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WARPED PRODUCT SUBMANIFOLDS IN KENMOTSU SPACE FORMS

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Abstract. Recently, Chen established a general sharp inequality for warped products in real space forms. As applications, he obtained obstructions to minimal isometric immersions of warped products into real space forms. Afterwards, Matsumoto and one of the present authors proved the Sasakian version of this inequality.

In the present paper, we obtain sharp estimates for the warping function in terms of the mean curvature for warped products isometrically immersed in Kenmotsu space forms. Some applications are derived.

1. Introduction

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f a positive differentiable function on M_1. The warped product of M_1 and M_2 is the Riemannian manifold

$$M_1 \times_f M_2 = (M_1 \times M_2, g),$$

where $g = g_1 + f^2 g_2$ (see, for instance, [5]).

It is well-known that the notion of warped products plays some important role in Differential Geometry as well as in Physics. For a recent survey on warped products as Riemannian submanifolds, we refer to [4].

Let $x : M_1 \times_f M_2 \to \widetilde{M}(c)$ be an isometric immersion of a warped product $M_1 \times_f M_2$ into a Riemannian manifold $\widetilde{M}(c)$ with constant sectional curvature c. We denote by h the second fundamental form of x and $H_i = \frac{1}{n_i}\text{trace}\, h_i$, where

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trace h_i is the trace of h restricted to M_i and $n_i = \dim M_i$ $(i = 1, 2)$. We call H_i $(i = 1, 2)$ the partial mean curvature vectors.

The immersion x is said to be mixed totally geodesic if $h(X, Z) = 0$, for any vector fields X and Z tangent to M_1 and M_2 respectively.

In [5], Chen established the following sharp relationship between the warping function f of a warped product $M_1 \times_f M_2$ isometrically immersed in a real space form $\widetilde{M}(c)$ and the squared mean curvature $\|H\|^2$.

Theorem 1.1. *Let $x : M_1 \times_f M_2$ be an isometric immersion of an n-dimensional warped product into an m-dimensional Riemannian manifold $\widetilde{M}(c)$ of constant sectional curvature c. Then*:

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1 c, \tag{1.1}$$

where $n_i = \dim M_i, i = 1, 2$, and Δ is the Laplacian operator of M_1.

Moreover, the equality case of (1.1) *holds if and only if x is a mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where $H_i, i = 1, 2,$ are the partial mean curvature vectors.*

As applications, the author obtained necessary conditions for a warped product to admit a minimal isometric immersion in a Euclidean space or in a real space form (see [5]). Examples of submanifolds satisfying the equality case of (1.1) are given.

In the present paper, we establish corresponding inequalities for warped product submanifolds tangent to the structure vector field ξ into Kenmotsu space forms. Certain applications are derived.

2. Kenmotsu Manifolds and Their Submanifolds

Tanno [10] has classified, into 3 classes, the connected almost contact Riemannian manifolds whose automorphisms groups have the maximum dimensions:

(1) homogeneous normal contact Riemannian manifolds with constant ϕ -holomorphic sectional curvature;

(2) global Riemannian products of a line or circle and a Kaehlerian space form;

(3) warped product spaces $L \times_f F$, where L is a line and F a Kaehlerian manifold.

Kenmotsu [6] studied the third class and characterized it by tensor equations. Later, such a manifold was called a Kenmotsu manifold.

A $(2m+1)$-dimensional Riemannian manifold $(\tilde{M}, g)$ is said to be a Kenmotsu manifold if it admits an endomorphism ϕ of its tangent bundle $T\tilde{M}$, a vector field ξ and a 1-form η, which satisfy:

$$
\begin{cases}
\phi^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \\
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \\
(\widetilde{\nabla}_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \\
\widetilde{\nabla}_X \xi = X - \eta(X)\xi,
\end{cases}
\tag{2.1}
$$

for any vector fields X, Y on $\tilde{M}$, where $\tilde{\nabla}$ denotes the Riemannian connection with respect to g.

