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ON THE DIRICHLET PROBLEM FOR THE EQUATION $-\Delta u = g(x,u) + \lambda f(x,u)$ WITH NO GROWTH CONDITIONS ON f

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Abstract. In this paper we establish some results concerning the existence of nonzero nonnegative and nonzero nonpositive solutions for a Dirichlet problem via variational methods. In particular, a general variational principle of B.Ricceri is applied.

1. Introduction

In this paper we study the following Dirichlet problem

$$(P_\lambda) \qquad \begin{cases} -\Delta u = g(x,u) + \lambda f(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where $\lambda \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded open set with boundary $\partial\Omega$ of class C^2, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions, and $\Delta(\cdot) = \mathrm{div}(\nabla(\cdot))$ is the Laplacian operator. Precisely, we establish a result concerning the existence of nonzero nonnegative and nonzero nonpositive strong solutions for problem (P_λ) for each λ belonging to an open real interval containing 0. We recall that a strong solution of problem (P_λ) is any $u \in W_0^{1,1}(\Omega) \cap W^{2,1}(\Omega)$ such that $-\Delta u(x) = g(x,u(x)) + \lambda f(x,u(x))$ for almost all $x \in \Omega$. While a weak solution of problem (P_λ) is any $u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla u(x)\nabla v(x)dx - \int_\Omega (g(x,u(x)) + \lambda f(x,u(x)))v(x)dx = 0$$

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for all $v \in W_0^{1,2}(\Omega)$. Hence, the weak solutions of problem (P_λ) are exactly the critical points of the energy functional

$$(P_\lambda) \qquad J_\lambda(u) = \frac{1}{2}\|u\|^2 - \int_\Omega (G(x, u(x)) + \lambda F(x, u(x)))\, dx$$

where

$$\|u\| = \left(\int_\Omega |\nabla u(x)|^2 dx\right)^{\frac{1}{2}}$$

is the usual norm in $W_0^{1,2}(\Omega)$ and

$$F(x, \xi) = \int_0^\xi f(x, t)dt, \quad G(x, \xi) = \int_0^\xi g(x, t)dt$$

for all $(x, \xi) \in \Omega \times \mathbb{R}$.

To prove our results, we use variational methods and, in particular, we make use of a critical point theorem obtained by B. Ricceri in [5] which is already been used by the author in [2] to establish a multiplicity theorem of weak solutions for an equation of the type $-\Delta u = g(x, u) + \lambda f(x, u)$ with Neumann boundary condition. Recently, problem (P_λ) was studied in [4] by means of quite different arguments and two multiplicity results was obtained. There the nonlinearities f, g are supposed continuous on $\overline{\Omega} \times \mathbb{R}$ and, in particular, this is the only assumption on f. Nevertheless, to obtain the multiplicity of the solutions, in [4] it is essential to require that the nonlinearity g is odd with respect to the second variable. In the present paper, we want to study problem (P_λ) where, as in [4], no growth conditions on f are assumed and g is not supposed to be odd with respect to the second variable. We proceed to establish, at first, a theorem concerning the existence of a nonzero nonnegative strong solution for problem (P_λ). Successively, with a straightforward change of the hypotheses, we establish a theorem concerning the existence of a nonzero nonpositive strong solution for problem (P_λ). Hence, combining the previous two results, we obtain a multiplicity theorem.

Before closing this section we introduce the following notations: for each $q \in [1, +\infty]$ we denote by $\|u\|_q$ and $\|\cdot\|_{W^{2,q}(\Omega)}$ the usual norms of $L^p(\Omega)$ and $W^{2,q}(\Omega)$ respectively. Further, by $< \cdot, \cdot >$ we denote the scalar product in $W_0^{1,2}(\Omega)$ which induces the norm, that is

$$< u, v > \stackrel{def}{=} \int_\Omega \nabla u \nabla v \; dx$$

for all $u, v \in W_0^{1,2}(\Omega)$.

2. The Results

In this section we state and proof the main results. Theorem 2.1 below gives the existence of a nonzero nonnegative solution to the problem (P_λ).

Theorem 2.1. *Let $s \in (1,2)$ $q > \frac{N}{2}$ and $a > 0$. Let $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ be two Carathéodory functions satisfying the following conditions*

(i) $\sup_{0 \le \xi \le r} |f(\cdot, \xi)| \in L^q(\Omega)$ *for all* $r > 0$ *;*

(ii) $f(x, 0) = 0$ *for a.e.* $x \in \Omega$*;*

(iii) $|g(x,t)| \le at^{s-1}$ *for all* $t \ge 0$ *and a.e* $x \in \Omega$*.*

(iv) *there exists a non empty open set* $D \subseteq \Omega$ *such that*

$$\liminf_{\xi \to 0^+} \frac{\inf_{x \in D} \int_0^\xi g(x,t)dt}{\xi^2} = +\infty$$

Then, there exist $\sigma, \overline{\lambda} > 0$ such that, for every $\lambda \in [-\overline{\lambda}, \overline{\lambda}]$, there exists a strong nonzero nonnegative solution $u_\lambda \in W_0^{1,2}(\Omega) \cap W^{2,q}(\Omega)$ of problem (P_λ) with $\|u_\lambda\|_{W^{2,q}(\Omega)} \le \sigma$.

