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TOPOLOGY INDUCED BY QUOTIENTS ON NEARRING MODULES

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Abstract. The aim of this paper is to study one special quotient in the class of nearring modules. Contrast to the well known noetherian quotient, this quotient is less noticed. Results and applications for this quotient are developed and an induced topology in the class of type-1 N-modules is introduced.

1. Introduction

Let G and N be two algebraic objects with N acting on G from the right hand side. This is usually denoted as $G \times N \to G$ with $an = b$ for $a, b \in G$ and $n \in N$. Two linear equations, say $ax = b$ for given $a, b \in G$ and $yn = b$ for given $n \in N$, $b \in G$, can be informally imposed. To solve the equation $ax = b$ on N, the noetherian quotient $(b : a)$ is introduced to denote the solution set in N. As the other equation $yn = b$, the quotient $[b : n]$ will be used to denote the solution set in G [7, p. 41].

Let $(G, +)$ be a group (not necessarily abelian) and $(N, +, \cdot)$ a left nearring. We call G a (right) N-module if $a(f + g) = af + ag$ and $a(fg) = (af)g$ for all $a \in G$ and all $f, g \in N$. Given subsets $X, Y \subseteq G$ and $U \subseteq N$. Two quotients can be defined as following:

$$(X : Y) := \{f \in N \mid af \in X \text{ for all } a \in Y\} \text{ and}$$
$$[X : U] := \{a \in G \mid af \in X \text{ for all } f \in U\}.$$

The first one is the usual noetherian quotient commonly used in the class of ring modules. The other while less studied is first defined in [7, p. 41] and will be called in this paper as the *inverse quotient of* X *by* U. When $X = \{0\}$, we write

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$(0 : Y)$ and $[0 : U]$ for $(\{0\} : Y)$ and $[\{0\} : U]$ respectively. The set $(0 : Y)$ is the *annihilator* of Y in N and is sometimes denoted as $\mathrm{Ann}(Y)$ or $\mathrm{Ann}_N(Y)$ in literatures. The set $[0 : U]$ is denoted as $Z(U)$ and called the *zero set* of U in G by Scott [10]. In fact, it is called an *affine algebraic variety* if we consider the polynomial ring $F[x_1, x_2, \cdots, x_n]$ acting on F^n where F is an algebraicly closed fields. Note that the set $[0 : U]$ had been used by Johnson [4, 5] to characterize the maximal right ideals in the class of transformation nearrings.

A topology had been introduced by Scott in [10] using zero sets $[0 : U]$ for 2-primitive nearrings and further studied the applications of this topology in the class of primary N-modules [11]. The attemp of this paper is to introduce a topology using inverse quotient in the class of N-modules. It is shown in this paper that: *Given N a 1-primitive nearring on a group G. If G is a nonabelian group or N_0 a nonring, then the collection of sets $G \setminus [0 : U]$ for all $U \subseteq N$ forms a topology in G (by Corollary 2.2 and Corollary 3.4), and all the mappings of G induced by elements in N are continuous with respect to this topology (Theorem 3.8).* Examples are given to demonstrate and delimit our results. For other terminologies on nearring theory not defined in this paper, please refer to [7, 9] but note that [9] using right nearrings instead of left nearrings. Terminologies on topology are referred to Munkres [8].

2. Inverse Quotient

Let G be a (right) N-module. If X is a subset of G and U is a subset of N, the following convention will be made for the inverse quotient of X by U throughout this paper. If $U = \{0\}$, we write $[X : 0]$ in place of $[X : \{0\}]$. A similar simplification will be used if X or U is a singleton. When X or U is an empty set, we make the following convention that $[X : \emptyset] = G$; $[\emptyset : U] = \emptyset$ and leave $[\emptyset : \emptyset]$ undefined.

Proposition 2.1. *Assume G is an N-module. Let X, U be subsets of G and N respectively. Suppose $\{X_i\}_{i\in I}$ is a collection of subsets of G and $\{U_j\}_{j\in J}$ is a collection of subsets of N. Then we have:*

(1) $\bigcap_{j\in J}[X : U_j] = [X : \bigcup_{j\in J} U_j]$. *In particular,* $[X : U] = \bigcap_{f\in U}[X : f]$.

(2) $\bigcup_{j\in J}[X : U_j] \subseteq [X : \bigcap_{j\in J} U_j]$. *If $\{U_j\}_{j\in J}$ is a chain by using set inclusion, then* $\bigcup_{j\in J}[X : U_j] = [X : \bigcap_{j\in J} U_j]$.

(3) $\bigcap_{i\in I}[X_i : U] = [\bigcap_{i\in I} X_i : U]$.

(4) $\bigcup_{i\in I}[X_i : U] \subseteq [\bigcup_{i\in I} X_i : U]$. *If $\{X_i\}_{i\in I}$ is a chain by using set inclusion, then* $\bigcup_{i\in I}[X_i : U] = [\bigcup_{i\in I} X_i : U]$.