We denote by ω the fundamental 2-form of $\tilde{M}$, i.e. $\omega(X, Y) = g(\phi X, Y)$, $\forall X, Y \in \Gamma(T\tilde{M})$. It was proved that the pairing (ω, η) defines a locally conformal cosymplectic structure, i.e.

$$d\omega = 2\omega \wedge \eta, \quad d\eta = 0.$$

A Kenmotsu manifold with constant ϕ-holomorphic sectional curvature c is called a Kenmotsu space form and is denoted by $\tilde{M}(c)$. Then its curvature tensor $\tilde{R}$ is expressed by [6]

$$
\begin{aligned}
4\tilde{R}(X, Y)Z = {} & (c-3)\{g(Y, Z)X - g(X, Z)Y\} + (c+1)[\{\eta(X)Y \\
& -\eta(Y)X\}\eta(Z) + \{g(X, Z)\eta(Y) - g(Y, Z)\eta(X)\}\xi \\
& +\omega(Y, Z)\phi X - \omega(X, Z)\phi Y - 2\omega(X, Y)\phi Z].
\end{aligned}
\tag{2.2}
$$

Let $\tilde{M}$ be a Kenmotsu manifold and M an n-dimensional submanifold tangent to ξ.

For any vector field X tangent to M, we put

$$\phi X = PX + FX, \tag{2.3}$$

where PX (resp. FX) denotes the tangential (resp. normal) component of ϕX. Then P is an endomorphism of tangent bundle TM and F is a normal bundle valued 1-form on TM.

The equation of Gauss is given by

$$
\begin{aligned}
\tilde{R}(X, Y, Z, W) = {} & R(X, Y, Z, W) \\
& +g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),
\end{aligned}
\tag{2.4}
$$

for any vectors X, Y, Z, W tangent to M.

We denote by H the mean curvature vector, i.e.

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$

where $\{e_1, ..., e_n\}$ is an orthonormal basis of the tangent space $T_pM, p \in M$.

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r), \quad i, j = 1, ..., n; r = n+1, ..., 2m+1,$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

We denote by

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j).$$

By analogy with submanifolds in a Kaehler manifold, different classes of submanifolds in a Kenmotsu manifold were considered (see, for example, [11]).

A submanifold M tangent to ξ is said to be invariant (resp. anti-invariant) if $\phi(T_pM) \subset T_pM, \forall p \in M$ (resp. $\phi(T_pM) \subset T_p^{\perp}M, \forall p \in M$).

A submanifold M tangent to ξ is called a contact CR-submanifold [11] if there exists a pair of orthogonal differentiable distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ on M, such that:

(i) $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \{\xi\}$, where $\{\xi\}$ is the 1-dimensional distribution spanned by ξ;

(ii) $\mathcal{D}$ is invariant by ϕ, i.e. $\phi(\mathcal{D}_p) \subset \mathcal{D}_p, \forall p \in M$;

(iii) $\mathcal{D}^{\perp}$ is anti-invariant by ϕ, i.e. $\phi(\mathcal{D}_p^{\perp}) \subset T_p^{\perp}M, \forall p \in M$.

In particular, if $\mathcal{D}^{\perp} = \{0\}$ (resp. $\mathcal{D} = \{0\}$), M is an invariant (resp. anti-invariant) submanifold.

We recall the following result of Chen for later use.

Lemma [3]. *Let* $n \geq 2$ *and* $a_1, ..., a_n, b$ *real numbers such that*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right)$$

Then $2a_1a_2 \geq b$*, with equality holding if and only if*

$$a_1 + a_2 = a_3 = ... = a_n.$$

3. Warped Product Submanifolds

Chen established a sharp relationship between the warping function f of a warped product $M_1 \times_f M_2$ isometrically immersed in a real space form $\widetilde{M}(c)$ and the squared mean curvature $\|H\|^2$ (see [5]). For other results on warped product submanifolds in complex space forms we refer to [8]. Similar inequalities for warped product submanifolds of a Sasakian space form were proved in [7].

In the present paper, we establish corresponding inequalities for warped product submanifolds in Kenmotsu space forms. We investigate warped product submanifolds tangent to the structure vector field ξ in a Kenmotsu space form $\widetilde{M}(c)$.

We distinguish 2 cases:

(a) ξ is tangent to M_1;

(b) ξ is tangent to M_2.