Proof. Define

$$g_0(x,t) = \begin{cases} 0 & \text{if } (x,t) \in \Omega \times (-\infty, 0) \\ g(x,t) & \text{if } (x,t) \in \Omega \times [0, \infty) \end{cases} .$$

From condition $iii)$ g_0 turns out to be a Carathéodory function. Moreover, put

$$\Psi(u) = \frac{1}{2}\|u\|^2 - \int_\Omega \left(\int_0^{u(x)} g_0(x,t)dt \right) dx.$$

By Rellich-Kondrachov Theorem and condition $iii)$ it easily seen that Ψ is coercive, sequentially weakly semicontinuous and (strongly) continuous. Moreover, from Proposition 41.10 of [7], Ψ is Gâteuaux differentiable on $W_0^{1,2}(\Omega)$. We want to prove that

$$\inf_{W_0^{1,2}(\Omega)} \Psi < 0. \tag{2.1}$$

To this end, choose a nonzero nonnegative function $v \in C_0^\infty(\Omega)$ with $\inf_B v > 0$ where B is a closed ball contained in D. Fixed $K > \frac{\|v\|^2}{2\int_\Omega v(x)^2 dx}$, by condition $iv)$, we find $\overline{\xi} > 0$ such that, for all $\xi \in (0, \overline{\xi}]$ one has

$$\inf_{x \in D} \int_0^\xi g_0(x,t)dt > K\xi^2. \tag{2.2}$$

Choose $\varepsilon > 0$ such that $\varepsilon \sup_\Omega v < \overline{\xi}$ and put $u_\varepsilon = \varepsilon v$. Then, we obtain

$$\begin{aligned}\Psi(u_\varepsilon) &= \frac{\varepsilon^2}{2}\|v\|^2 - \int_\Omega \left(\int_0^{\varepsilon v(x)} g_0(x,t)dt\right)dx \\ &\leq \varepsilon^2 \left(\frac{1}{2}\|v\|^2 - K\int_\Omega v(x)^2 dx\right) < 0\end{aligned}$$

and so (2.1) holds. At this point, we fix

$$t \in (\inf_{W_0^{1,2}(\Omega)} \Psi, 0). \tag{2.3}$$

Then, by coercivity of Ψ, the set $\Psi^{-1}((-\infty,0))$ is contained in a closed ball of $W_0^{1,2}(\Omega)$ and this latter, thanks to the reflexivity of $W_0^{1,2}(\Omega)$, turns out to be a weakly sequentially compact set. By Theorem 8.16 of [3], there exists a constant $C_0 = C_0(N,q,\Omega)$ such that, for each $h \in L^q(\Omega)$ and for each weak solution $u \in W_0^{1,2}(\Omega)$ of the equation $-\Delta u = h$ on Ω, one has $\|u\|_\infty \leq C_0\|h\|_q$. Now, fix $C > (aC_0)^{\frac{1}{2-s}} m(\Omega)^{\frac{1}{q(2-s)}}$, where $m(\Omega)$ is the Lebesgue measure of Ω, define

$$f_0(x,\xi) = \begin{cases} f(x,\xi) & \text{if} \quad (x,\xi) \in \Omega \times [0,C] \\ f(x,C) & \text{if} \quad (x,\xi) \in \Omega \times [C,+\infty) \\ 0 & \quad\text{otherwise} \end{cases}$$

and put $\Phi(u) = -\int_\Omega \left(\int_0^{u(x)} f_0(x,t)dt\right)dx$ for all $u \in W_0^{1,2}(\Omega)$. From condition $ii)$, f_0 is a Carathéodory function. Hence, by Rellich-Kondrachov Theorem it easily seen that Φ is sequentially weakly continuous and moreover, by condition $i)$ and Proposition 41.10 of [7], Φ turns out to be Gâteuaux differentiable on $W_0^{1,2}(\Omega)$. At this point, we are able to apply Theorem 2.1 of [5] to the functionals Ψ, Φ. Hence, we find a positive real number ρ^* such that, for all $\rho \geq \rho^*$, the restriction of functional the $\rho\Psi + \Phi$ to the set $\Psi^{-1}((-\infty,t))$ has a global minimum $u^{(\rho)}$ which, in turn, is a critical point of $\rho\Psi + \Phi$ belonging to $\Psi^{-1}((-\infty,t))$. Therefore, if we put $\overline{\lambda}_1 = \frac{1}{\rho^*}$, then for each $\lambda \in [0,\overline{\lambda}_1]$ there exists a critical point u_λ for the functional $\Psi + \lambda\Phi$ which belongs to the set $\Psi^{-1}((-\infty,t))$. So, in particular one has