Proof. (1) Observe that

$$\begin{aligned}[X : \bigcup_{j\in J} U_j] &= \{a \in G \mid af \in X \text{ for all } f \in \bigcup_{j\in J} U_j\} \\ &= \bigcap_{j\in J}\{a \in G \mid af \in X \text{ for all } f \in U_j\} = \bigcap_{j\in J}[X : U_j]\,.\end{aligned}$$

(2) If $\bigcap_{j\in J} U_j = \emptyset$, then it is clear that

$$G = \left[X : \bigcap_{j\in J} U_j\right] \supseteq \bigcup_{j\in J}[X : U_j]\,.$$

If $\bigcap_{j\in J} U_j \neq \emptyset$, then

$$\begin{aligned}\bigcup_{j\in J}[X : U_j] &= \bigcup_{j\in J}\{a \in G \mid af \in X \text{ for all } f \in U_j\} \\ &\subseteq \{a \in G \mid af \in X \text{ for all } f \in \bigcap_{j\in J} U_j\} = \left[X : \bigcap_{j\in J} U_j\right].\end{aligned}$$

It is clear that if $\{U_j\}_{j\in J}$ is a chain then the above inclusion becomes equality.

(3) If $\bigcap_{i\in I} X_i = \emptyset$, it is clear by definition that our assertion holds. We now suppose $\bigcap_{i\in I} X_i \neq \emptyset$.

$$\begin{aligned}\left[\bigcap_{i\in I} X_i : U\right] &= \{a \in G \mid af \in \bigcap_{i\in I} X_i \text{ for all } f \in U\} \\ &= \bigcap_{i\in I}\{a \in G \mid af \in X_i \text{ for all } f \in U\} = \bigcap_{i\in I}[X_i : U]\,.\end{aligned}$$

(4) Observe that

$$\begin{aligned}\bigcup_{i\in I}[X_i : U] &= \bigcup_{i\in I}\{a \in G \mid af \in X_i \text{ for all } f \in U\} \\ &\subseteq \{a \in G \mid af \in \bigcup_{i\in I} X_i \text{ for all } f \in U\} = \left[\bigcup_{i\in I} X_i : U\right].\end{aligned}$$

It is clear that if $\{X_i\}_{i\in I}$ is a chain then the above inclusion becomes equality. ■

Let X be a fixed subset of G. A subset Y of G will be called *X-closed* if there exists a subset U of N such that $[X : U] = Y$. Since $G = [X : \emptyset]$, the group

G is considered as an X-closed set for all specified nonempty set X. Hereafter, the empty subset $\emptyset$ of G shall also be regarded as an X-closed set. The following results are immediate from Proposition 2.1.

Corollary 2.2. *Any intersection of X-closed subsets of an N-module G is X-closed.*

Corollary 2.3. *Let U_1, U_2 be subsets of N and X a subset of G. If $U_1 \subseteq U_2$, then $[X : U_1] \supseteq [X : U_2]$.*

Proposition 2.4. *Let X, Y be nonempty subsets of an N-module G. Then $Y \subseteq [X : (X : Y)]$. Furthermore, $Y = [X : (X : Y)]$ if and only if Y is X-closed.*

Proof. If $(X : Y)$ is empty, then it is clear that $Y \subseteq [X : (X : Y)] = G$. If $(X : Y)$ is not empty, then $Y(X : Y) \subseteq X$ and so $Y \subseteq [X : (X : Y)]$.

Suppose $Y = [X : (X : Y)]$, then Y is clearly X-closed. On the other hand, suppose that Y is X-closed, say $Y = [X : U]$ for some subset U of N. If U is empty then $Y = G$, so we have $[X : (X : Y)] \subseteq Y$. If U is not empty, then $YU \subseteq X$ and so $U \subseteq (X : Y)$. Corollary 2.3 then implies that $[X : (X : Y)] \subseteq [X : U] = Y$. Hence $[X : (X : Y)] = Y$. ∎

Corollary 2.5. *Let X, W be subsets of an N-module G. Suppose Y is an X-closed subset of G. Then we have $(X : Y) \subseteq (X : W)$ if and only if $W \subseteq Y$.*

Proof. The necessary part is trivial. We need to show that it is sufficient. From Corollary 2.3, it follows that $[X : (X : W)] \subseteq [X : (X : Y)]$. Therefore $W \subseteq [X : (X : W)] \subseteq [X : (X : Y)] = Y$ by Proposition 2.4. ∎

The assumption that Y is X-closed in G in Corollary 2.5 is not superfluous. For instance, let $G = (\mathbb{Z}_3, +) = (\{0, 1, 2\}, +)$ be the group of order three written additively and $N = M_c(G) = \{\theta_0, \theta_1, \theta_2\}$ the nearring of constant mappings on G. Let $X = \{0\}$, $Y = \{1\}$, $W = \{2\}$ be subsets of $\mathbb{Z}_3$. Observe that $\mathbb{Z}_3$ and $\emptyset$ are the only 0-closed sets, and thus Y is not X-closed in this case. However, $(X : Y) = (0 : 1) = \{\theta_0\} \subseteq (X : W) = (0 : 2) = \{\theta_0\}$ but $\{2\} \not\subseteq \{1\}$.