Lemma 3.1. *Let $x : M_1 \times_f M_2$ be an isometric immersion of an n-dimensional warped product into a $(2m+1)$-dimensional Kenmotsu space form $\widetilde{M}(c)$, such that ξ is tangent to M_1. Then*:

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1\frac{c-3}{4} + \left(\frac{3}{n_2}\sum_{j=1}^{n_1}\sum_{t=n_1+1}^{n} g^2(Pe_j, e_t) - 1\right)\frac{c+1}{4}, \tag{3.1}$$

where $n_i = \dim M_i, i = 1, 2$, and Δ is the Laplacian operator on M_1.

Moreover, the equality case of (3.1) *holds if and only if x is a mixed totally geodesic immersion and $n_1 H_1 = n_2 H_2$, where $H_i, i = 1, 2$, are the partial mean curvature vectors.*

Proof. Let $M_1 \times_f M_2$ be a warped product submanifold into a Kenmotsu space form $\widetilde{M}(c)$ of constant ϕ-sectional curvature c, such that ξ is tangent to M_1.

Since $M_1 \times_f M_2$ is a warped product, it is easily seen that

$$\nabla_X Z = \nabla_Z X = \frac{1}{f}(Xf)Z, \tag{3.2}$$

for any vector fields X, Z tangent to M_1, M_2, respectively.

If X and Z are unit vector fields, it follows that the sectional curvature $K(X \wedge Z)$ of the plane section spanned by X and Z is given by

$$K(X \wedge Z) = g(\nabla_Z \nabla_X X - \nabla_X \nabla_Z X, Z) = \frac{1}{f}\{(\nabla_X X)f - X^2 f\}. \tag{3.3}$$

We choose a local orthonormal frame $\{e_1, ..., e_n, e_{n+1}, ..., e_{2m+1}\}$, such that $e_1, ..., e_{n_1}$ are tangent to M_1, $e_{n_1} = \xi$, $e_{n_1+1}, ..., e_n$ are tangent to M_2 and e_{n+1} is parallel to the mean curvature vector H.

Then, using (3.3), we get

$$\frac{\Delta f}{f}=\sum_{j=1}^{n_1}K(e_j\wedge e_s), \tag{3.4}$$

for each $s\in\{n_1+1,...,n\}$.

From the equation of Gauss, we have

$$n^2\|H\|^2=2\tau+\|h\|^2-n(n-1)\frac{c-3}{4}-(3\|P\|^2-2n+2)\frac{c+1}{4}, \tag{3.5}$$

where τ denotes the scalar curvature of $M_1\times_f M_2$, that is,

$$\tau=\sum_{1\le i<j\le n}K(e_i\wedge e_j).$$

We set

$$\delta=2\tau-n(n-1)\frac{c-3}{4}-(3\|P\|^2-2n+2)\frac{c+1}{4}-\frac{n^2}{2}\|H\|^2. \tag{3.6}$$

Then, (3.5) can be written as

$$n^2\|H\|^2=2(\delta+\|h\|^2). \tag{3.7}$$

With respect to the above orthonormal frame, (3.7) takes the following form:

$$\left(\sum_{i=1}^{n}h_{ii}^{n+1}\right)^2=2\left\{\delta+\sum_{i=1}^{n}(h_{ii}^{n+1})^2+\sum_{i\neq j}(h_{ij}^{n+1})^2+\sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\right\}.$$

If we put $a_1=h_{11}^{n+1}, a_2=\sum_{i=2}^{n_1}h_{ii}^{n+1}$ and $a_3=\sum_{t=n_1+1}^{n}h_{tt}^{n+1}$, the above equation becomes

$$\left(\sum_{i=1}^{3}a_i\right)^2=2\left\{\delta+\sum_{i=1}^{3}a_i^2+\sum_{1\le i\neq j\le n}(h_{ij}^{n+1})^2+\sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\right.$$

$$\left.-\sum_{2\le j\neq k\le n_1}h_{jj}^{n+1}h_{kk}^{n+1}-\sum_{n_1+1\le s\neq t\le n}h_{ss}^{n+1}h_{tt}^{n+1}\right\}.$$

Thus a_1,a_2,a_3 satisfy the Lemma of Chen (for $n=3$), i.e.

$$\left(\sum_{i=1}^{3}a_i\right)^2=2\left(b+\sum_{i=1}^{3}a_i^2\right),$$

with

$$b = \delta + \sum_{1\le i\ne j\le n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n} (h_{ij}^r)^2 - \sum_{2\le j\ne k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} - \sum_{n_1+1\le s\ne t\le n} h_{ss}^{n+1}h_{tt}^{n+1}.$$

Then $2a_1a_2 \ge b$, with equality holding if and only if $a_1 + a_2 = a_3$.