$$\Psi(u_\lambda) < t < 0 \tag{2.3}$$

and thus u_λ is nonzero. Moreover, we have that u_λ is a weak solution of the problem

$$\begin{cases} -\Delta u = g_0(x,u) + \lambda f_0(x,u_\lambda) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where, thanks to condition $iii)$, one has

$$|g_0(x,t) + \lambda f_0(x, u_\lambda(x))| \leq \alpha(x)(1 + |t|)$$

for all $(x,t) \in \Omega \times \mathbb{R}$, with

$$\alpha(\cdot) := \left(\sup_{t \in \mathbb{R}} \frac{1 + |t|^{s-1}}{1 + |t|} \right) \left(a + \overline{\lambda}_1 \sup_{0 \leq \xi \leq C} |f_0(\cdot, \xi)| \right).$$

Observe that, by condition $i)$, $\alpha \in L^q(\Omega)$. Then, by Lemma B.3 of [6], we have $u_\lambda \in L^m(\Omega)$ for all $m \in [1, +\infty[$. Now, note that u_λ is a weak solution of the following linear Dirichlet problem

$$\begin{cases} -\Delta u = g_0(x, u_\lambda) + \lambda f_0(x, u_\lambda) & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \partial\Omega \end{cases}$$

where one has $g_0(\cdot, u_\lambda(\cdot)) + \lambda f_0(\cdot, u_\lambda(\cdot)) \in L^q(\Omega)$. Consequently, applying Theorem 8.16 and Theorem 8.30 of [3] we have $u_\lambda \in C^0(\overline{\Omega})$ and

$$\begin{aligned} \|u_\lambda\|_\infty &\leq C_0(\|g_0(\cdot, u_\lambda(\cdot))\|_q + \lambda \| \sup_{0 \leq \xi \leq C} |f(\cdot, \xi)| \|_q) \\ &\leq a C_0 m(\Omega)^{\frac{1}{q}} \|u_\lambda\|_\infty^{s-1} + \lambda C_0 \| \sup_{0 \leq \xi \leq C} |f(\cdot, \xi)| \|_q). \end{aligned} \tag{2.5}$$

Since $C > (aC_0)^{\frac{1}{2-s}} m(\Omega)^{\frac{1}{q(2-s)}}$, a simple calculation shows that, in view of from (2.5), there exists $\overline{\lambda} \in (0, \overline{\lambda}_1)$ such that, for all $\lambda \in [0, \overline{\lambda}]$ one has $\|u_\lambda\|_\infty < C$. Now, we claim that u_λ is nonnegative in Ω. Assume the contrary. Then, being $u_\lambda \in C^0(\overline{\Omega})$, the set $A := \{x \in \Omega : u_\lambda(x) < 0\}$ is nonempty and open. Further, being u_λ a critical point of $\Psi + \lambda \Phi$, for all $v \in C_0^\infty(A)$ we have

$$\int_A \nabla u_\lambda(x) \nabla v(x) dx - \int_A g_0(x, v_\lambda(x)) + \lambda f_0(x, v_\lambda(x))) v(x) dx = 0$$

from which

$$\int_A \nabla u_\lambda(x) \nabla v(x) dx = 0.$$

Since $u_\lambda \mid_A \in W_0^{1,2}(\Omega)$ and being $C_0^\infty(A)$ dense in $W_0^{1,2}(\Omega)$, by the previous equality, with $v = u_\lambda \mid_A$, we obtain $\int_A |\nabla u_\lambda(x)|^2 dx = 0$, which is absurd. The previous argument permits us to conclude that

$$0 \leq u_\lambda(x) \leq C \tag{2.6}$$

for all $x \in \Omega$. Consequently, for all $\lambda \in [0, \overline{\lambda}]$ we find a weak nonzero nonnegative solution v_λ of the Dirichlet problem

$$\begin{cases} -\Delta u = g(x,u) + \lambda f(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

Repeating the same arguments carried out up to now but considering the functionals Ψ and $-\Phi$ and choosing eventually $\overline{\lambda}$ sufficiently small, we have that the same above conclusion holds for every $\lambda \in [-\overline{\lambda}, 0[$. To conclude the proof, we observe that by Theorem 8.2' of [1], there exists a constant C_1 such that u_λ is a strong solution of (P_λ), $u_\lambda \in W^{2,q}(\Omega)$ and

$$\|u_\lambda\|_{W^{2,q}(\Omega)} \leq C_1(\|g(\cdot, u_\lambda(\cdot)) + \lambda f(\cdot, u_\lambda(\cdot))\|_q + \|u_\lambda\|_q).$$

