TAIWANESE JOURNAL OF MATHEMATICS
Vol. 10, No. 4, pp. 1025-1045, June 2006
This paper is available online at http://www.math.nthu.edu.tw/tjm/

MULTI-VALUED OPERATORS AND FIXED POINT THEOREMS IN BANACH ALGEBRAS I

B. C. Dhage

Abstract. In this paper the multi-valued versions of some well-known hybrid fixed point theorems of Dhage [6, 7] in Banach algebras are proved. As an application, an existence theorem for a certain integral inclusion in Banach algebras is proved.

1. Introduction

Krasnoselskii initiated the study of hybrid fixed point theorems in Banach spaces by combining the well-known metric fixed point theorem of Banach [21] with the topological fixed point theorem of Schauder [21] which is known as the Krasnoselskii fixed point theorem in nonlinear analysis. There are several extensions and generalizations of Krasnoselskii fixed point theorem in the course of time. See Burton [3], O'Regan [18] and the references therein. Similarly another hybrid fixed point theorem similar to that of Krasnoselskii is proved by the present author [6] in Banach algebras and since then, several extensions and generalizations of this fixed point theorem have been proved. See Dhage [6-8] and the references therein. It is known that these hybrid fixed theorems have some nice applications to nonlinear integral equations of mixed type that arise in the inversion of certain nonlinear perturbed differential equations. See for example, Krasnoselskii [12], Zeidler [22], Dhage [8] and the references therein.

The fixed point theory for multi-valued mappings is an important topic of set-valued analysis. Several well-known fixed point theorems of single-valued mappings such as Banach and Schauder have been extended to multi-valued mappings in Banach spaces. The hybrid fixed point theorems of single-valued mappings are not exception. Very recently the multi-valued analogue of Krasnoselskii fixed point theorem is obtained by Petrusel [16]. In the present paper we shall prove some multi-valued analogues of a hybrid fixed point theorem of the present author [6].

Received July 17, 2004, revised December 8, 2004.
Communicated by Sen-Yen Show.
2000 *Mathematics Subject Classification*: 47H10, 34A60.
Key words and phrases: Multi-valued operator, Fixed point theorem and integral inclusion.

2. Preliminaries

Before proving our main hybrid fixed point theorems for multi-valued operators in Banach algebras, we give some useful definitions.

Definition 2.1. A mapping $T : X \to X$ is called **Lipschitz** if there exists a constant $\alpha > 0$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\| \tag{0.1}$$

for all $x, y \in X$.

Note that if $\alpha < 1$, then T is called a contraction on X with the contraction constant α, and if $\alpha = 1$, then T is called a nonexpansive mapping on X.

Let X be a metric space and let $T : X \to X$. Then T is called a **totally compact** operator if $\overline{T(X)}$ is a compact subset of X. T is called a **compact** operator if $\overline{T(S)}$ is a compact subset of X for any bounded subsets of X. Again T is called **totally bounded** if for any bounded subset S of X, $T(S)$ is a totally bounded subset of X. Further T is called **completely continuous** if it is continuous and compact. Note that every compact operator is totally bounded, but the converse may not be true. However, these two notions are equivalent on a bounded subset of a complete metric space X.

We shall be interested in the multi-valued analogues of the following hybrid fixed point theorem involving the product of two operators in Banach algebras.

Theorem 2.1. (Dhage [6]) *Let S be a closed, convex and bounded subset of a Banach algebra X and let $A : X \to X, B : S \to X$ be two operators such that*

- (a) *A is Lipschitz with a Lipschitz constant α,*
- (b) *B is completely continuous, and*
- (c) *$AxBy \in S$ for all $x, y \in S$.*

Then the operator equation

$$AxBx = x \tag{2.2}$$

has a solution, whenever $\alpha M < 1$, where $M = \|B(S)\| = \sup\{\|Bx\| : x \in S\}$.

Note that the above fixed point theorem involves the hypothesis of the complete continuity of the operator T, however, in the case of multi-valued operators we have the different types of continuities, namely, lower semi-continuity and upper semi-continuity. Here in this present work, we shall formulate the fixed point theorems for each of these continuity criteria. Below we give some preliminaries of the multi-valued analysis which will be needed in the sequel.

Let X be a Banach space and let $\mathcal{P}(X)$ denote the class of all subsets of X. Denote

$$\mathcal{P}_f(X) = \{A \subset X \mid A \text{ is non-empty and has a property } f\}.$$

Thus $\mathcal{P}_{bd}(X), \mathcal{P}_{cl}(X), \mathcal{P}_{cv}(X)$, $\mathcal{P}_{cp}(X)$, $\mathcal{P}_{cl,bd}(X)$, $\mathcal{P}_{cp,cv}(X)$ denote the classes of bounded, closed, convex , compact , closed-bounded and compact-convex subsets of X respectively. Similarly $\mathcal{P}_{cl,cv,bd}(X)$ and $\mathcal{P}_{cp,cv}(X)$ denote the classes of closed, convex and bounded and compact, convex subsets of X respectively. A correspondence $T : X \to \mathcal{P}_f(X)$ is called a multi-valued operator or multi-valued mapping on X. A point $u \in X$ is called a fixed point of T if $u \in Tu$. The multi-valued operator T is called **lower semi-continuous** (in short l.s.c.)if G is any open subset of X, then

$$T^{-1(w)}(G) = \{x \in X \mid Tx \cap G \neq \emptyset\}$$

is an open subset of X. Similarly the multi-valued operator T is called **upper semi-continuous** (in short u.s.c.) if the set

$$T^{-1}(G) = \{x \in X \mid Tx \subset G\}$$

is open in X for every open set G in X. Finally T is called continuous if it is lower as well as upper semi-continuous on X. A multi-valued map $T : X \to \mathcal{P}_{cp}(X)$ is called **totally compact** if $\overline{T(X)}$ is a compact subset of X. $T : X \to \mathcal{P}_{cp}(X)$ is called **compact** if $\overline{T(S)}$ is a compact subset of X for any bounded subsets of X. T is called **totally bounded** if for any bounded subset S of X, $T(S) = \bigcup_{x\in S} Tx$ is a totally bounded subset of X. It is clear that every compact multi-valued operator is totally bounded, but the converse may not be true. However the two notions are equivalent on a bounded subset of X. Finally T is called **completely continuous** if it is upper semi-continuous and compact on X.

