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PARTIAL INVERSE SEMIGROUP C^*-ALGEBRA

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Abstract. The notion of partial group C^*-algebra of a discrete group introduced by R. Exel in [3] is generalized to an idempotent unital inverse semigroup, and the partial inverse semigroup C^*-algebra is defined. By using the algebras of multipliers of ideals of an associative algebra, we can prove some theorem in the C^*-algebra context without using the approximate identity.

1. Introduction

Concepts of partial action of groups and action of inverse semigroups appeared in the theory of operator algebras roughly ten years ago. Together with the notion of partial actions a generalization of the concept of crossed product appeared in that theory (see [1-3, 8, 10]). The theory of partial crossed products by groups is well developed. In [3] R. Exel investigated *partial group C^*-algebra* of a discrete group as a partial crossed product of a special C^*-algebra by a partial action. We are going to follow his footsteps constructing the crossed product of a specific C^*-algebra and an idempotent unital inverse semigroup by an action and define the *partial inverse semigroup C^*-algebra*.

Another relevant concept which we will consider in this paper is the property of associativity of $A \times_\alpha S$ where A is an algebra and S is an idempotent unital inverse semigroup. The associativity of the group ring $A \times_\alpha G$ in which A is a C^*-algebra has been proved by R. Exel in [1] where he employed special properties of C^*-algebras i.e. approximate identity. For a general algebra A, the question of associativity of $A \times_\alpha G$ was open since then. Recently R. Exel answered this question for an algebra whose ideals are *idempotent* [4]. In this paper we will prove

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the associativity of $A \times_\alpha S$, in which A is a special C^*-algebra, without using approximate identity.

By a *unital inverse semigroup* we mean a semigroup S with the unit element e such that for each s in S there exists a unique element s^* in S with the following properties:

(i) $ss^*s = s$,

(ii) $s^*ss^* = s^*$.

See [7] for more information about inverse semigroups.

Before we define the action of an inverse semigroup on a C^*-algebra A, we need to know about *partial automorphism*.

Definition 1.1. Let A be a C^*-algebra. *A partial automorphism* of A is a triple (α, I, J) in which I and J are closed two-sided ideals in A and

$$\alpha : I \longrightarrow J$$

is a C^*-algebra isomorphism.

For a given C^*-algebra A let $PAut(A)$ be the set of all partial automorphisms of A. Using the fact that ideals of ideals of a C^*-algebra are, themselves, ideals of that C^*-algebra and the intersection of two ideals are equal to their products, we see that $PAut(A)$ is a unital inverse semigroup with the identity (i, A, A) in which i is the identity map on A.

Note that $(\alpha, I, J)^* = (\alpha^{-1}, J, I)$ and if (α, I, J) and (β, K, L) are two partial automorphisms of A, then their product is the partial atuomorphism $(\alpha\beta, \beta^{-1}(I), \alpha\beta(\beta^{-1}(I)))$ in which $\alpha\beta$ is the composition of α and β with the largest possible domain.

Multiplier Algebra 1.2. Let A be a unital associative algebra. By a *multiplier* of A we mean a pair (L, R) of bounded linear transformations on A such that for all $a, b \in A$

$$L(ab) = L(a)b, R(ab) = aR(b) \text{ and } R(a)b = aL(b).$$

The set of all multipliers of A will be denoted by $M(A)$. One checks that $M(A)$ is itself an associative algebra under the following operations:

$$(L, R) + (L', R') = (L + L', R + R');$$

$$r(L, R) = (rL, rR), \qquad r \in \mathbb{C};$$

$$(L, R)(L', R') = (L \circ L', R' \circ R);$$

where, as usual, the symbol "$\circ$" between operators means the composition of them. With these operations, $M(A)$ is called the *multiplier algebra of* A. The algebra $M(A)$ has the unit element (i, i), where $i : A \longrightarrow A$ is the identity map. Let I be a two-sided ideal in A. To each $x \in A$ associated two linear transformations $L_x : I \longrightarrow I$ and $R_x : I \longrightarrow I$ defined by $L_x(a) = xa$ and $R_x(a) = ax$ respectively. $L = L_x$ is a *left multiplier* and $R = R_x$ is a *right multiplier* of I, and

$$M(I) = \{(L_x, R_x) : x \in A\}.$$

([5],[6] and [9] are good references for multiplier algebras.)