Corollary 2.6. *Let X be a subset of an N-module G. Suppose Y_1 and Y_2 are X-closed subsets of G. Then $Y_1 = Y_2$ if and only if $(X : Y_1) = (X : Y_2)$.*

Proof. This follows immediately from Corollary 2.5. ∎

Corollary 2.6 reveals that there is a one-one correspondence between X-closed subsets of G and their noetherian quotient with respect to X. This correspondence is lattice reversing as indicated in Corollary 2.5.

It is also interesting to see that most of the conclusions above have "dual" results when the role of noetherian quotient and inverse quotient interchanged. We put some of them in the following proposition.

Proposition 2.7. *Let X, Y_1, Y_2 be nonempty subsets of an N-module G and U, U_1, U_2 nonempty subsets of N.*

(1) *If $Y_1 \subseteq Y_2$, then $(X : Y_1) \supseteq (X : Y_2)$.*

(2) *The set $U \subseteq (X : [X : U])$. Furthermore, $U = (X : [X : U])$ if and only if $U = (X : Y)$ for some $Y \subseteq G$.*

(3) *Assume $U_1 = (X : Y)$ for some $Y \subseteq G$. Then $[X : U_1] \subseteq [X : U_2]$ if and only if $U_2 \subseteq U_1$.*

(4) *Suppose $U_1 = (X : Y_1)$ and $U_2 = (X : Y_2)$. Then $U_1 = U_2$ if and only if $[X : U_1] = [X : U_2]$.*

Proof.

(1) This result follows from the identity that $\cap_{j \in J}(X : Y_j) = (X : \cup_{j \in J} Y_j)$.

(2) It suffices to show the case when $[X : U] \neq \emptyset$. Observe that $[X : U]U \subseteq X$, and thus $U \subseteq (X : [X : U])$.

Furtheremore, if $U = (X : [X : U])$, then simply let $Y = [X : U] \subseteq G$. On the other hand, assume $U = (X : Y)$ for some $Y \subseteq G$. If $Y = \emptyset$, then $U = N$ and thus $(X : [X : U]) \subseteq U$. If $Y \neq \emptyset$, then $YU \subseteq X$ and so $Y \subseteq [X : U]$. Therefore $(X : [X : U]) \subseteq (X : Y) = U$ by (1). Hence result.

(3) Assume $[X : U_1] \subseteq [X : U_2]$ where $U_1 = (X : Y)$ for some $Y \subseteq G$. Observe that $(X : [X : U_1]) \supseteq (X : [X : U_2])$ by (1) and $U_2 \subseteq (X : [X : U_2])$ by (2). Therefore $U_2 \subseteq (X : [X : U_2]) \subseteq (X : [X : U_1]) = U_1$ by (2) again.

On the other hand, assume $U_2 \subseteq U_1$. Let $a \in [X : U_1]$. Then $a\mu \in X$ for all $\mu \in U_1$. Consequently $a\mu \in X$ for all $\mu \in U_2$ or $a \in [X : U_2]$. Therefore $[X : U_1] \subseteq [X : U_2]$.

(4) This result follows immediately from (3). ■

Proposition 2.8. *Let X, Y be subsets of an N-module G and $f \in N$. If Y is X-closed, then $[Y : f] = [X : f(X : Y)]$.*

Proof. If $[Y : f] = \emptyset$, then clearly $[Y : f] \subseteq [X : f(X : Y)]$. If $(X : Y) = \emptyset$, then $[X : f(X : Y)] = G$ and so $[Y : f] \subseteq [X : f(X : Y)]$. Suppose that both $[Y : f] \neq \emptyset$ and $(X : Y) \neq \emptyset$. Since $[Y : f]f \subseteq Y$, we have $[Y : f]f(X : Y) \subseteq Y(X : Y) \subseteq X$. Therefore $[Y : f] \subseteq [X : f(X : Y)]$.