For any $A, B \in P_f(X)$, let us denote

$$A \pm B = \{a \pm b \mid a \in A, b \in B\},$$

$$A \cdot B = \{ab \mid a \in A, b \in B\},$$

and

$$\lambda A = \{\lambda a \mid a \in A\}$$

for $\lambda \in \mathbb{R}$. Similarly denote

$$|A| = \{|a| \mid a \in A\}$$

and

$$\|A\| = \sup\{|a| \mid a \in A\}.$$

Let $A, B \in \mathcal{P}_{cl}(X)$ and let $a \in A$. Then by

$$D(a, B) = \inf\{\|a - b\| \mid b \in B\}$$

and

$$\rho(A, B) = \sup\{D(a, B) \mid a \in A\}.$$

The function $H : \mathcal{P}_{cl}(X) \times \mathcal{P}_{cl,bd}(X) \to \mathbf{R}^+$ defined by

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\}$$

is metric and is called the Hausdorff metric on X. It is clear that

$$H(0, C) = \|C\| = \sup\{\|c\| \mid c \in C\}$$

for any $C \in \mathcal{P}_{cl}(X)$.

Definition 2.2. Let $T : X \to \mathcal{P}_{cl}(X)$ be a multi-valued operator. Then T is called a multi-valued Lipschitz operator if there exists a constant $k > 0$ such that for each $x, y \in X$ we have

$$H(T(x), T(y)) \leq k\|x - y\|.$$

The constant k is called a Lipschitz constant of T. In particular if $k < 1$, then T is called a multi-valued contraction on X.

The following fixed point theorem for multi-valued contraction mappings appears in Covitz and Nadler [4].

Theorem 2.2. *Let (X, d) be a complete metric space and let $T : X \to \mathcal{P}_{cl}(X)$ be a multi-valued contraction. Then T has a fixed point.*

3. Multi-valued Fixed Point Theory

Before going to the main fixed point results, we state some lemmas useful in the sequel.

Lemma 3.1. (Lim [14] *Let (X, d) be a complete metric space and $T_1, T_2 : X \to \mathcal{P}_{bd,cl}(X)$ be two multi-valued contractions with the same contraction constant k. Then*

$$\rho(Fix(T_1), Fix(T_2) \leq \frac{1}{1-k} \sup_{x \in X} \rho(T_1(x), T_2(x)).$$

Lemma 3.2. *If* $A, B \in \mathcal{P}_{bd,cl}(X)$*, then* $H(AC, BC) \leq H(0, C)H(A, B)$

Proof. Let $x \in AC$ and $y \in BC$ be arbitrary. Then there exist $a \in A, b \in B$, and $c_1, c_2 \in C$ such that $x = ac_1$ and $y = bc_2$. Now

$$\begin{aligned} D(x, BC) &= \inf\{\|x - y\| \mid y \in BC\} \\ &= \inf\{\|x - bc_2\| \mid b \in B, c_2 \in C_2\} \\ &= \inf\{\|ac_1 - bc_2\| \mid b \in B, c_2 \in C_2\} \\ &\leq \inf\{\|ac_1 - bc_1\| + \|bc_1 - bc_2\| \mid b \in B, c_2 \in C_2\} \\ &\leq \inf\{\|a - b\|\,\|c_1\| + \|b\|\|c_1 - c_2\| \mid b \in B, c_2 \in C_2\} \\ &= \inf\{\|a - b\|\,\|c_1\| \mid b \in B\} \\ &= D(a, B)\|c_1\|. \end{aligned}$$

Again

$$\begin{aligned} \rho(AC, BC) &= \sup\{D(x, BC) \mid x \in A\} \\ &= \sup\{D(a, B)\|c_1\| \mid a \in A, c_1 \in C\} \\ &\leq \sup\{D(a, B)\|C\| \mid a \in A\} \\ &= \rho(A, B)\|C\| \\ &= \rho(A, B)H(0, C). \end{aligned}$$

Similarly

$$\rho(BC, AC) = \rho(B, A)H(0, C).$$

Hence

$$\begin{aligned} H(AC, BC) &= \max\{\rho(AC, BC), \rho(BC, AC)\} \\ &\leq \max\{\rho(A, B)H(0, C), \rho(B, A)H(0, C)\} \\ &= H(0, C)\max\{\rho(A, B), \rho(B, A)\} \\ &= H(0, C)H(A, B). \end{aligned}$$

The proof of the lemma is complete. ■

Lemma 3.3. *Let* $A : X \to \mathcal{P}_{bd}(X)$ *be a multi-valued Lipschitz operator. Then for any bounded subset* S *of* X*,* $A(S)$ *is bounded.*

Proof. Let S be a bounded subset of the Banach space X. Then there is constant $r > 0$ such that $\|x\| \leq r$ for all $x \in S$. Since A is Lipschitz, we have

$$\begin{aligned}\|Ax\| &\le H(Ax,0)\\ &\le H(Ax,A0)+H(A0,0)\\ &\le k\|x\|+\|A0\|\\ &\le kr+\|A0\|\\ &=\delta\end{aligned}$$

for all $x \in S$. Hence $A(S)$ is bounded. ■

Now we state a key result which is useful in the sequel.

Theorem 3.1. (Rybinski [20]) *Let S be a nonempty and closed subset of a Banach space X and let Y be a metric space. Assume that the multi-valued operator $F : S \times Y \to \mathcal{P}_{cl,cv}(S)$ satisfies*

(a) $H(F(x_1,y),F(x_2,y)) \le k\|x_1-x_2\|$, for each $(x_1,y),(x_2,y) \in S \times Y$,

(b) for every $x \in S$, $F(x,\cdot)$ is lower semi-continuous (briefly l.s.c.) on Y.

Then there exists a continuous mapping $f : S \times Y \to S$ such that $f(x,y) \in F(f(x,y),y)$ for each $(x,y) \in S \times Y$.

Theorem 3.2. *Let S be a closed convex and bounded subset of the Banach space X and let $A : S \to \mathcal{P}_{cl,cv,bd}(X)$, $B : S \to \mathcal{P}_{cp,cv}(X)$ be two multi-valued operators such that*

(a) A is multi-valued Lipschitz operator with Lipschitz constant k,

(b) B is l.s.c. and compact,

(c) $AxBy$ is a convex subset of S for each $x,y \in S$, and

(d) $Mk < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| \mid x \in S\}$.

