigcup V_1'$ is a dominating set of G, and

$$\begin{aligned} \sigma(G) + \alpha_T(G) &\leq |V_1| + |V_1'| + |A| \\ &= 2|E_T| + 2|V_1'| + |V_2'| \\ &\leq 2|E_T| + |V_T| + |V_0| = p. \end{aligned}$$

If G is 1-regular, then $\sigma(G) + \alpha_T(G) = p$.

Theorem 4. *Let $G(V, E)$ be a graph of order p. Then*

$$\sigma_1(G) + \alpha_T(G) \leq p.$$

Moreover, the upper bounds is sharp.

Proof. Let V' be the set of the isolated vertices of G. Then the minimum degree of $G - V' = G'$ is at least 1. Let A be the minimum total covering of G' satisfying Lemma 2. Let M be a matching from V_1' to V_0 that saturated v_1' by Lemma 2. Then $A \bigcup V'$ is a minimum total covering of G and $E_T \bigcup M$ is an edge dominating set of G. Hence

$$\begin{aligned} \sigma_1(G) + \alpha_T(G) &\leq |E_T| + |M| + |V'| + |V_T| + |E_T| \\ &\leq 2|E_T| + |V'| + |V_T| + |V_0| = p. \end{aligned}$$

It is easy to see if G is 1-regular or $E = \phi$ then $\sigma_1(G) + \alpha_T(G) = p$.

Theorem 5. *Let $G(V, E)$ be a graph of order p. Then*

$$\lceil \frac{p+3}{2} \rceil \leq \sigma(\overline{G}) + \beta_T(G) \leq \lceil \frac{3p}{2} \rceil.$$

Moreover, the bounds are sharp.

Proof. At first, we prove the inequality on the right hand side. Let $A = V_T \bigcup E_T$ be a maximum total independent set of G such that $|E_T|$ is as large as possible. By the proof of Theorem 1, we have $V_T \bigcup V(E_T) = V$.

If $V_T = \emptyset$, then $\beta_T(G) = \frac{p}{2}$. Since $\sigma(\overline{G}) \leq p$, we have

$$\sigma(\overline{G}) + \beta_T(G) \leq \lceil \frac{3p}{2} \rceil.$$

If $E_T = \emptyset$, then by the proof of Theorem 1, we have $G = \overline{K_p}$ and $\overline{G} = K_p$. Hence

$$\sigma(\overline{G}) + \beta_T(G) = 1 + p \leq \lceil \frac{3p}{2} \rceil.$$

Assume that $V_T \neq \emptyset$ and $E_T \neq \emptyset$. Since V_T is independent, $\overline{G}[V_T] = K_{|V_T|}$. Hence

$$\sigma(\overline{G}) \leq 1 + \sigma(\overline{G}[V(E_T)]) \leq 1 + 2|E_T| = 1 + p - |V_T|,$$

$$\sigma(\overline{G}) + \beta_T(G) \leq 1 + p - |V_T| + |V_T| + |E_T| = 1 + p + |E_T|$$

$$= 1 + p + \tfrac{p-|V_T|}{2} = \tfrac{3p}{2} + 1 - \tfrac{|V_T|}{2} \leq \tfrac{3p}{2} + \tfrac{1}{2}.$$

Since $\sigma(\overline{G}) + \beta_T(G)$ is an integer, hence

$$\sigma(\overline{G}) + \beta_T(G) \leq \lfloor \frac{3p+1}{2} \rfloor = \lceil \frac{3p}{2} \rceil.$$

If $G = K_p$, then $\sigma(\overline{G}) = p$, $\beta_T(G) = \lceil \frac{p}{2} \rceil$ and the right holds the equality.

Now, we prove the left hand inequality. Let M be a maximum matching of G and $V_1 = V \setminus V(M)$. Then $M \bigcup V_1$ is a total independent set of G and $\beta_T(G) \geq |M \bigcup V_1| = \frac{p+|V_1|}{2}$.

If G has an isolated vertex, then $\sigma(\overline{G}) = 1$ and $|V_1| \geq 1$. Hence

$$\sigma(\overline{G}) + \beta_T(G) \geq \frac{p + |V_1| + 2}{2} \geq \frac{p+3}{2}.$$

Assume $\delta(G) > 0$. Then $\sigma(\overline{G}) > 1$ and $\beta_T(G) \geq \frac{p}{2}$. Hence

$$\sigma(\overline{G}) + \beta_T(G) \geq \frac{p}{2} + 2 \geq \lceil \frac{p+3}{2} \rceil.$$

If $G = K_{p-1} \bigcup K_1$, then $\sigma(\overline{G}) = 1$, $\beta_T(G) = 1 + \lceil \frac{p-1}{2} \rceil$ and

$$\sigma(\overline{G}) + \beta_T(G) = 2 + \lceil \frac{p-1}{2} \rceil = \lceil \frac{p+3}{2} \rceil.$$

Theorem 6. *Let $G(V, E)$ be a graph of order p. Then*

$$\lceil \frac{p}{2} \rceil \le \sigma_1(\overline{G}) + \beta_T(G) \le \lfloor \frac{3p}{2} \rfloor.$$

Moreover, the bounds are sharp.