On the other hand, if $[X : f(X : Y)] = \emptyset$, then clearly $[X : f(X : Y)] \subseteq [Y : f]$. Suppose that $[X : f(X : Y)] \neq \emptyset$ and $v \in [X : f(X : Y)]$. Then $vf(X : Y) \in X$ and so $(X : Y) \subseteq (X : vf)$. Thus $vf \in Y$ by Corollary 2.5 and hence $v \in [Y : f]$ or $[X : f(X : Y)] \subseteq [Y : f]$. Hence result. ∎

Let N be a nearring, G an N-module. A subgroup H of G such that $af \in H$ for all $a \in H$, $f \in N$ is called an *N-submodule* of G. A nearring N is called a *distributively generated nearring* (abbre. *d. g. nearring*) if $(N, +)$ is generated as a group by a semigroup $(S, \cdot)$ of distributive elements in N. This d. g. nearring is generally denoted (N, S) or $N[S]$. An N-module G is called a *d. g. module* or $N[S]$-*module* provided that $N[S]$ is a d. g. nearring and that $(a+b)s = as+bs$ for all $a, b \in G$ and all $s \in S$. In general, there is no particular algebraic structure for inverse quotient in an N-module. However, some can be said in an $N[S]$-module.

Proposition 2.9. *Suppose G is an $N[S]$-module. Let K be a subset of G and U a subset of S.*

(1) *If K is a subgroup of G and $f \in S$, then $[K : f]$ is a subgroup of G.*

(2) *If K is a subgroup of G, then $[K : U]$ is a subgroup of G.*

(3) *If K is an N-submodule of G and U is contained in the multiplicative center of the semigroup $(S, \cdot)$, then $[K : U]$ is an N-submodule of G.*

(4) *If K is a subgroup of G and $SU \subseteq U$, then $[K : U]$ is an N-submodule of G. In particular, $[K : S]$ is an N-submodule of G for any subgroup K of G.*

Proof.

(1) Since $f \in S$, f induces an endomorphism on G. By corresponding theorem for groups and our assumption that K is a group, it is routine to see that $[K : f]$ is a subgroup of G.

(2) From Proposition 2.1 and (1) above.

(3) From (2), we know that $[K : U]$ is a subgroup of G. We need to show that $[K : U]f \subseteq [K : U]$ for all $f \in N$. Let $a \in [K : U]$. Then $a\alpha \in K$ for all $\alpha \in U$. Now $(af)\alpha = (a\alpha)f \in Kf \subseteq K$, so $af \in [K : U]$ for all $f \in N$. Therefore $[K : U]$ is an N-submodule of G.

(4) Let $a \in [K : U]$, $\alpha \in U$ and $f \in N$. Since N is d. g., we can write f as $f = \sum_{i=1}^{n} \varepsilon_i \alpha_i$ where $\varepsilon_i \in \{1, -1\}$ and $\alpha_i \in S$ for $i \in \{1, 2, \cdots, n\}$. Now we have

$$(af)\alpha = \left(a \sum_{i=1}^{n} \varepsilon_i \alpha_i\right) \alpha = \sum_{i=1}^{n} \varepsilon_i (a\alpha_i)\alpha = \sum_{i=1}^{n} \varepsilon_i a(\alpha_i \alpha) \in K.$$

Therefore $af \in [K : U]$ and so $[K : U]$ is an N-submodule of G. ∎

3. Topology Induced by Inverse Quotient

The purpose of this section is to introduce a topology on an N-module G using X-closed sets. Let N be a nearring, G an N-module. A normal subgroup K of G such that $(k+m)n - mn \in K$ for all $k \in K$; $m, n \in N$ is called an ***N*-ideal** of G. An N-ideal K of an N-module G is called $abelian$, if $(K, +)$ is an abelian group and $(x+y)\alpha = x\alpha + y\alpha$ for all $x, y \in K$ and for all $\alpha \in N_0$ where N_0 is the zero-symmetric subnearring of N.

The following lemma is modified from [10, Proposition 2.1] and detailed proof will be included for easier reference.

Lemma 3.1. *Let N be a nearring, G an N-module. Assume H_1, H_2 and W are N-ideals of G such that G is a direct sum of N-ideals $H_1 \oplus H_2$. If H_2 is minimal and nonabelian, then either $W \subseteq H_1$ or $H_2 \subseteq W$.*

Proof. Assume on the contrary that $W \not\subseteq H_1$ and $H_2 \not\subseteq W$. Since H_1 is a maximal N-ideal of G, $H_1 + W = G$. Notice that $kn = (k+0)n - 0n \in K$ for all k in an arbitrary N-ideal K and for all $n \in N_0$. It follows that any N-ideal K is both an N_0-submodule and N_0-ideal of G. By Wielandt's Lemma [7, Lemma 3.22] and the minimality of H_2, the N-ideal

$$[(H_1 + W) \cap (H_2 + W)]/[(H_1 \cap H_2) + W] \simeq (H_2 + W)/W \simeq H_2$$

is abelian, contradicting the hypothesis that H_2 is nonabelian. Hence result. ■

An N-module G is called $monogenic$ if there exists $g \in G$ such that $gN = G$. A monogenic N-module G is said to be ***type-0*** if G has no nontrivial proper N-ideals. A monogenic N-module G is called ***strongly monogenic*** if $gN = 0$ or $gN = G$ for all $g \in G$. A strongly monogenic nontrivial N-module G is said to be of ***type-1*** if G has no nontrivial proper N-ideals. It is said to be of ***type-2*** if G has no nontrivial proper N_0-ideals. A nearring N is called ***ν-primitive*** on G if G is a faithful type-ν N-module. A nearring which is not a ring is called a ***nonring***.