Then the operator inclusion $x \in AxBx$ has a solution in S.

Proof. Define a multi-valued operator $T : S \times S \to \mathcal{P}_{cl,cv}(S)$ by

$$T(x,y) = AxBy, \tag{3.1}$$

for $x,y \in S$.. We show that $T(x,y)$ is multi-valued contraction in x for each fixed $y \in X$. Let $x_1,x_2 \in X$ be arbitrary. Then by Lemma 3.2,

$$\begin{aligned}H(T(x_1,y),T(x_2,y)) &= H(A(x_1)B(y),A(x_2)B(y))\\ &\le H(A(x_1),A(x_2))\,H(0,By)\\ &\le k\|x_1-x_2\|\;\|B(S)\|\\ &\le kM\|x_1-x_2\|.\end{aligned}$$

This shows that the multi-valued operator $T_y(.) = T(., y)$ is a contraction on S with a contraction constant kM. Hence an application of Covitz-Nadler fixed point theorem yields that the fixed point set

$$Fix(T_y) = \{x \in S|\ x \in A(x)B(y)\}$$

is nonempty and closed subset of S for each $y \in S$.

Now the operator $T(x, y)$ satisfies all the conditions of Theorem 3.1 and hence an application of it yields that there exists a continuous mapping $f : S \times S \to S$ such that $f(x, y) \in A(f(x, y)) + B(y)$. Let us define $C(y) = Fix(T_y)$, $C : S \to \mathcal{P}_{cl}(S)$. Let us consider the single-valued operator $c : S \to S$ defined by $c(x) = f(x, x)$, for each $x \in S$. Then c is a continuous mapping having the property that

$$c(x) = f(x, x) \in A(f(x, x)) + B(x) = A(c(x)) + B(x), \tag{3.2}$$

for each $x \in S$.

Now, we will prove that c is compact on S. To do this, it is sufficient to show that C is compact on S. By Lemma 3.3, there exists a constant $\delta > 0$ such that $\|Ax\| \le \delta$ for all $x \in S$. Let $\epsilon > 0$. Since B is compact on S, $B(S)$ is compact. Then there exists $Y = \{y_1, \ldots, y_n\} \subset X$ such that

$$\begin{aligned} B(S) &\subset \{w_1, \ldots, w_n\} + \mathcal{B}\Big(0, \frac{1 - Mk}{\delta}\epsilon\Big) \\ &\subset \bigcup_{i=1}^{n} B(y_i) + \mathcal{B}\Big(0, \frac{1 - Mk}{\delta}\epsilon\Big), \end{aligned}$$

where $w_i \in B(y_i)$, for each $i = 1, 2, \ldots, n$; and $\mathcal{B}(a, r)$ is an open ball in X centered at $a \in X$ of radius r. It, therefore, follows that, for each $y \in S$,

$$B(y) \subset \bigcup_{i=1}^{n} B(y_i) + \mathcal{B}\left(0, \frac{1 - Mk}{\delta}\epsilon\right)$$

and hence there exists an element $y_k \in Y$ such that

$$\rho(B(y), B(y_k)) < \frac{1 - Mk}{\delta}\epsilon.$$

Then

$$\begin{aligned}\rho(C(y), C(y_k)) &= \rho(Fix(T_y, Fix(T_{y_k}))) \\ &\leq \frac{1}{1-Mk} \sup_{x\in S} \rho(T_y(x), T_{y_k}(x)) \\ &= \frac{1}{1-Mk} \sup_{x\in S} \rho(A(x)B(y), A(x)B(y_k)) \\ &\leq \frac{1}{1-Mk} \sup_{x\in S} \rho(0, Ax)\rho(B(y), B(y_k)) \\ &< \frac{\delta}{(1-Mk)} \frac{(1-Mk)}{\delta}\epsilon \\ &= \epsilon.\end{aligned}$$

Thus for each $u \in C(y)$ there exists $v_k \in C(y_k)$ such that $\|u - v_k\| < \epsilon$. Hence, for each $y \in Y$, $C(y) \subset \bigcup_{1}^{n} \mathcal{B}(v_i, \epsilon)$, where $v_i \in C(y_i)$, $i = 1, 2, \ldots, n$. This further implies that $c(S) \subset C(S) \subset \bigcup_{1}^{n} \mathcal{B}(v_i, \epsilon)$, and so, c is a compact operator on S.

Finally, note that the mapping $c : S \to S$ satisfies all the assumptions of Schauder's fixed point theorem and hence c has a fixed point, that is, there is a point $u \in S$ such that $u = c(u)$. From (3.2) it follows that $u = c(u) \in A(c(u))Bu = AuBu$ This completes the proof. ■

Theorem 3.3. *Let S be a closed convex and bounded subset of the Banach space X and let $A : X \to \mathcal{P}_{bd,cl,cv}(X)$, $B : S \to \mathcal{P}_{cp,cv}(X)$ be two multi-valued operators such that*

(a) *A is multi-valued contraction ,*

(b) *B is l.s.c. and compact,*

(c) $AxBy$ is convex subset of X and $x \in AxBy \Rightarrow x \in S$ for all $y \in S$, and

(d) *$Mk < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| \mid x \in S\}$.*

Then the operator inclusion $x \in AxBx$ has a solution in S.

Proof. Let $y \in S$ be fixed and define the multi-valued operator $T_y : X \times S \to X$ by

$$T_y(x) = AxBy, \quad x \in X.$$

Then proceeding as in the proof of Theorem 3.2, it can be proved that T_y is a multi-valued contraction on X. Now an application of Theorem 2.2 yields that the fixed point set F_{T_y} of T_y is non-empty and closed in X. Thus we have

$$F_{T_y} = \{u \in X \mid u \in AuBy\} \subset X$$

is nonempty and closed for each $y \in S$. From hypothesis (c) it follows that $F_{T_y} \subset S$ for all $y \in S$.