The following proposition is pivotal for our purpose.

Proposition 1.3. *If A is an associative algebra and I an ideal of A, then we have*

(i) $(R' \circ L)(ab) = (L \circ R')(ab)$ *for every* $(L, R), (L', R') \in M(I)$ *and* $a, b \in I$.

(ii) *If I is an idempotent ideal of A then* $(R' \circ L)(a) = (L \circ R')(a)$ *for all* $a \in I$. *(An ideal I of an algebra A is called idempotent if $I^2 = I$.)*

Proof. (i) Since (L, R) and (L', R') are multipliers of I, by the properties of multiplier we have

$$\begin{aligned}(R' \circ L)(ab) &= R'(L(ab)) = R'(L(a)b)\\ &= L(a)R'(b) = L(aR'(b)) = L(R'(ab))\\ &= (L \circ R')(ab).\end{aligned}$$

(ii) For arbitrary $a \in I$, since $I = I^2$, we have $a \in I^2 = \overline{span}\{xy : x \in I, y \in I\}$. Therefore it suffices to prove the statement for $a = xy$ in which $x, y \in I$.

Now by (i) $(R' \circ L)(a) = (R' \circ L)(xy) = (L \circ R')(xy) = (L \circ R')(a)$. ■

2. Partial Inverse Semigroup C^*-Algebra

This section starts with the definition of action of an inverse semigroup on a C^*-algebra, then we construct an auxiliary C^*-algebra and consequently we consider the partial crossed product of this C^*-algebra by that action. The major new results of this section are 2.4, 2.6 and theorem 2.7.

Definition 2.1. Let S be a unital inverse semigroup with the identity e and A be a C^*-algebra. *An action of S on A* is a semigroup homomorphism:

$$\begin{aligned}\beta \;:\; & S \longrightarrow PAut(A)\\ & s \mapsto (\beta_s, E_{s^*}, E_s)\end{aligned}$$

such that $E_e = A$.

An element s of an inverse semigroup S is called *idempotent* if $s^2 = s$. And S is called an *idempotent semigroup* if $s^2 = s$ for all s in S.

Lemma 2.2. *Let S be a unital inverse semigroup, A be a C^*-algebra, β an action of S on A and $s \in S$, then $\beta_{s^*} = \beta_s^{-1}$, β_e is the identity map on A and if s is an idempotent, then β_s is the identity map on $E_{s^*} = E_s$.*

Proof. Since β is a homomorphism, we have

$$\beta_s = \beta(s) = \beta(ss^*s) = \beta(s)\beta(s^*)\beta(s) = \beta_s\beta_{s^*}\beta_s$$

on the other hand $\beta_s^{-1} = \beta_s\beta_s^{-1}\beta_s$. So, by the uniqueness of inverse in inverse semigroups, we conclude that $\beta_{s^*} = \beta_s^{-1}$.

Moreover

$$\beta_e\beta_s = \beta_{es} = \beta_s = \beta_{se} = \beta_s\beta_e$$

therefore $\beta_e = i_A$. If s is an idempotent, since $s^2 = s$ we have $sss = s^2 = s$ and $ss^*s = s$ so by uniqueness of inverse of s we conclude that $s = s^*$ and $\beta_s = \beta_{s^*}$. On the other hand $(\beta_s)^2 = \beta_s\beta_s = \beta_s\beta_{s^*} = \beta_e = i$.