Lemma 3.2. *Let Y_1, Y_2 be 0-closed subsets of a type-1 N-module G. If one of the following conditions holds, then $Y_1 \cup Y_2$ is 0-closed in G.*

(1) *G is a nonabelian group.*

(2) *$N_0/(0 : G)$ is a nonring.*

Proof. Without lost of generality, we may suppose that both Y_1 and Y_2 are not empty. Let U_1 and U_2 be subsets of N such that $Y_1 = [0 : U_1]$ and $Y_2 = [0 : U_2]$. Let $U = (0 : Y_1 \cup Y_2)$. From Proposition 2.4, we have

$$Y_1 \cup Y_2 \subseteq [0 : (0 : Y_1 \cup Y_2)] = [0 : U].$$

Let $a \in [0 : U]$. As $a\mu = 0$ for all $\mu \in U$, we have $U \subseteq (0 : a)$. If $(0 : Y_1) \subseteq (0 : a)$, then we have $a \in Y_1$ by Corollary 2.5. Thus $[0 : U] \subseteq Y_1$. It follows that $[0 : U] = Y_1 \cup Y_2$ and $Y_1 \cup Y_2$ is 0-closed.

We may now assume that $(0 : Y_1) \not\subseteq (0 : a)$. Recall that Y_1 is not empty. If $a = 0$ then $(0 : Y_1) \not\subseteq (0 : 0) = N_0$ implies the existence of a nonzero constant map $\theta_c \in N$ such that $Y_1\theta_c = 0$, a contradiction. So $a \neq 0$. Since G is strongly monogenic, $aN = 0$ or $aN = G$. If $aN = 0$ then $(0 : a) = N$, consequently, $(0 : Y_1) \subseteq (0 : a)$, a contradiction. If $aN = G$ then $N/(0 : a) \simeq G$ as a simple N-module. It follows that $(0 : a)$ is a maximal right ideal of N, and so $N = (0 : a) + (0 : Y_1)$. Let $M = (0 : a) \cap (0 : Y_1)$. Then N/M can be written as a direct sum as $N/M = (0 : a)/M \oplus (0 : Y_1)/M$, and so $(0 : Y_1)/M$ is a minimal N-ideal of N/M. Now we have

$$(0 : Y_1)/M \simeq_N (N/M)/((0 : a)/M) \simeq_N N/(0 : a) \simeq_N aN = G.$$

From the hypothesis, if G is a nonabelian group, then clearly $(0 : Y_1)/M$ is nonabelian. Note that $(0 : G)$ is zero-symmetric and is an ideal in N_0. It is then routine to see that G is a faithful $N_0/(0 : G)$-module. If $N_0/(0 : G)$ is a nonring, then from [7, Lemma 3.7.], we can conclude that $(0 : Y_1)/M$ is nonabelian either. Hence $(0 : Y_1)/M$ is a nonabelian minimal N-ideal of N/M.

By Lemma 3.1, we have either

$$(0 : Y_1)/M \subseteq (M + (0 : Y_2))/M$$

or

$$(M + (0 : Y_2))/M \subseteq (0 : a)/M.$$

This implies that

$$\text{either } (0 : Y_1) \subseteq M + (0 : Y_2) \text{ or } M + (0 : Y_2) \subseteq (0 : a).$$

If $(0 : Y_1) \subseteq M + (0 : Y_2) = (0 : a) \cap (0 : Y_1) + (0 : Y_2)$, then

$$(0 : Y_1) = (0 : a) \cap (0 : Y_1) + (0 : Y_2) \cap (0 : Y_1).$$

Moreover, $U = (0 : Y_1 \cup Y_2) = (0 : Y_1) \cap (0 : Y_2)$, and $U \subseteq (0 : a)$, we thus have $(0 : Y_1) \subseteq (0 : a)$. It follows that $a \in Y_1$ by Corollary 2.5. On the other hand, if $M + (0 : Y_2) \subseteq (0 : a)$, it is clear that $(0 : Y_2) \subseteq (0 : a)$ or $a \in Y_2$ by Corollary 2.5. Therefore $[0 : U] \subseteq Y_1 \cup Y_2$. Hence $Y_1 \cup Y_2 = [0 : U]$ is 0-closed in G. ■

The following example shows that assuming G is a type-1 N-module is not superfluous in Lemma 2.2.