Note that the function $T(x,y)$ satisfies all the conditions of Theorem 3.1 and hence an application of it yields that there is a continuous function $f : X \times S \to S$ satisfying

$$f(x,y) \in T(f(x,y),y) = A(f(x,y))By$$

for each $y \in S$. Now define a multi-valued operator $C : S \to S$ by $C(y) = F_{T_y}$. Consider the single-valued mapping $c : S \to X$ by

$$c(y) = f(y,y) \in A(f(y,y))Bx = A(c(y))By.$$

Clearly c is continuous and maps S into itself. Obviously $c(y) \in C(y)$ for each $y \in S$. Again proceeding with the arguments as in the proof of Theorem 3.2, it is proved that c is compact on S. Now the desired conclusion follows by an application of Schauder's fixed point principle to the mapping c on S. This completes the proof. ■

A Hausdorff measure of noncompactness χ of a bounded set S in X is a nonnegative real number $\chi(S)$ defined by

$$\chi(A) = \inf\left\{ r > 0 : A = \bigcup_{i=1}^{n} \mathcal{B}(x_i, r),\ x_i \in A \right\}. \tag{3.3}$$

The function χ enjoys the following properties:

(χ_1) $\chi(S) = 0 \Leftrightarrow S$ is precompact.

(χ_2) $\chi(S) = \chi(\overline{S}) = \chi(\overline{co}\, S)$, where $\overline{S}$ and $\overline{co}\, S$ denote respectively the closure and the closed convex hull of S.

(χ_3) $S_1 \subset S_2 \Rightarrow \chi(S_1) \leq \chi(S_2)$

(χ_4) $\chi(S_1 \cup S_2) = \max\{\chi(S_1), \chi(S_2)\}$.

(χ_5) $\chi(\lambda S) = |\lambda| \chi(S), \forall \lambda \in \mathbf{R}$.

(χ_6) $\alpha(S_1 + S_2) \leq \chi(S_1) + \chi(S_2)$.

The details of measures of noncompactness and their properties appear in Deimling [5] and Zeidler [22].

Definition 3.1. A mapping $T : X \to X$ is called **condensing** if for any bounded subset A of X, $T(A)$ is bounded and

$$\chi(T(S)) < \chi(A),\ \ \chi(S) > 0.$$

Note that contraction and completely continuous mappings are condensing but the converse may not be true. The following fixed point theorem for condensing multi-valued mappings is well-known. See Hu and Papageorgiou [10] and the references therein.

Theorem 3.4. *Let S be a closed convex and bounded subset of a Banach algebra X and let $T : S \to \mathcal{P}_{cl,cv}(S)$ be a upper semi-continuous and χ-condensing multi-valued operator. Then T has a fixed point point.*

We need the following result useful in the sequel.

Lemma 3.4. (Banas [2]) *If $S_1, S_2 \in \mathcal{P}_{bd}(X)$, then*

$$\chi(S_1.S_2) \leq \chi(S_1)\|S_2\| + \chi(S_2)\|S_1\|.$$

Theorem 3.5. *Let X be a Banach algebra and let $A : S \to \mathcal{P}_{cl,cv,bd}(X)$ and $B : S \to \mathcal{P}_{cp,cv}(X)$ be two multi-valued operators satisfying*

(a) *A is Lipschitz with a Lipschitz constant k,*

(b) *B is compact and upper semi-continuous,*

(c) *$AxBx$ is a convex subset of S for each $x \in S$, and*

(d) *$Mk < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| \mid x \in S\}$.*

Then the operator inclusion $x \in Ax\,Bx$ has a solution.

Proof. Define a mapping $T : S \to \mathcal{P}_{cl,cv,bd}(S)$ by

$$Tx = Ax\,Bx,\ x \in S. \tag{3.4}$$

We shall show that T satisfies all the conditions of Theorem 2.2 on S.

Step I. First we claim that T defines a multi-valued map $T : S \to \mathcal{P}_{cp,cv}(S)$. Obviously Tx is convex subset of S for each $x \in S$. From Lemma 3.4 it follows that

$$\chi(Tx) = \chi(Ax \cdot Bx) \leq \chi(Ax).\|B(x)\| + \chi(Bx).\|A(x)\| = 0$$

for every $x \in S$ and the claim follows.

Step II. Now we shall show that the mapping T is an upper semi-continuous on S. Since S is bounded set in X and A is multi-valued Lipschitz operator, we have by hypothesis (a), there exists a constant $\delta > 0$ such that $\|Ax\| \leq \delta$ for all $x \in S$.

Let $\{x_n\}$ be a sequence in S converging to the point $x^* \in S$ and let $\{y_n\}$ be sequence defined by $y_n \in Tx_n$ converging to the point y^*. It is enough to prove that $y^* \in Tx^*$. Now for any $x, y \in S$, we have

$$
\begin{aligned}
H(Tx, Ty) &= H(AxBx, AyBy) \\
&\leq H(AxBx, AyBx) + H(AyBx, AyBy) \\
&\leq H(Ax, Ay)H(0, Bx) + H(0, Ay)H(Bx, By) \\
&\leq k\|x - y\|\|B(S)\| + \delta H(Bx, By) \\
&\leq Mk\|x - y\| + \delta H(Bx, By)
\end{aligned} \tag{3.5}
$$

Since B is u.s.c., it is H-upper semi-continuous and consequently one has

$$H(Bx_n, Bx^*) \to 0 \quad \text{whenever} \quad x_n \to x^*.$$

Therefore

$$
\begin{aligned}
D(y^*, Tx^*) &\leq \lim_{n\to\infty} D(y_n, Tx^*) \\
&\leq H(Tx_n, Tx^*) \\
&\leq Mk\|x_n - x^*\| + \delta H(Bx_n, Bx^*) \\
&\longrightarrow 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

This shows that the multi-valued operator T is an upper semi-continuous on S.