In the case under consideration, this means

$$\begin{aligned}&\sum_{1\le j<k\le n_1} h_{jj}^{n+1}h_{kk}^{n+1} + \sum_{n_1+1\le s<t\le n} h_{ss}^{n+1}h_{tt}^{n+1}\\ &\ge \frac{\delta}{2} + \sum_{1\le\alpha<\beta\le n} (h_{\alpha\beta}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{\alpha,\beta=1}^{n} (h_{\alpha\beta}^r)^2.\end{aligned} \tag{3.8}$$

Equality holds if and only if

$$\sum_{i=1}^{n_1} h_{ii}^{n+1} = \sum_{t=n_1+1}^{n} h_{tt}^{n+1}. \tag{3.9}$$

Using again Gauss equation, we have

$$n_2\frac{\Delta f}{f} = \tau - \textstyle\sum_{1\le j<k\le n_1} K(e_j\wedge e_k) - \sum_{n_1+1\le s<t\le n} K(e_s\wedge e_t)$$

$$\begin{aligned}&= \tau - \frac{n_1(n_1-1)(c-3)}{8}\\ &- \left[3\sum_{1\le j<k\le n_1-1} g^2(Pe_j,e_k) - n_1 + 1\right]\frac{c+1}{4}\\ &- \sum_{r=n+1}^{2m+1}\sum_{1\le j<k\le n_1} (h_{jj}^r h_{kk}^r - (h_{jk}^r)^2)\\ &\frac{n_2(n_2-1)(c+3)}{8} - \frac{3}{4}(c+1)\sum_{n_1+1\le s<t\le n} g^2(Pe_s,e_t)\\ &- \sum_{r=n+1}^{2m+1}\sum_{n_1+1\le s<t\le n} (h_{ss}^r h_{tt}^r - (h_{st}^r)^2).\end{aligned} \tag{3.10}$$

Combining (3.8) and (3.10), we obtain

$$
\begin{aligned}
n_2\frac{\Delta f}{f} \le\ & \tau - \frac{n(n-1)(c-3)}{8} + n_1 n_2\frac{c-3}{4} - \frac{\delta}{2} \\
& - \left[3\sum_{1\le j<k\le n_1-1} g^2(Pe_j,e_k) + 3\sum_{n_1+1\le s<t\le n} g^2(Pe_s,e_t) - n_1 + 1\right]\frac{c+1}{4} \\
& - \sum_{r=n+1}^{2m+1}\sum_{1\le j<k\le n_1}(h^r_{jj}h^r_{kk} - (h^r_{jk})^2) \\
& - \sum_{r=n+1}^{2m+1}\sum_{n_1+1\le s<t\le n}(h^r_{ss}h^r_{tt} - (h^r_{st})^2),
\end{aligned} \tag{3.11}
$$