Example 3.3. There exists an N-module G (not type-1) such that G is a nonabelian group and $N= N_0$ is a nonring but the union of 0-closed sets is not 0-closed.

Let $G = S_3 = \{0, a, b, c, d, e\}$ be the symmetric group of degree 3, $N = E(S_3)$ the nearring generated additively by the group endomorphisms of S_3. This nearring is of order 54 and a detailed description of all its elements is in [6, p. 70]. Consider S_3 as an $E(S_3)$-module. The alternating group $A_3 = \{0, d, e\}$ is an $E(S_3)$-ideal of S_3. Thus the $E(S_3)$-module S_3 is not type-1. All the 0-closed sets are $\emptyset$, $\{0\}$, $\{0, a\}$, $\{0, b\}$, $\{0, c\}$, $\{0, d, e\}$, $\{0, a, b, c\}$, $\{0, a, d, e\}$, $\{0, c, d, e\}$, $\{0, b, d, e\}$ and S_3. The set $\{0, a, b\} = \{0, a\} \cup \{0, b\}$ is a union of 0-closed sets but it is not 0-closed.

The following corollary is a generalization of Scott's result [10, Theorem 2.2] on 2-primitive nearrings.

Corollary 3.4. *Let N be a 1-primitive nearring on a group G. If G is a nonabelian group or N_0 is a nonring, then the union of 0-closed subsets in G is 0-closed.*

Lemma 3.5. *Let G be an N-module and K an N-ideal of G. Let $\rho\colon G \to G/K$ be the canonical N-homomorphism. Then there is a one-one correspondence between the*

Proof. Let Y be a nonempty K-closed subset of G, and let $U \subseteq N$ such that $Y = [K : U]$. Clearly $Y\rho \subseteq [0_{G/K} : U]$. If $a + K \in [0_{G/K} : U]$, then $(a + K)\mu = 0_{G/K}$ or $a\mu \in K$ for all $\mu \in U$. That is $a \in Y$ and $a + K \in Y\rho$. Hence $Y\rho = [0_{G/K} : U]$ and $Y\rho$ is 0-closed in the N-module G/K. On the other hand, let W be a nonempty 0-closed subset of G/K over N, say $W = [0_{G/K} : U]$ for some $U \subseteq N$. Claim that $W\rho^{-1} = [K : U]$. Since K is an N-ideal, we have $(g + k)n - gn \in K$ for all $k \in K$, $g \in G$, $n \in N$.

Now let $a \in W\rho^{-1}$ and $\mu \in U$. From our assumption, we have

$$a\mu + K = (a + K)\mu = (a\rho)\mu \in WU = 0_{G/K}.$$

Therefore $a\mu \in K$ and $a \in [K : U]$. Hence $W\rho^{-1} \subseteq [K : U]$.

If $a \in [K : U]$, then $a\mu \in K$ for all $\mu \in U$ and so we get $(a+K)\mu = a\mu+K = K$ or $a\rho \in [0_{G/K} : U] = W$. Hence $a \in W\rho^{-1}$ and $[K : U] \subseteq W\rho^{-1}$. ∎

Theorem 3.6. *Let G be an N-module and K an N-ideal of G such that G/K is a type-1 N-module. If G/K is a nonabelian group or $N_0/((K : G) \cap N_0)$ is a nonring, then the finite union of K-closed subsets of N-module G are K-closed.*

Proof. This follows immediately from Lemma 3.2 and Lemma 3.5 and the fact that $(0_{G/K} : G/K) = (K : G)$. ■

From Corollary 2.2 and Theorem 3.6, a topology exists for a certain type-1 N-module G with respect to a particular N-ideal K. This topology is referred as *inverse* K*-topology* in this paper. However, this topology is usually non-Hausdoff [8, p. 98] as demonstrated in the following example.

Example 3.7. Let $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ and $N = \{f \in M_0(G) \mid 3f = 0\}$. This group G is a faithful type-2 N-module [7, Example 3.36]. The nearring N is a 0-symmetric nonring of order 16. Hence the collection of all 0-closed sets in G forms a topology τ. There are five 0-closed sets in this topology, namely, $\mathbb{Z}_4$, $\{0, 1, 3\}$, $\{0, 2, 3\}$, $\{0, 3\}$, $\emptyset$. Therefore the collection of all open sets is $\tau = \{\emptyset, \{2\}, \{1\}, \{1, 2\}, \mathbb{Z}_4\}$. This topology is not Hausdoff for the singletons $\{0\}$ and $\{3\}$ can not be seperated by open sets.