Step III. Finally we show that that T is χ-condensing on S. Since B is compact, $B(S)$ is a precompact subset of X. Let $\epsilon > 0$ be given. Then there exists a set $Z = \{x_1, \ldots, x_n\}$ in X such that

$$
\begin{aligned}
B(S) &\subset \bigcup_{i=1}^{n} \mathcal{B}\left(y_i, \frac{\epsilon}{\delta}\right) \\
&\subset \{y_1, \ldots, y_n\} + \mathcal{B}\left(0, \frac{\epsilon}{\delta}\right) \\
&\subset \bigcup_{i=1}^{n} \mathcal{B}\left(Bx_i, \frac{\epsilon}{\delta}\right)
\end{aligned}
$$

for some $y_i \in Bx_i$, for $i = 1, \ldots, n$. Therefore for every $x \in S$, there exists an $x_i \in Z$ such that

$$\rho(Bx, Bx_i) < \frac{\epsilon}{\delta}.$$

Let $\chi(S) = r$. Then we have

$$S \subset \bigcup_{i=1}^{m} \mathcal{B}(x_i, r + \epsilon).$$

Now for any $x \in S$ we have

$$\rho(Tx, Tx_i) \leq H(Tx, Tx_i) \tag{3.6}$$
$$\leq Mk\|x - x_i\| + \delta H(Bx, Bx_i)$$
$$< Mk\|x - x_i\| + \frac{\epsilon}{2}$$
$$\leq Mk(r + \epsilon) + \frac{\epsilon}{2}$$

for each $i = 1, \dots, n$. Again each Tx_i is compact for each $i =, \dots, n$, there are $y_1^i, \dots, y_{n(i)}^i$ in Tx_i such that

$$Tx_i \subset \bigcup_{j=1}^{n(i)} \mathcal{B}\left(y_j^i, \frac{\epsilon}{2}\right).$$

Now from (3.5) it follows that

$$T(S) \subset \bigcup_{i=1}^{n} \left\{ \bigcup_{j=1}^{n(i)} \left\{ \mathcal{B}(y_j^i, Mk(r+\epsilon)+\epsilon) \right\} \right\}.$$

Therefore

$$\chi(T(S)) < Mk(r + \epsilon) + \epsilon.$$

Since ϵ is arbitrary, one has

$$\chi(T(S)) \leq Mk\,r = Mk\,\chi(S) < \chi(S)$$

whenever $\chi(S) > 0$. This shows that T is χ-condensing on S into itself. Now an application of Theorem 3.4 yields that T has a fixed point. This further implies that the operator inclusion $x \in Ax\,Bx$ has a solution. This completes the proof. ∎

4. Differential Inclusions

In this section we consider the differential inclusions for proving the existence theorems by an application of the abstract results of the previous section under generalized Lipschitz and Carathéodory conditions.

Given a closed and bounded interval $J = [0, a]$ in $\mathbb{R}$ for some $a \in \mathbb{R}$, $a > 0$, consider the differential inclusion (in short DI)

$$\begin{cases} \left(\dfrac{x(t)}{f(t, x(t))}\right)' \in G(t, x(t)) \quad \text{a.e.} \quad t \in J \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{4.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R} - \{0\}$ is continuous and $G : J \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$.

By a solution to DI (4.1) we mean a function $x \in AC(J, \mathbb{R})$ that satisfies

(i) the function $t \mapsto \frac{x(t)}{f(t,x(t))}$ is differentiable, and

(ii) $\left(\frac{x(t)}{f(t,x(t))}\right)' = v(t),\ t \in J$ for some $v \in L^1(J, \mathbb{R})$ satisfying $v(t) \in G(t, x(t))$ a.e. $t \in J$,

where $AC(J, \mathbb{R})$ is the space of all absolutely continuous real-valued functions on J.

The DI (4.1) is new in the theory of differential inclusions and the special cases of it have been discussed in the literature extensively. For example, if $f(t, x) = 1$, then the DI (4.1) reduces to DI

$$\left\{ \begin{array}{l} x' \in G(t, x) \quad \text{a.e.} \quad t \in J \\ x(0) = x_0 \in \mathbb{R}. \end{array} \right. \tag{4.2}$$

There is a considerable work available in the literature for the DI (4.2). See Aubin and Cellina [1], Deimling [5] and Hu and Papageorgiou [10] etc. Similarly, in the special case when $G(t, x) = \{g(t, x)\}$, we obtain the differential equation

$$\left\{ \begin{array}{l} \left(\dfrac{x(t)}{f(t,x(t))}\right)' = g(t,x) \quad \text{a.e.} \quad t \in J \\ x(0) = x_0 \in \mathbb{R}. \end{array} \right. \tag{4.3}$$

The differential equation (4.3) has been studied recently in Dhage and O'Regan [18] and Dhage [9] for the existence of solutions. Therefore it of interest to discuss the the DI (4.3) for various aspects of its solution under some suitable conditions. In this section we shall prove the existence of the solution of DI (4.3) under the mixed generalized Lipschitz and Carathéodoty conditions.

Define a norm $\|\cdot\|$ in $C(J, \mathbb{R})$ by $\|x\| = \sup_{t \in J} |x(t)|$. Again define a multiplication " $\cdot$ " by $(x.y)(t) = x(t)y(t)$ for all $t \in J$. Then $C(J, \mathbb{R})$ is a Banach algebra with respect to the above norm and multiplication in it.

We need the following definitions in the sequel.

Definition 4.1. A multi-valued map $F : J \to \mathcal{P}_f(\mathbb{R})$ is said to be measurable if for any $y \in X$, the function $t \to d(y, F(t)) = inf\{|y - x| : x \in F(t)\}$ is measurable.

Definition 4.2. A measurable multi-valued function $F : J \to \mathcal{P}_{cp}(\mathbb{R})$ is said to be integrably bounded if there exists a function $h \in L^1(J, \mathbb{R})$ such that $\|v\| \leq h(t)$ a.e. $t \in J$ for all $v \in F(t)$.

Remark 4.1. It is known that if $F : J \to \mathcal{P}_{cp}(\mathbf{R})$ is a an integrably bounded multi-valued operator , then the set S^1_F of all Lebesgue integrable selections of F is closed and non-empty. See Hu and Papageorgiou [10].

Definition 4.3. A multi-valued function $\beta : J \times \mathbf{R} \to \mathcal{P}_{cp}(\mathbf{R})$ is called Carathéodory if

(i) $t \to \beta(t, x)$ is measurable for each $x \in E$, and

(ii) $x \to \beta(t, x)$ is an upper semi-continuous almost everywhere for $t \in J$.

Definition 4.4. A Carathéodory multi-valued operator $\beta(t, x)$ is called L^1_X-Carathéodory if there exists a function $h \in L^1(J, \mathbf{R})$ such that

$$\|\beta(t, x)\| \leq h(t) \quad \text{a.e.} \quad t \in J$$

for all $x \in \mathbf{R}$, and the function h is called a growth function of β on $J \times \mathbf{R}$.