or equivalently

$$
\begin{aligned}
n_2\frac{\Delta f}{f} \le\ & \tau - \frac{n(n-1)(c-3)}{8} + n_1 n_2\frac{c-3}{4} - \frac{\delta}{2} \\
& - \left[3\sum_{1\le j<k\le n_1-1} g^2(Pe_j,e_k) + 3\sum_{n_1+1\le s<t\le n} g^2(Pe_s,e_t) - n_1 + 1\right]\frac{c+1}{4} \\
& - \sum_{j=1}^{n_1}\sum_{t=n_1+1}^{n}(h^{n+1}_{jt})^2 - \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{\alpha,\beta=1}^{n}(h^r_{\alpha\beta})^2 \\
& + \sum_{r=n+2}^{2m+1}\sum_{1\le j<k\le n_1}((h^r_{jk})^2 - h^r_{jj}h^r_{kk}) + \sum_{r=n+2}^{2m+1}\sum_{n_1+1\le s<t\le n}((h^r_{st})^2 - h^r_{ss}h^r_{tt}) \\
=\ & \tau - \frac{n(n-1)(c-3)}{8} + n_1 n_2\frac{c-3}{4} - \frac{\delta}{2} \\
& - \left[3\sum_{1\le j<k\le n_1-1} g^2(Pe_j,e_k) + 3\sum_{n_1+1\le s<t\le n} g^2(Pe_s,e_t) - n_1 + 1\right]\frac{c+1}{4} \\
& - \sum_{r=n+1}^{2m+1}\sum_{j=1}^{n_1}\sum_{t=n_1+1}^{n}(h^r_{jt})^2 - \frac{1}{2}\sum_{r=n+2}^{2m+1}\left(\sum_{j=1}^{n_1}h^r_{jj}\right)^2 - \frac{1}{2}\sum_{r=n+2}^{2m+1}\left(\sum_{t=n_1+1}^{n}h^r_{tt}\right)^2 \\
\le\ & \tau - \frac{n(n-1)(c-3)}{8} + n_1 n_2\frac{c-3}{4} - \frac{\delta}{2} \\
& - \left[3\sum_{1\le j<k\le n_1-1} g^2(Pe_j,e_k) + 3\sum_{n_1+1\le s<t\le n} g^2(Pe_s,e_t) - n_1 + 1\right]\frac{c+1}{4}.
\end{aligned}
$$

Taking account of (3.4), one derives

$$n_2\frac{\Delta f}{f}\leq\frac{n^2}{4}\|H\|^2+n_1n_2\frac{c-3}{4}+\left(3\sum_{j=1}^{n_1}\sum_{t=n_1+1}^{n}g^2(Pe_j,e_t)-n_2\right)\frac{c+1}{4}, \tag{3.12}$$

which is the inequality to prove.

We see that the equality sign of (3.12) holds if and only if

$$h^r_{jt}=0,\quad 1\leq j\leq n_1, n_1+1\leq t\leq n, n+1\leq r\leq 2m, \tag{3.13}$$

and

$$\sum_{i=1}^{n_1}h^r_{ii}=\sum_{t=n_1+1}^{n}h^r_{tt}=0,\quad n+2\leq r\leq 2m. \tag{3.14}$$

Obviously (3.13) is equivalent to the mixed totally geodesy of the warped product $M_1\times_f M_2$ and (3.9) and (3.14) implies $n_1H_1=n_2H_2$.

The converse statement is straightforward. ■

We apply the above Lemma to Kenmotsu space forms having $c<-1$, $c=-1$ and $c>-1$, respectively.

Proposition 3.2. *Let $x: M_1\times_f M_2$ be an isometric immersion of an n-dimensional warped product into a $(2m+1)$-dimensional Kenmotsu space form $\widetilde{M}(c)$, with $c<-1$, such that ξ is tangent to M_1. Then*:

$$\frac{\Delta f}{f}\leq\frac{n^2}{4n_2}\|H\|^2+n_1\frac{c-3}{4}-\frac{c+1}{4}. \tag{3.15}$$

Moreover, the equality case of (3.15) *holds if and only if x is a mixed totally geodesic immersion, the partial mean curvature vectors satisfy $n_1H_1=n_2H_2$ and $\phi(TM_1)$ and TM_2 are orthogonal.*

Remark. On a contact CR-warped product submanifold $M_1\times_f M_2$, $\phi(TM_1)$ and TM_2 are orthogonal. The converse statement is not always true.

Proposition 3.3. *Let $x: M_1\times_f M_2$ be an isometric immersion of an n-dimensional warped product into a $(2m+1)$-dimensional Kenmotsu space form $\widetilde{M}(-1)$, such that ξ is tangent to M_1. Then*:

$$\frac{\Delta f}{f}\leq\frac{n^2}{4n_2}\|H\|^2-n_1. \tag{3.16}$$

Moreover, the equality case of (3.16) *holds if and only if* x *is a mixed totally geodesic immersion and the partial mean curvature vectors satisfy* $n_1 H_1 = n_2 H_2$.