Proposition 2.7 shows that result related to inverse quotient often has a dual one in noetherian quotient and vice versa. In this example, the collection of all sets in the form $(0 : Y)$ for some $Y \subseteq G$ are: $N = (0 : 0)$; $\{(0000), (0010), (0020), (0030)\} = (0 : 1)$; $\{(0000), (0100), (0200), (0300)\} = (0 : 2)$; $\{(0000)\} = (0 : \{1, 2\})$. Here the notation $(abcd)$ represents the mapping in N which maps $0, 1, 2, 3$ to a, b, c, d respectively. The union $(0 : 1) \cup (0 : 2)$ is a set of order 7 which cannot be the annihilator of any subsets of G. Therefore a possible dual for inverse 0-topology does not exist in this case.

The following result can be expected from the definition of K-closed set.

Theorem 3.8. *Let G be an N-module and K an N-ideal of G such that G/K is a type-1 N-module. If G/K is a nonabelian group or $N_0/((K : G) \cap N_0)$ is a nonring then the maps induced by the elements of N are continuous from G to G with respect to*

Proof. Let Y be a K-closed subset of G and $f \in N$. From Proposition 2.8, it follows that $Yf^{-1} = [Y : f] = [K : f(K : Y)]$ which is clearly K-closed in G and so f is continuous. ■

As indicated in Proposition 2.1(1), any K-closed set $[K : U]$ can be generated as the intersection of K-closed sets $[K : f]$ for all $f \in U$. In other words, arbitrary open sets in the inverse K-topology can be generated by union of open sets of the form $G \setminus [K : f]$ for $f \in N$. Explicitly,

$$G \setminus [K : U] = G \setminus (\cap_{f \in U}[K : f]) = \cup_{f \in U}(G \setminus [K : f]).$$

Let $\beta_K(G_N)$ denote the set $\{G \setminus [K : f] \mid \text{for all } f \in N\}$. Then $\beta_K(G_N)$ is indeed a *base* [8, p. 78] for the inverse K-topology in an N-module G. We write this as the following.

Proposition 3.9. *Let G be an N-module with inverse K-topology defined on G where K is an N-ideal of G. Let $\beta_K(G_N) = \{G \setminus [K : f] \mid \text{for all } f \in N\}$. Then $\beta_K(G_N)$ is a base for this topology.*

Proof. Assume O is a nonempty open set in the inverse K-topology. Say $O = G \setminus [K : U]$ for some $U \subseteq N$. Since O is nonempty, U cannot be an empty set. Let $a \in O$ be arbitrary. Then $a \in G \setminus [K : f]$ for some $f \in U$. Let $C = G \setminus [K : f] \in \beta_K(G_N)$. Since $[K : U] \subseteq [K : f]$ by Corollary 2.3. It follows that $a \in C \subseteq O$. Thus $\beta_K(G_N)$ is a base for the inverse K-topology by [8, Lemma 13.2]. ∎

Proposition 3.9 will be helpful in finding examples to demonstate the inverse K-topology. In the following, examples will be given to motivate further stuty in this topic.

Example 3.10. (Near modules over nearring of polynomials) Let $G = \mathbb{R}$ be the group of real numbers, and $N = (\mathbb{R}[x], +, \circ)$ the nearring of polynomials using the usual addition of polynomials and substitution as multiplication [1, 2]. Consider $\mathbb{R}$ as an $\mathbb{R}[x]$-module. The linear polynomial $\frac{b}{a}x \in \mathbb{R}[x]$ will map any nonzero a to any chosen $b \in \mathbb{R}[x]$. Thus $\mathbb{R}$ is a type-2 $\mathbb{R}[x]$-module. The zero-symmetric part $\mathbb{R}_0[x]$ is a nonring, so the hypothesis in Theorem 3.6 hold.

Observe that $[0 : -a + x] = \{a\}$ for all $a \in \mathbb{R}$. That is all singletons in $\mathbb{R}$ are 0-closed. Thus the inverse 0-topology is the *finite complement topology* [8, p. 77] on $\mathbb{R}$. Note that this is also the *Zariski topology* [3, p. 427] on $\mathbb{R}$.

Further, consider the reals $\mathbb{R}$ as an $\mathbb{R}_0[x]$-module. Using similar arguments as above, $\mathbb{R}$ is a type-2 $\mathbb{R}_0[x]$-module. Observe that $[0 : ax - x^2] = \{0, a\}$ for all $a \in \mathbb{R}$. Thus all finite subsets containing $\{0\}$ in $\mathbb{R}$ are 0-closed. In other words, the open sets in $\mathbb{R}$ are either $\mathbb{R}$, $\emptyset$ or those infinite subsets of $\mathbb{R}$ not containg $\{0\}$ and having finite complement.

Both topologies defined on $\mathbb{R}$ are T_1 but not T_2 and *irreducible* in the sense that any two nonempty open sets has nonempty intersection. In other words, the closure of any nonempty open set is $\mathbb{R}$.