Lemma 2.3. *If β is an action of the unital inverse semigroup S on a C^*-algebra A, then $\beta_t(E_{t^*}E_s) = E_{ts}$ for all s, t in S.*

Proof. Since E_{t^*} and E_s are ideals in the C^*-algebra A we have $E_{t^*}E_s = E_{t^*} \cap E_s$. So

$$\begin{aligned}\beta_t(E_{t^*}E_s) = \beta_t(E_{t^*} \cap E_s) &= \text{image}(\beta_t\beta_s)\\ &= \text{image}(\beta(t)\beta(s))\\ &= \text{image}\beta(ts)\\ &= \text{image}(\beta_{ts}) = E_{ts}.\end{aligned}$$

■

An auxiliary C^*-algebra and an action of a semigroup on it 2.4. Let S be a *unital idempotent inverse semigroup* with unit e. The auxiliary C^*-algebra in our study is the universal C^*-algebra denoted by A_e, defined via generators and relations as follows. The set of generators consists of symbol P_E in which E is a finite subset of S such that $e \in E$, and the relations are:

(i) $P_EP_F = P_{E\cup F}$

(ii) $P_E^* = P_E$

for all possible choices of E and F. Since $P_E^2 = P_E P_E = P_{E \cup E} = P_E$ and $P_E^* = P_E$, each P_E will be a projection. Obviously A_e is an abelian C^*-algebra with the identity element $P_{\{e\}}$ and is nothing but the closed linear span of elements of the form P_E.

In order to define an action of S on the above C^*-algebra we need to construct a collection of partial automorphisms of A_e. Therefore we define the collection $\{(\beta_t, E_{t^*}, E_t) : t \in S\}$ as follows:

For $t \in S$ let $E_t = \overline{span}\{P_E : t, e \in E\}$ and $\beta_t : E_{t^*} \longrightarrow E_t$ defined by $\beta_t(P_E) = P_{tEt \cup \{e\}}$ where $tEt = \{tst : s \in E\}$. It is clear that each E_t is a closed two-sided ideal in A_e and each β_t is a C^*-isomorphism from E_{t^*} onto E_t.

It should be noted that in the definition of β_t from E_{t^*} onto E_t the situation is more delicated than it may appear at first glance. That is, t^*, $e \in E$ where $P_E \in E_{t^*}$, and our definition of β_t should imply that t and e be elements of some F where $P_F \in E_t$. Clearly $t = tt^*t \in \beta_t(E_{t^*})$, $e \in \beta_t(E_{t^*})$ and also since S is an idempotent semigroup we have $t = t^2 = tet \in \beta_t(E_{t^*})$. Therefore

$$\beta : S \longrightarrow PAut(A_e)$$

defined by

$$t \mapsto (\beta_t, E_{t^*}, E_t)$$

is an *action* of S on A_e.

A close look to the construction of ideals E_t shows that each E_t is an idempotent ideal in A_e, simply because each P_E is a projection.

The following proposition shows that, for given partial automorphism (β_s, E_{s^*}, E_s) we can make an element of $M(E_s)$.

Proposition 2.5. *Let A be a C^*-algebra, S a unital inverse semigroup and (β_s, E_{s^*}, E_s) be a partial automorphism of A for $s \in S$. If L and R are left and right multiplier of E_{s^*} respectively, then $(\beta_s \circ L \circ \beta_{s^*}, \beta_s \circ R \circ \beta_{s^*}) \in M(E_s)$.*

Proof. Since $\beta_{s^*} : E_s \longrightarrow E_{s^*}$ is an *algebra* isomorphism, $L : E_{s^*} \longrightarrow E_{s^*}$ is a left multiplier and $\beta_s : E_{s^*} \longrightarrow E_s$ is an *algebra isomorphism* we conclude that

$$\beta_s \circ L \circ \beta_{s^*} : E_s \longrightarrow E_s$$

and similarly

$$\beta_s \circ R \circ \beta_{s^*} : E_s \longrightarrow E_s$$

are linear transformations on E_s. In order to see that $(\beta_s \circ L \circ \beta_{s^*}, \beta_s \circ R \circ \beta_{s^*})$ is a multiplier of E_s, let $\beta_s \circ L \circ \beta_{s^*} = L'$ and $\beta_s \circ R \circ \beta_{s^*} = R'$. Now for $a, b \in E_s$, $x \in A_e$ we have