Denote

$$S^1_\beta(x) = \{v \in L^1(J, E) \mid v(t) \in \beta(t, x(t)) \text{ a.e. } t \in J\}.$$

Then we have the following lemmas due to Lasota and Opial [13].

Lemma 4.1. *Let E be a Banach space. If dim$(E) < \infty$ and $\beta : J \times E \to \mathcal{P}_{cp}(E)$ is L^1-Carathéodory, then $S^1_\beta(x) \neq \emptyset$ for each $x \in E$.*

Lemma 4.2. *Let E be a Banach space, β a Carathéodory multi-valued operator with $S^1_\beta \neq \emptyset$ and let $\mathcal{L} : L^1(J, E) \to C(J, E)$ be a continuous linear mapping. Then the operator*

$$\mathcal{L} \circ S^1_\beta : C(J, E) \to \mathcal{P}_{bd,cl}(C(J, E))$$

is a closed graph operator on $C(J, E) \times C(J, E)$.

We consider the following hypotheses in the sequel.

(H_1) The function f is bounded on $J \times \mathbf{R} \to \mathbf{R}$ with bound K.

(H_2) The function $f : J \times \mathbf{R} \to \mathbf{R} - \{0\}$ is continuous and there exists a bounded function $\ell : J \to \mathbf{R}$ with bound $\|\ell\|$ satisfying

$$|f(t, x) - f(t, y)| \leq \ell(t)|x - y| \quad \text{a.e.} \quad t \in J$$

for all $x, y \in \mathbf{R}$.

(H_3) The multi-valued operator $G : J \times \mathbf{R} \to \mathcal{P}_{cp,cv}(\mathbf{R})$ is L^1_X-Carathéodory with growth function h.

Theorem 4.1. *Assume that the hypotheses (H_1)-(H_3) hold. Further if*

$$\|\ell\| \left(\left| \frac{x_0}{f(0, x_0)} \right| + \|h\|_{L^1} \right) < 1, \tag{4.4}$$

then the DI (4.1) has a solution on J.

Proof. Let $X = C(J, \mathbf{R})$. Define a subset S of X by

$$S = \{x \in X \mid \|x\| \leq M\}, \tag{4.5}$$

where

$$M = K \left(\left| \frac{x_0}{f(0, x_0)} \right| + \|h\|_{L^1} \right).$$

Clearly S is a closed, convex and bounded subset of the Banach algebra X. Consider the two multi-valued mappings A and B on S defined by

$$Ax(t) = f(t, x(t)) \tag{4.6}$$

and

$$Bx(t) = \left\{ u \in X \mid u(t) = \frac{x_0}{f(0, x_0)} + \int_0^t v(s)\, ds, \quad v \in S_G^1(x) \right\} \tag{4.7}$$

for all $t \in J$.

Then the DI (4.1) is equivalent to the operator inclusion

$$x(t) \in Ax(t)Bx(t), \ \ t \in J. \tag{4.8}$$

We will show that the multi-valued operators A and B satisfy all the conditions of Theorem 3.2. Clearly the operator B is well defined since $S_G^1(x) \neq \emptyset$ for each $x \in X$.

Step I. We first show that the operators A and B define the multi-valued operators $A : S \to \mathcal{P}_{cl,cv,bd}(X)$ and $B : S \to \mathcal{P}_{cp,cv}(X)$. The claim concerning A is obvious, since it is a single-valued operator on S. First, we show that B has compact values on S. Observe that the operator B is equivalent to the composition $\mathcal{L} \circ S_G^1$ of two operators on $L^1(J, \mathbf{R})$, where $\mathcal{L} : L^1(J, \mathbf{R}) \to X$ is the continuous operator defined by

$$\mathcal{L}v(t) = \frac{x_0}{f(0, x_0)} + \int_0^t v(s)\, ds. \tag{4.9}$$

To show B has compact values, it then suffices to prove that the composition operator $\mathcal{L} \circ S_G^1$ has compact values on S. Let $x \in S$ be arbitrary and let $\{v_n\}$

be a sequence in $S^1_G(x)$. Then, by the definition of S^1_G, $v_n(t) \in G(t, x(t))$ a.e. for $t \in J$. Since $G(t, x(t))$ is compact, there is a convergent subsequence of $v_n(t)$ (for simplicity call it $v_n(t)$ itself) that converges in measure to some $v(t)$, where $v(t) \in G(t, x(t))$ a.e. for $t \in J$. From the continuity of $\mathcal{L}$, it follows that $\mathcal{L}v_n(t) \to \mathcal{L}v(t)$ pointwise on J as $n \to \infty$. In order to show that the convergence is uniform, we first show that $\{\mathcal{L}v_n\}$ is an equi-continuous sequence. Let t, $\tau \in J$; then

$$\begin{aligned} |\mathcal{L}v_n(t) - \mathcal{L}v_n(\tau)| &\leq \left| \int_0^t v_n(s)\, ds - \int_0^\tau v_n(s)\, ds \right| \\ &\leq \left| \int_t^\tau |v_n(s)|\, ds \right|. \end{aligned} \tag{4.10}$$

Since $v_n \in L^1(J, \mathbf{R})$, so the right hand side of 3.14 tends to 0 as $t \to \tau$. Hence, the sequence $\{\mathcal{L}v_n\}$ is equi-continuous, and an application of the Ascoli theorem implies that there is a uniformly convergent subsequence. We then have $\mathcal{L}v_{n_j} \to \mathcal{L}v \in (\mathcal{L} \circ S^1_G)(x)$ as $j \to \infty$, and so $(\mathcal{L} \circ S^1_G)(x)$ is compact. Therefore, B is a compact-valued multi-valued operator on X.