Example 3.11. (Near modules over transformation nearrings) Let G be a group with order at least 3. Then the nearring of group mappings $M_0(G) = \{f : G \to G \mid 0f = 0\}$ is a nonring. Observe that the mapping $\alpha_b : G \to G$ such that $0\alpha_b = 0$ and $a\alpha_b = b$ for all $a \in G \setminus \{0\}$ is in $M_0(G)$. Thus $aM_0(G) = G$ for any nonzero $a \in G$. It follows that G is both a type-2 $M(G)$-module and $M_0(G)$-module. For any nonempty set C of G, define the mapping $\alpha_C\colon G \to G$ via $a\alpha_C = 0$ for all $a \in C$ and $b\alpha_C = d$ for some fixed nonzero $d \in G$ and for all $b \in G \setminus C$. Then $\alpha_C \in M(G)$ and $[0 : \alpha_C] = C$. Thus the inverse 0-topology in the $M(G)$-module G is the *discrete topology*. Further, observe that $\alpha_C \in M_0(C)$ if and only if $0 \in C$. Thus the nonempty 0-closed set in the $M_0(G)$-module G is a subset containing 0 and vice versa. When G is a simple nonabelian group, it is a type-2 $M_c(G)$-module. In this case, the inverse 0-topology is the *trivial topology*.

Proposition 3.9 shows that $\beta_K(G_N)$ is a base for the inverse K-topology. If the nearring N is generated by a subsemigroup S additively, we may ask: Is the set $\tau_K(G_N) = \{G \setminus [K : \alpha] \mid \text{for all } \alpha \in S\}$ a *subbase* for this topology? This is not true in general. Let G be a finite simple nonabelian group. Consider G as an $M_0(G)$-module, Example 3.11 shows that the nonempty 0-open sets are those subsets not containing 0. Observe that $M_0(G) = I(G)$ is generated additively by Inn(G), the group of inner automorphisms of G. But $[0 : \alpha] = \{0\}$ for all $\alpha \in$ Inn(G). Therefore $\tau_0(G_{M_0(G)}) = \{0\}$ which can not be the subbase for the inverse 0-topology.

Proposition 3.12. *Let N be a 0-symmetric nearring with unity. Assume G is a unital N-module with inverse 0-topology defined. Let A be a nonempty 0-closed subset of G. Then A contains no nonempty proper 0-closed subset if and only if $(0 : A)$ is a maximal right ideal of N.*

Proof. Since N contains unity, the right ideal $(0 : A) \neq N$. Let R be a right ideal of N containing $(0 : A)$. Then $[0 : R] \subseteq [0 : (0 : A)] = A$ by Proposition 2.4. Observe that $0 \in [0 : R]$ for N is 0-symmetric. Since A contains no nonempty proper 0-closed subset, $[0 : R] = A$. It follows that $R \subseteq (0 : A)$ and thus $(0 : A)$ is maximal.

Conversely, if A contains a nonempty proper 0-closed subset B, then $(0 : A) \subseteq (0 : B)$ by Corollary 2.5. If $(0 : A) = (0 : B)$, then $B = [0 : (0 : B)] = [0 : (0 : A)] = A$ by Proposition 2.4. Thus $(0 : B)$ properly contains $(0 : A)$. Since the N-module is unital, $1 \notin (0 : B)$. Consequently, $(0 : B)$ is a proper ideal of N and $(0 : A)$ can not be maximal. Hence result. ∎

According to Proposition 3.12, we call a nonempty 0-closed set *principle* if it contains no nonempty proper 0-closed subset. Recall that $a \in G$ is called a *generator* if $aN = G$. The following corollary provide a partial converse for Theorem 3.5.

Corollary 3.13. *Suppose N is a 0-symmetric nearring with unity, G a unital monogenic N-module with inverse 0-topology defined. Let $a \in G$ be a generator. Then the following are equivalent.*

(1) *The doubleton* $\{0, a\}$ *is principlly closed.*

(2) *The annihilator* $(0 : \{0, a\})$ *is a maximal right ideal of* N.

(3) *The annihilator* $(0 : a)$ *is a maximal right ideal of* N.

(4) *The* N*-module* G *is simple* $($*i.e.,* G *has no nontrivial* N*-ideals*$)$.

(5) G *is a type-0* N*-module.*

Proof. The equivalence of (1) and (2) follows from Proposition 3.12. Further, it is clear that $(0 : \{0, a\}) \subseteq (0 : a)$. If $f \in (0 : a)$, then $af = 0$ and $0f = 0$ for N is 0-symmetric. Thus $f \in (0 : \{0, a\})$. Hence $(0 : \{0, a\}) = (0 : a)$. This shows the equivalence of (2) and (3). Since a is a generator, $G \simeq N/(0 : a)$ as N-module. The equivalence of (3) and (4) follows immediately. For the last equivalence of (4) and (5), this follows the definition of type-0 N-module. ∎

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