$$L'_x(ab) = \beta_s \circ L_x(\beta_{s^*}(ab)) = \beta_s(x\beta_{s^*}(ab)) = \beta_s(x)(ab)$$

and

$$\begin{aligned} L'(a)b &= [\beta_s \circ L_x(\beta_{s^*}(a))]b \\ &= [\beta_s(x\beta_{s^*}(a))]b = [\beta_s(x)a]b \\ &= \beta_s(x)(ab) \quad \text{i.e.} \quad L'(ab) = L'(a)b. \end{aligned}$$

Also

$$\begin{aligned} R'(ab) &= (\beta_s \circ R_x \circ \beta_{s^*})(ab) = \beta_s \circ R_x(\beta_{s^*}(ab)) \\ &= \beta_s(\beta_{s^*}(ab)x) = (ab)\beta(x), \\ & \quad \text{and} \quad aR'(b) = a(\beta_s \circ R_x \circ \beta_{s^*})(b) \\ &= a(\beta_s \circ R_x(\beta_{s^*}(b))) = a\beta_s(\beta_{s^*}(b)x) \\ &= a(b\beta_s(x)) = (ab)\beta_s(x), \end{aligned}$$

therefore $R'(ab) = aR'(b)$.

And

$$\begin{aligned} R'(a)b &= [(\beta_s \circ R_x \circ \beta_{s^*})(a)]b = [\beta_s \circ R_x(\beta_{s^*}(a))]b \\ &= [\beta_s(\beta_{s^*}(a)x)]b = [a\beta_s(x)]b. \end{aligned}$$

On the other hand

$$\begin{aligned} aL'(b) &= a[(\beta_s \circ L_x \circ \beta_{s^*})b] = a[\beta_s \circ L_x(\beta_{s^*}(b))] \\ &= a\beta_s(x\beta_{s^*}(b)) = a\beta_{s^*}(x)b = [a\beta_{s^*}(x)]b \end{aligned}$$

i.e. $R'(a)b = aL'(b)$, and these facts show $(\beta_s \circ L \circ \beta_{s^*}, \beta_s \circ R \circ \beta_{s^*})$ is a multiplier of E_s. ∎

Here we are ready to give the definition of *Partial inverse semigroup* C^*-*algebra*.

Definition 2.6. Let $\beta : S \longrightarrow PAut(A_e)$ be the action of the unital idempotent inverse semigroup S on the C^*-algebra A_e which is discussed in 2.4. The partial inverse semigroup C^*-algebra $C_p^*(S)$ given by the crossed product [11] of A_e by β, that is,

$$C_p^*(S) = A_e \times_\beta S.$$

Examples.

(a) Let $A = \mathbb{C}$ be the C^*-algebra of complex numbers and $S = \{0, 1\}$ be the unital inverse semigroup with the identity element 1 and 0 is its idempotent element. We have $1^* = 1$ and $0^* = 0$. Let β_s be the identity map on A for all $s \in S$. Obviously, for all $s \in S$ we have $E_s = \mathbb{C}$ and (β_s, E_{s^*}, E_s) is a parital automorphism of A. Now

$$\beta : S \to PAut(A)$$

$$s \longmapsto (\beta_s, E_{s^*}, E_s)$$

is an action of S on A. Since S is an idempotent inverse semigroup, by [11, 4.6] we see that $\mathbb{C} \times_\beta S$ is isomorphic to $\mathbb{C}$.

(b) The set of all pairs with complex coordinates, $\mathbb{C}^2$, is a C^*-algebra with norm, multiplication and involution defined as follow:

$$||(c_1, c_2)|| = max\{|c_1|, |c_2|\};$$

$$(c_1, c_2)(c_1', c_2') = (c_1c_1', c_2c_2');$$

$$(c_1, c_2)^* = (\overline{c_1}, \overline{c_2}).$$

The group of integers, $\mathbb{Z}$, is a unital inverse semigroup.