Proof. At first, we prove the right hand inequality. Let $V'_T \bigcup E'_T$, $V_T \bigcup E_T$ be the maximum total independent sets of $\overline{G}$ and G, respectively; such that $V'_T \bigcup V(E'_T) = V = V_T \bigcup V(E_T)$. Then

$$\beta_T(G) = \frac{p + |V_T|}{2}, \quad \beta_T(\overline{G}) = \frac{p + |V'_T|}{2}.$$

By Theorem 2,

$$\sigma_1(\overline{G}) + \beta_T(\overline{G}) = p.$$

Therefore

$$\sigma_1(\overline{G}) + \beta_T(G) = p + \beta_T(G) - \beta_T(\overline{G})$$

$$= p + \frac{|V_T| - |V'_T|}{2} \le \frac{3p}{2}.$$

Hence $\sigma_1(\overline{G}) + \beta_T(G) \le \lfloor \frac{3p}{2} \rfloor$, and $\sigma_1(\overline{G}) + \beta_T(G) = \lfloor \frac{3p}{2} \rfloor$ if $G = \overline{K}_p$.

Since $\beta_T(G) \ge \lceil \frac{p}{2} \rceil$, hence the left hand inequality is trivial, and $\sigma_1(\overline{G}) + \beta_T(G) = \lceil \frac{p}{2} \rceil$ if $G = K_p$.

Theorem 7. *Let $G(V, E)$ be a graph of order p. Then*

$$\sigma(\overline{G}) + \alpha_T(G) \le \lceil \frac{3p}{2} \rceil.$$

Moreover, the bound is sharp.

Proof. Let $B = V_T \bigcup E_T$ be a minimum total covering of G and satisfying the conditions of Lemma 2.

If G is disconnected, then $\overline{G}$ has a spanning complete bipartite subgraph. Hence $\sigma(\overline{G}) \le 2$, and $\sigma(\overline{G}) = 1$ if $p = 2$. By $\alpha_T(G) \le p$ and $\sigma(\overline{G}) \le \lceil \frac{p}{2} \rceil$, we have $\alpha_T(G) + \sigma(\overline{G}) \le p + \lceil \frac{p}{2} \rceil \le \lceil \frac{3p}{2} \rceil$.

Assume G is connected. By Lemma 2, V_2' is independent, $N_G(V_2') \subseteq V(E_T)$, and $|V_0| \geq 2|V_1'|$.

Case 1. If $V_2' \neq \emptyset$, $\sigma(\overline{G}) \leq 1 + \sigma(\overline{G}[V(E_T)]) \leq 1 + 2|E_T|$. Therefore

$$\begin{aligned}\sigma(\overline{G}) + \alpha_T(G) &\leq 1 + 3|E_T| + |V_T| \\ &= 1 + \frac{3(p - |V_T| - |V_0|)}{2} + |V_T| \\ &= \frac{3p}{2} + 1 - \frac{3|V_0|}{2} - \frac{|V_T|}{2} \leq \frac{3p+1}{2}.\end{aligned}$$

Case 2. If $V_2' = \emptyset$, then

$$\alpha_T(G) = \frac{p - |V_T| - |V_0|}{2} + |V_T| = \frac{p + |V_T| - |V_0|}{2} \leq \frac{p}{2}.$$

Therefore $\sigma(\overline{G}) + \alpha_T(G) \leq p + \alpha_T(G) \leq \frac{3p}{2} \leq \lceil \frac{3p}{2} \rceil$.

If $G = K_p$, then $\sigma(\overline{G}) + \alpha_T(G) = \lceil \frac{3p}{2} \rceil$.

Theorem 8. *Let $G(V, E)$ be a graph of order p. Then*

$$\lceil \frac{p}{2} \rceil \leq \sigma_1(\overline{G}) + \alpha_T(G) \leq \lfloor \frac{3p}{2} \rfloor.$$

Moreover, the bounds are sharp.

Proof. At first, we prove the right hand inequality. By $\sigma_1(\overline{G}) \leq \lfloor \frac{p}{2} \rfloor$ and $\alpha_T(G) \leq p$, we have

$$\sigma_1(\overline{G}) + \alpha_T(G) \leq \lfloor \frac{3p}{2} \rfloor.$$

And $\sigma_1(\overline{G}) + \alpha_T(G) = \lfloor \frac{3p}{2} \rfloor$ if $G = \overline{K}_p$.

Now, we prove the left hand inequality. By Lemma 1, let S be a minimal independent edge dominating set of $\overline{G}$ and $V_1 = V \setminus V(S)$. Then V_1 is an independent set of $\overline{G}$ and $G[V_1] = K_{|V_1|}$, $\alpha_T(G) \geq \alpha_T(G[V_1]) = \lceil \frac{|V_1|}{2} \rceil$. Hence

$$\sigma_1(\overline{G}) + \alpha_T(G) \geq |S| + \lceil \frac{|V_1|}{2} \rceil = |S| + \lceil \frac{p - 2|S|}{2} \rceil = \lceil \frac{p}{2} \rceil.$$

And $\sigma_1(\overline{G}) + \alpha_T(G) = \lceil \frac{p}{2} \rceil$ if $G = K_p$.

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