Proposition 3.4. *Let* $x : M_1 \times_f M_2$ *be an isometric immersion of an* n-*dimensional warped product into a* $(2m+1)$-*dimensional Kenmotsu space form* $\widetilde{M}(c)$, *with* $c > -1$, *such that* ξ *is tangent to* M_1. *Then*:

$$\frac{\Delta f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1\frac{c-3}{4} + \left(\frac{3}{n_2}\|P\|^2 - 1\right)\frac{c+1}{4}. \tag{3.17}$$

Moreover, the equality case of (3.17) *holds if and only if* x *is a mixed totally geodesic immersion, the partial mean curvature vectors satisfy* $n_1 H_1 = n_2 H_2$ *and both* M_1 *and* M_2 *are anti-invariant submanifolds in* $\widetilde{M}(c)$.

As applications, we derive certain obstructions to the existence of minimal warped product submanifolds in Kenmotsu space forms.

Corollary 3.5. *Let* $M_1 \times_f M_2$ *be a warped product whose warping function* f *is harmonic. Then* $M_1 \times_f M_2$ *admits no minimal immersion into a Kenmotsu space form* $\widetilde{M}(c)$ *with* $c \leq -1$, *such that* ξ *be tangent to* M_1.

Proof. Assume f is a harmonic function on M_1 and $M_1 \times_f M_2$ admits a minimal immersion into a Kenmotsu space form $\widetilde{M}(c)$ with $c \leq -1$, such that ξ is tangent to M_1. Then, the inequalities (3.15) and (3.16) become impossible.

Corollary 3.6. *If the warping function* f *of a warped product* $M_1 \times_f M_2$ *is an eigenfunction of the Laplacian on* M_1 *with corresponding eigenvalue* $\lambda > 0$, *then* $M_1 \times_f M_2$ *does not admit a minimal immersion in a Kenmotsu space form* $\widetilde{M}(c)$ *with* $c \leq -1$, *such that* ξ *be tangent to* M_1.

Assume now that $M_1 \times_f M_2$ is a warped product submanifold of a Kenmotsu space form $\widetilde{M}(c)$ such that ξ is tangent to M_2.

If we put $Z = \xi$ in (3.2), the last equation (2.1) leads to a contradiction. Thus we may state the following.

Proposition 3.7. *There do not exist warped product submanifolds* $M_1 \times_f M_2$ *in a Kenmotsu space form* $\widetilde{M}(c)$ *such that* ξ *is tangent to* M_2.

References

1. K. Arslan, R. Ezentas, I. Mihai, C. Murathan and C. Ozgur, Ricci curvature of submanifolds in Kenmotsu space forms, *Int. J. Math. Math. Sci.*, **29** (2002), 719-726.

2. D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math. 509, Springer, Berlin, 1976.

3. B. Y. Chen, *Some pinching and classification theorems for minimal submanifolds*, Arch. Math. **60** (1993), 568-578.

4. B. Y. Chen, Geometry of warped products as Riemannian submanifolds and related problems, *Soochow J. Math.*, **28** (2002), 125-156.

5. B. Y. Chen, On isometric minimal immersions from warped products into real space forms, *Proc. Edinburgh Math. Soc.*, **45** (2002), 579-587.

6. K. Kenmotsu, A class of almost contact Riemannian manifolds, *Tohoku Math. J.*, **24** (1972), 93-103.

7. K. Matsumoto and I. Mihai, Warped product submanifolds in Sasakian space forms, *SUT J. Math.*, **38** (2002), 135-144.

8. A. Mihai, Warped product submanifolds in complex space forms. *Acta Sci. Math. (Szeged)*, **70** (2004), 419-427.

9. S. Nölker, Isometric immersions of warped products, *Differential Geom. Appl.*, **6** (1996), 1-30.

10. S. Tanno, The automorphisms groups of almost contact metric manifolds, *Tohoku Math. J.*, **21** (1969), 21-38.

11. K. Yano and M. Kon, Structures on Manifolds, World Scientific, Singapore, 1984.

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