Again let $u_1, u_2 \in Bx$. Then there are $v_1, v_2 \in S^1_G(x)$ such that

$$u_1(t) = \frac{x_0}{f(0, x_0)} + \int_0^t v_1(s)\, ds,\ t \in J,$$

and

$$u_2(t) = \frac{x_0}{f(0, x_0)} + \int_0^t v_2(s)\, ds,\ t \in J.$$

Now for any $\gamma \in [0, 1]$,

$$\begin{aligned} \gamma u_1(t) + (1-\gamma)u_2(t) &= \gamma \left(\frac{x_0}{f(0, x_0)} + \int_0^t v_1(s)\, ds \right) \\ &\quad + (1-\gamma) \left(\frac{x_0}{f(0, x_0)} + \int_0^t v_2(s)\, ds \right) \\ &= \frac{x_0}{f(0, x_0)} + \int_0^t [\gamma v_1(s) + (1-\gamma)v_2(s)]\, ds \\ &= \frac{x_0}{f(0, x_0)} + \int_0^t v(s)\, ds \end{aligned}$$

where $v(t) = \gamma v_1(t) + (1-\gamma)v_2(s) \in G(t, x)$ for all $t \in J$. Hence $\gamma u_1 + (1-\gamma)u_2 \in Bx$ and consequently Bx is convex for each $x \in X$. As a result B defines a multi-valued operator $B : X \to \mathcal{P}_{cp,cv}(X)$.

Step II. Next we show A a multi-valued Lipschitz operator on S. Let $x, y \in S$. Then

$$\begin{aligned}\|Ax - Ay\| &= \sup_{t\in J} |Ax(t) - Ay(t)| \\ &= \sup_{t\in J} |f(t, x(t)) - f(t, y(t))| \\ &\leq \sup_{t\in J} \ell(t)|x(t) - y(t)| \\ &\leq \|\ell\|\|x - y\|,\end{aligned}$$

showing that A is a multi-valued Lipschitz operator on S.

Step III. Next we show that B is completely continuous on S. Let S be a bounded subset of X. Then there is a constant $r > 0$ such that $\|x\| \leq r$ for all $x \in S$. First we prove that B is compact on S. To do this, it is enough to prove that $B(S)$ is a uniformly bounded and equi-continuous set in X. To see this, let $u \in B(S)$ be arbitrary. Then there is a $v \in S^1_G(x)$ such that

$$u(t) = \frac{x_0}{f(0, x_0)} + \int_0^t v(s)\, ds.$$

for some $x \in S$. Hence

$$\begin{aligned}|u(t)| &\leq \left|\frac{x_0}{f(0, x_0)}\right| + \int_0^t |v(s)|\, ds \\ &\leq \left|\frac{x_0}{f(0, x_0)}\right| + \int_0^t \|G(s, x(s))\|\, ds \\ &\leq \left|\frac{x_0}{f(0, x_0)}\right| + \int_0^t h(s) ds \\ &= \left|\frac{x_0}{f(0, x_0)}\right| + \|h\|_{L^1}\end{aligned}$$

for all $x \in S$ and so $B(S)$ is a uniformly bounded set in X. Again as in step I , it is proved that

$$|u(t) - u(\tau)| \leq |p(t) - p(\tau)|$$

where $p(t) = \displaystyle\int_0^t h(s) ds$.

Notice that p is a continuous function on J, so it is uniformly continuous on J. As a result we have that

$$|u(t) - u(\tau)| \to 0 \text{ as } t \to \tau.$$

This shows that $B(S)$ is a equi-continuous set in X. Next we show that B is a upper semi-continuous multi-valued mapping on X. Let $\{x_n\}$ be a sequence in S such that $x_n \to x_*$. Let $\{y_n\}$ be a sequence such that $y_n \in Bx_n$ and $y_n \to y_*$. We shall show that $y_* \in Bx_*$. Since $y_n \in Bx_n$, there exists a $v_n \in S^1_G(x_n)$ such that

$$y_n(t) = \frac{x_0}{f(0, x_0)} + \int_0^t v_n(s)\, ds, \;\; t \in J.$$

We must prove that there is a $v_* \in S^1_G(x_*)$ such that

$$y_*(t) = \frac{x_0}{f(0, x_0)} + \int_0^t v_*(s)\, ds, \;\; t \in J.$$

Consider the continuous linear operator $\mathcal{K} : L^1(J, \mathbf{R}) \to C(J, \mathbf{R})$ defined by

$$\mathcal{K}y(t) = \int_0^t v(s)\, ds, \;\; t \in J.$$

Now

$$\left\| \left(y_n - \frac{x_0}{f(0, x_0)} \right) - \left(y_* - \frac{x_0}{f(0, x_0)} \right) \right\| \to 0 \text{ as } n \to 0.$$

From lemma 4.2, it follows that $\mathcal{K} \circ S^1_G$ is a closed graph operator. Also from the definition of K we have

$$y_n(t) - \frac{x_0}{f(0, x_0)} \in \mathcal{K} \circ S^1_G(x_n).$$

Since $y_n \to y_*$, there is a point $v_* \in S^1_G(x_*)$ such that

$$y_*(t) = \frac{x_0}{f(0, x_0)} + \int_0^t v_*(s) ds, \;\; t \in J.$$

This shows that B is a upper semi-continuous operator on X. Thus B is an upper semi-continuous and compact and hence completely continuous multi-valued operator on X.

Step IV. Here we show that $AxBx$ is a convex subset of S for each $x \in S$. Let $x \in S$ be arbitrary and let $w, y \in S$. Then there are $u, v \in S_G(x)$ such that

$$w = [f(t, x(t))] \left(\frac{x_0}{f(0, x_0)} + \int_0^t u(s)\, ds \right)$$

and

$$y = [f(t, x(t))] \left(\frac{x_0}{f(0, x_0)} + \int_0^t v(s)\, ds \right).$$

Now for any $\lambda \in [0,1]$,

$$\begin{aligned}\lambda y + (1-\lambda)w &= \lambda[f(t,x(t))]\left(\frac{x_0}{f(0,x_0)} + \int_0^t v(s)\,ds\right)\\ &\quad +(1-\lambda)[f(t,x(t))]\left(\frac{x_0}{f(0,x_0)} + \int_0^t v(s)\,ds\right)\\ &= [f(t,x(t))]\left(\lambda\frac{x_0}{f(0,x_0)} + \int_0^t \lambda v(s)\,ds\right)\\ &\quad +[f(t,x(t))]\left((1-\lambda)\frac{x_0}{f(0,x_0)} + \int_0^t (1-\lambda)v(s)\,ds\right)\\ &= [f(t,x(t))]\left(\frac{x_0}{f(0,x_0)} + \int_0^t [\lambda u(s) + (1-\lambda)v(s)]\,ds\right).\end{aligned}$$