Consequently, taking into account of (2.6), we have

$$\|u_\lambda\|_{W^{2,q}(\Omega)} \leq \sigma$$

with

$$\sigma = aC_1 m(\Omega)^{\frac{1}{q}} C^{s-1} + \overline{\lambda} C_1 \| \sup_{0 \leq \xi \leq C} |f(\cdot, \xi)| \|_q + C_1 m(\Omega)^{\frac{1}{q}} C$$

for all $\lambda \in [-\overline{\lambda}, \overline{\lambda}]$. ∎

Theorem 2.2 below gives the existence of a nonzero nonpositive solution to a problem (P_λ).

Theorem 2.2. *Let $s \in (1,2)$ $q > \frac{N}{2}$ and $a > 0$. Let $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ be two Carathéodory functions satisfying the following conditions*

(i) $\sup_{-r \leq \xi \leq 0} |f(\cdot, \xi)| \in L^q(\Omega)$ *for all* $r > 0$ *;*

(ii) $f(x, 0) = 0$ *for a.e.* $x \in \Omega$*;*

(iii) $|g(x, -t)| \leq a(t)^{s-1}$ *for all* $t \geq 0$ *and a.e* $x \in \Omega$.

(iv) *there exists a non empty open set* $D \subseteq \Omega$ *such that*

$$\liminf_{\xi \to 0^-} \frac{\inf_{x \in D} \int_0^\xi g(x,t)dt}{\xi^2} = +\infty$$

Then, there exist $\sigma, \overline{\lambda} > 0$ such that, for every $\lambda \in [-\overline{\lambda}, \overline{\lambda}]$, there exists a strong nonzero nonpositive solution $u_\lambda \in W_0^{1,2}(\Omega) \cap W^{2,q}(\Omega)$ of problem (P_λ) with $\|u_\lambda\|_{W^{2,q}(\Omega)} \leq \sigma$.

Proof. We apply Theorem 2.1 considering the functions $-g(x,-t)$, $-f(x,-t)$. Then, there exist $\sigma, \overline{\lambda} > 0$ such that, for every $\lambda \in [-\overline{\lambda}, \overline{\lambda}]$, there exists a nonzero nonnegative strong solution $v_\lambda \in W_0^{1,2}(\Omega) \cap W^{2,q}\Omega)$ of the problem

$$\begin{cases} -\Delta u = -g(x,-u) - \lambda f(x,-u) & \text{in} \quad \Omega \\ u = 0 & \text{in} \quad \partial\Omega \end{cases}$$

with $\|u_\lambda\|_{W^{2,q}\Omega} \leq \sigma$. Hence, it is enough to take $u_\lambda = -v_\lambda$. ∎

Clearly, combining Theorem 2.1 and Theorem 2.2 we obtain the following result

Theorem 2.3. *Let* $s \in (1,2)$ $q > \frac{N}{2}$ *and* $a > 0$. *Let* $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ *be two Carathéodory functions satisfying the following conditions*

(i) $\sup\limits_{|\xi| \leq r} |f(\cdot,\xi)| \in L^q(\Omega)$ *for all* $r > 0$ *;*

(ii) $f(x,0) = 0$ *for a.e.* $x \in \Omega$*;*

(iii) $|g(x,t)| \leq a|t|^{s-1}$ *for all* $t \in \mathbb{R}$ *and a.e* $x \in \Omega$*.*

(iv) *there exists a non empty open set* $D \subseteq \Omega$ *such that*

$$\liminf_{\xi \to 0} \frac{\inf_{x \in D} \int_0^\xi g(x,t)dt}{\xi^2} = +\infty$$

Then, there exist $\sigma, \overline{\lambda} > 0$ *such that, for every* $\lambda \in [-\overline{\lambda}, \overline{\lambda}]$, *there exist a strong nonzero nonnegative solution* $u_\lambda \in W_0^{1,2}(\Omega) \cap W^{2,q}(\Omega)$ *and a strong nonzero nonpositive solution* $v_\lambda \in W_0^{1,2}(\Omega) \cap W^{2,q}(\Omega)$ *of problem* (P_λ) *with* $\max\{\|u_\lambda\|_{W^{2,q}(\Omega)}, \|v_\lambda\|_{W^{2,q}(\Omega)}\} \leq \sigma$*.*

Among the existing results which give the same conclusion of Theorem 2.3 we stress Theorem 6.2 of [6]. We note that in the latter a subcritical growth condition is imposed on the nonlinearity. In our case, condition $i)$ permits us to consider nonlinearities without such a condition. In particular we see that condition $i)$ is less restrictive than assuming $f \in C^0(\overline{\Omega} \times \mathbb{R})$ (observe that, as previously said, this latter condition is required in [4]).

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