Take $A = \mathbb{C}^2$ and $S = \mathbb{Z}$. Define ideals $E_0 = A, E_{-1} = \{(a, 0) : a \in A\}, E_1 = \{(0, a) : a \in A\}$ and $E_n = \{(0, 0)\}$ for all n, except for $n = -1, 0, 1$. Let β_0 be the identity map on A. Also $\beta_1((a, 0)) = (0, a)$ is the forward shift and $\beta_n = \beta_1^n$ for all $n \neq 0$. Obviously, (β_n, E_{-n}, E_n) is a partial automorphism of A. Therefore by [8, 2.5], $A \times_\beta S$ is isomorphic to the matrix algebra M_2.

Sieben in [11], before the definition of crossed product showed that

$$L = \{x \in l^1(S, A) : x(s) \in E_s\}$$

with the multiplication

$$(x * y)(s) = \sum_{rt=s} \beta_r[\beta_{r^*}(x(r))y(t)],$$

the involution

$$x^*(s) = \beta_s(x(s^*)^*),$$

the norm, scalar multiplication, and addition inherited from $l^1(S, A)$ is a Banach $*$-algebra.

Since $x(r) \in E_r$ and $\beta_{r^*} : E_r \longrightarrow E_{r^*}$ is C^*-algebra isomorphism we conclude that $\beta_{r^*}(x_r) \in E_{r^*}$. $y(t) \in E_t$, therefore $\beta_{r^*}(x_r)y(t) \in E_{r^*}E_t = E_{r^*} \cap E_t$. By Lemma 2.3, $\beta_r(E_{r^*}E_t) = E_{rt}$, this shows that

$$\beta_r(\beta_{r^*}(x_r)y(t)) \in E_{rt} = E_s. \qquad \text{i.e.}$$

$(x * y)(s) \in L$. Similarly $x(s^*) \in E_{s^*}$ and since E_{s^*} is self-adjoint, we have $(x(s^*))^* \in E_{s^*}$. $\beta_s : E_{s^*} \longrightarrow E_s$ is a C^*-algebra isomorphism, therefore $\beta_s((x(s^*))^*) \in E_s$, this shows that L is closed with respect to the above involution.

Simple calculations show that

$$\|x^*\| = \|x\|, \|x * y\| \leq \|x\|\|y\|, \qquad (x+y)^* = x^* + y^*,$$

$$(x * y)^* = y^* * x^* \quad \text{and} \quad (\lambda x)^* = \bar{\lambda} x^*$$

for all x, y in L and $\lambda \in \mathbb{C}$. These facts show that L will be a Banach $*$-algebra if we can prove that

$$(x * y) * z = x * (y * z) \tag{1}$$

for all x, y, z in L.

In the following *main theorem* we will prove the equality in (1) without using the approximate identity properties.

Theorem 2.7. *Let A be a C^*-algebra, S a unital inverse semigroup, β an action of S on A and $l^1(S, A)$ be the Banach algebra of A-valued functions on S. If*

$$L = \{x \in l^1(S, A) : x(s) \in E_s\}$$

then the operation $*$ defined by

$$(x * y)(s) = \sum_{rt=s} \beta_r[\beta_{r^*}(x(r))y(t)]$$

is associative on L.

Proof. Let $a_s\delta_s$ be the function in L which takes the value a_s at the point s of S and zero at every other element of S. We have $x = \sum_{r \in S} a_r\delta_r$, $y = \sum_{s \in S} a_s\delta_s$ and $z = \sum_{t \in S} a_t\delta_t$. Obviously the theorem is proved if we can show that

$$(a_r\delta_r * a_s\delta_s) * a_t\delta_t = a_r\delta_r * (a_s\delta_s * a_t\delta_t)$$

for arbitrary $a_r \in E_r$, $a_s \in E_s$ and $a_t \in E_t$. Notice that $a_r\delta_r * a_s\delta_s = \beta_r(\beta_{r^*}(a_r)a_s)\delta_{rs}$. As a consequence we have