Since $G(t,x(t))$ is convex, $z = \lambda y + (1-\lambda)w \in G(t,x(t))$ for all $t \in J$ and so $z \in S^1_G(x)$. As a result $\lambda y + (1-\lambda)w \in AxBx$. Hence $AxBx$ is a convex subset of X. Again we have

$$\begin{aligned}|w(t)| &= |f(t,x(t))|\left(\left|\frac{x_0}{f(0,x_0)}\right| + \int_0^t |v(s)|\,ds\right)\\ &\le K\left(\left|\frac{x_0}{f(0,x_0)}\right| + \int_0^t h(s)\,ds\right)\\ &\le M,\end{aligned}$$

for all $t \in J$ and so, $w \in S$. Therefore $AxBx$ is a convex subset of S for each $x \in S$.

Step V. Finally from condition (4.1) it follows that

$$Mk = \|\ell\|\left(\left|\frac{x_0}{f(0,x_0)}\right| + \|h\|_{L^1}\right) < 1.$$

Thus A and B satisfy all the conditions of Theorem 4.1 and hence an application of it yields that the operator inclusion $x \in AxBx$ has a solution. Consequently the DI (4.1) has solution on J. This completes the proof. ■

Example 4.1. Let $J = [0,1]$ denote a closed and bounded interval in $\mathbb{R}$ and define a function $f : J \times \mathbb{R} \to \mathbb{R}$ by

$$f(t,x) = \begin{cases} 1 & \text{if} \quad -\infty < x \le 0,\\ 1+x & \text{if} \quad 0 \le x \le 1,\\ 2 & \text{if} \quad x \ge 1.\end{cases}$$

for all $t \in J$. Now consider the DI

$$\begin{cases} \left(\dfrac{x(t)}{f(t, x(t))}\right)' \in G(t, x(t)) \text{ a.e. } \quad t \in J, \\ x(0) = \dfrac{1}{2} \end{cases} \tag{4.11}$$

where $p : J \to \mathbb{R}$ is Lebesgue integrable, and $G : J \times \mathbb{R} \to P_f(\mathbb{R})$ is given by

$$G(t, x) = \begin{cases} p(t) \text{ if } x < 0 \\ [e^{-x}p(t), \, p(t)] \text{ if } x \geq 0 \end{cases}$$

Clearly the function $f(t, x)$ is continuous and bounded on $J \times \mathbb{R}$ with bound 2. Again it is also Lipschitz with a Lipschitz constant 1. Also it follows that G is L^1_X-Carathéodory with $h(t) = p(t), \ t \in J$. Therefore if $\|p\|_{L^1} < \frac{1}{2}$, then the DI (4.9) has a solution on J.

Acknowledgment

The author is thankful to the referee for giving some useful suggestions for the improvement of this paper.

References

1. J. Aubin and A. Cellina, *Differential Inclusions*, Springer-Verlag, 1984.
2. J. Banas and M. Lecho, Fixed points of the product of operators in Banach algebras, *PanAmerican Math. J.*, **12** (2002), 101-109.
3. T. A. Burton , A fixed point theorem of Krasnoselskii, *Appl. Math. Lett.*, **11** (1998), 85-88.
4. H. Covitz, S. B. Nadler jr., Multivalued contraction mappings in generalized metric spaces, *Israel J. Math.*, **8** (1970), 5-11.
5. K. Deimling, *Multivalued differential equations*, W. de Gruyter, 1992.
6. B. C. Dhage, On some variants of Schauder's fixed point principle and applications to nonlinear integral equations, *J. Math. Phy. Sci.*, **25** (1988), 603-611.
7. B. C. Dhage, On a fixed point theorem in Banach algebras with aplications, *Appl. Math. Lett.*, **18** (2005), 273-280.
8. B. C. Dhage, A hybrid fixed point theorem in Banach algebras with applications, *Comm. Appl. Nonlinear Anal.*, **13** (2006), 71-84.
9. B. C. Dhage, Multi-valued mappings and fixed points I, *Nonlinear Functional Anal. & Appl.*, **10(3)** (2005), 359-378.

10. S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis , Vol. I: Theory*, Kluwer Academic Publishers, Dordrechet/Boston/London, 1997.

11. M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer Acad. Publ., Dordrecht, 1991.

12. M. A. Krasnoselskii, *Topological Methods in The Theory of Nonlinear Integral Equations*, Pergamon Press, 1964.

13. A. Lasota and Z. Opial, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations, *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys.*, **13** (1965), 781-786.

14. T. C. Lim, On fixed point stability for set-valued contraction mappings with applications to generalized differential equations, *J. Math. Anal. Appl.*, **110** (1985), 436-441.

15. A. Petruşel, A generalization of Krasnoselskii's fixed point theorem, *Proc. Sci. Comm. Meeting of "Aurel Vlaicu" Univ. Arad*, **14A** (1996), 109-112.

16. A. Petruşel, Multivalued operators and fixed points, *P.U.M.A.*, **9** (1998), 165-170.

17. W. V. Petryshyn and P. M. Fitzpatrtik, A degree theory, fixed point theorems, and mappings theorems for multi-valued noncompact mappings, *Trans. Amer. Math. Soc.*, **194** (1974), 1-25.

18. D. O'Regan, New fixed point results for 1-set contractive set-valued maps, *Computes Math. Appl.*, **35(4)** (1998), 27-34.

19. I. A. Rus, Technique of the fixed point structures for multivalued mappings, *Math. Japonica*, **38** (1993), 289-296.

20. L. Rybinski, An application of the continuous selection theorem to the study of the fixed points of multivalued mappings, *J. Math. Anal. Appl.*, **153** (1990), 391-396.

21. D. R. Smart, *Fixed Point Theorems*, Cambridge Univ. Press, 1980.

22. E. Zeidler, *Nonlinear Functional Analysis and Its Applications: Part I,* Springer-Verlag, 1985.

B. C. Dhage
Kasubai, Gurukul Colony,
Ahmedpur-413 515,
Dist: Latur, Maharashtra,
India
E-mail: bcd20012001@yahoo.co.in