$$\begin{aligned}
(a_r\delta_r * a_s\delta_s) * a_t\delta_t &= \beta_r(\beta_{r^*}(a_r)a_s)\delta_{rs} * a_t\delta_t \\
&= \beta_{rs}\{\beta_{(rs)^*}[\beta_r(\beta_{r^*}(a_r)a_s)]a_t\}\delta_{rst} \\
&= \beta_{rs}\{\beta_{s^*}[\beta_{r^*}(a_r)a_s]a_t\}\delta_{rst} \\
&= \beta_r\{\beta_s(\beta_{s^*}[\beta_{r^*}(a_r)a_s]a_t)\}\delta_{rst}
\end{aligned}$$

It should be noted that since $a_r \in E_r$ and $\beta_{r^*} : E_r \longrightarrow E_{r^*}$ is an isomorphism, we have $\beta_{r^*}(a_r) \in E_{r^*}$ and $\beta_{r^*}(a_r)a_s \in E_{r^*} \cap E_s$. By Lemma 2.3, $\beta_r(E_{r^*} \cap E_s) = E_r \cap E_{rs}$, therefore $\beta_r(\beta_{r^*}(a_r)a_s) \in E_r \cap E_{rs}$, hence we can split $\beta_{s^*r^*}$ i.e. $\beta_{s^*r^*}(\cdot) = \beta_{s^*}(\beta_{r^*}(\cdot))$. Similar argument shows that we can split β_{rs} as $\beta_r \circ \beta_s$, consequently

$$(a_r\delta_r * a_s\delta_s) * a_t\delta_t = \beta_r\{\beta_s(\beta_{s^*}[\beta_{r^*}(a_r)a_s]a_t)\}\delta_{rst}. \tag{2}$$

And

$$\begin{aligned} a_r\delta_r * (a_s\delta_s * a_t\delta_t) &= a_r\delta_r * [\beta_s(\beta_{s^*}(a_s)a_t]\delta_{st} \\ &= \beta_r\{\beta_{r^*}(a_r)[\beta_s(\beta_{s^*}(a_s)a_t)]\}\delta_{rst}. \end{aligned} \tag{3}$$

If we apply β_{r^*} on the right hand sides of (2) and (3) we conclude that equality holds if and only if

$$\beta_s(\beta_{s^*}[\beta_{r^*}(a_r)a_s]a_t) = \beta_{r^*}(a_r)[\beta_s(\beta_{s^*}(a_s)a_t)]. \tag{4}$$

Since $\beta_{r^*} : E_r \longrightarrow E_{r^*}$ is an isomorphism, $\beta_{r^*}(a_r)$ runs over E_{r^*} and consequently (4) is equivalent to

$$\beta_s[\beta_{s^*}(aa_s)a_t] = a[\beta_s(\beta_{s^*}(a_s)a_t)]$$

for arbitrary element a in E_{r^*}, $a_s \in E_s$ and $a_t \in E_t$. By taking $r = t = e$ we have $E_r = E_t = E_e = A$. Therefore it suffices we prove

$$\beta_s[\beta_{s^*}(aa_s)a_t] = a[\beta_s(\beta_{s^*}(a_s)a_t)] \tag{5}$$

for a_r, a_t in A, s in S and $a_s \in E_s$.

Consider R_{a_t} as a right multiplier of E_{s^*} and L_a as a left multiplier of E_s, by Proposition 2.5, $\beta_s \circ R_{a_t} \circ \beta_{s^*}$ is a right multiplier of E_s. Since all ideal E_s are idempotent by (i) of Proposition 1.3 we have

$$(\beta_s \circ R_{a_t} \circ \beta_{s^*}) \circ L_a = L_a \circ (\beta_s \circ R_{a_t} \circ \beta_{s^*}) \tag{6}$$

on E_s. Applying both sides of (6) on a_s we conclude (5). ■

Beyond the purpose of proving associativity property without using the approximate identity, we belive there is a considerable amount of interesting information which can be obtained from this method.

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